Multibump Solutions for Duffing-like Systems

Simonetta Abenda

Paolo Caldiroli

Piero Montecchiari (*)

Sommario. - *Si studia il problema di esistenza di soluzioni omocline di un sistema Hamiltoniano del secondo ordine asintoticamente periodico: trovare* $q \in C^2(\mathbf{R}, \mathbf{R}^N) \setminus \{0\}$ *tale che:*

$$\ddot{q} = q - V'(t,q) \ , \quad q(t) \to 0 \ \ e \ \dot{q}(t) \to 0 \ \ per \ \ t \to \pm\infty \qquad \text{(HS)}$$

dove si assume che l'origine è un massimo locale per il corrispondente potenziale, uniformemente nel tempo, e che V' *è asintotico per* $t \to \pm\infty$ *a delle funzioni* $V'_{\pm}$ *periodiche e superquadratiche. Proviamo, via metodi variazionali che se le varietà stabile e instabile associate all'origine di uno dei problemi all'infinito hanno intersezione numerabile allora il problema (HS) ha infinite soluzioni omocline di tipo multibump.*

Summary. - *We study the problem of existence of homoclinic solutions of a second order asymptotically periodic Hamiltonian system: find* $q \in C^2(\mathbf{R}, \mathbf{R}^N) \setminus \{0\}$ *such that:*

$$\ddot{q} = q - V'(t,q) \ , \quad q(t) \to 0 \ \ \text{and} \ \dot{q}(t) \to 0 \ \ \text{as} \ \ t \to \pm\infty \qquad \text{(HS)}$$

where it is assumed that the origin is a local maximum for the corresponding potential, uniformly in time, and that V' *is asymptotic, as* $t \to \pm\infty$*, to time periodic and superquadratic functions* $V'_{\pm}$*. We prove via variational methods that if the stable and unstable manifolds associated to the origin of one of the systems at infinity have countable intersection then the problem (HS) has infinitely many homoclinic solutions of multibump type.*

(*) Indirizzo degli Autori: S. Abenda: Dipartimento di Matematica, Università di Bologna, Piazza Porta S. Donato 5, 40126 Bologna (Italy). P. Caldiroli: S. I. S. S. A, International School for Advanced Studies, Via Beirut 2-4, 34100 Trieste (Italy). P. Montecchiari: Dipartimento di Scienze Matematiche, Piazzale Europa 1, 34100 Trieste (Italy).

Key words: Hamiltonian systems, homoclinic orbits, multibump solutions, minimax argument.

1. Introduction

In recent years, starting with [10] and [14], variational methods have been successfully applied to study the existence of homoclinic solutions to Hamiltonian systems having a hyperbolic rest point.

In this paper we apply variational methods, inspired by those developed in [30] (see also [12, 15, 29]), to study a class of asymptotically periodic Hamiltonian systems.

To present our results we start by describing them in a very particular case which we think interesting in its own. We consider the following Duffing-like equation:

$$\ddot{q} = q - a(t)\,(1 + \epsilon\cos(\omega(t)t))\,q^3 \tag{1.1}$$

where $\epsilon \in \mathbf{R}$, $a(t)$ and $\omega(t)$ are smooth real functions. The dynamics of the equation (1.1) is well known in the periodic case, i.e. when $a(t) \equiv a_0 > 0$ and $\omega(t) \equiv \omega_0 \neq 0$, and can be exhaustively described using a perturbative approach based on the Melnikov theory and on the Smale–Birkhoff homoclinic theorem (see [16, 23, 34]).

The results contained in our work apply in the asymptotically periodic case, when $a(t)$ is bounded and $a(t) \to a_+ > 0$, $\omega(t) \to \omega_+ \neq 0$ as $t \to +\infty$. In particular we get existence of infinitely many homoclinic orbits of (1.1), namely classical solutions to (1.1) satisfying the further conditions

$$q(t) \to 0 \quad \text{and} \quad \dot{q}(t) \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{1.2}$$

Indeed we prove the following result.

THEOREM 1.3. *If $a(t)$ and $\omega(t)$ are real functions of class C^1 with $a(t)$ bounded, $a(t) \to a_+ > 0$ and $\omega(t) \to \omega_+ \neq 0$ as $t \to +\infty$, then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ there is a homoclinic orbit v_+ for the system at infinity:*

$$\ddot{q} = q - a_+(1 + \epsilon\cos(\omega_+ t))\,q^3 \tag{1.4}$$

for which the following holds: for any $r > 0$ there are $M, p \in \mathbf{N}$ such that for every sequence $(p_j)_{j\in\mathbf{N}} \subset \mathbf{N}$ satisfying $p_1 \geq p$ and $p_{j+1} - p_j \geq M \quad (j \in \mathbf{N})$, and for every sequence $\sigma = (\sigma_j)_{j\in\mathbf{N}} \in \{0,1\}^{\mathbf{N}}$ there is a solution v_σ to (1.1) such that

$$|v_\sigma(t) - \sigma_j v_+(t - p_j T_+)| < r \quad \text{and} \quad |\dot{v}_\sigma(t) - \sigma_j \dot{v}_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1}+p_j)T_+, \frac{1}{2}(p_j+p_{j+1})T_+]$ *and* $j \in \mathbf{N}$*, where* $p_0 = -\infty$ *and* $T_+ = \frac{2\pi}{\omega_+}$*. In addition any* v_σ *also satisfies* $v_\sigma(t) \to 0$ *and* $\dot v_\sigma(t) \to 0$ *as* $t \to -\infty$ *and it actually is a homoclinic orbit if* $\sigma_j = 0$ *definitively.*

Remark 1.5. The solutions given by theorem 1.3 are known as multibump solutions because they behave in this way: they remain in a small neighbourhood of the origin for a suitable large time and then leave it a finite or infinite number of times (according that $\sigma_j = 0$ definitively or not) staying near translates of v_+.

Remark 1.6. In the case $\epsilon = 0$ and $a(t)$ smooth, bounded and strictly monotone, the equation (1.1) does not have non zero homoclinic orbits. In fact if $q(t)$ satisfies (1.1) and (1.2) and $H(q(t)) = \frac{1}{2}|\dot q(t)|^2 - \frac{1}{2}|q(t)|^2 + \frac{1}{4}a(t)|q(t)|^4$ denotes the energy of $q(t)$, then

$$0 = \int_{\mathbf{R}} \frac{dH(q(t))}{dt}\,dt = \int_{\mathbf{R}} \dot a(t)\frac{|q|^4}{4}\,dt$$

and this implies $q \equiv 0$. We also see that for $\epsilon = 0$ the stable and unstable manifolds coincide and so their intersection is uncountable. This is the reason for which the argument used to prove theorem 1.3 fails in this case.

The class of systems which we study in this paper is shaped on (1.1). In fact we deal with second order Hamiltonian systems in $\mathbf{R}^N$

$$\ddot q = -U'(t,q) \tag{HS}$$

where $U'(t,q)$ denotes the gradient with respect to q of a smooth potential $U : \mathbf{R}\times\mathbf{R}^N \to \mathbf{R}$ having a strict local maximum at the origin.

Precisely we assume:

(U1) $U \in C^1(\mathbf{R}\times\mathbf{R}^N, \mathbf{R})$ with $U'(t,\cdot)$ locally Lipschitz continuous uniformly with respect to $t \in \mathbf{R}$;

(U2) $U(t,0) = 0$ and $U'(t,q) = L(t)q + o(|q|)$ as $q \to 0$ uniformly with respect to $t \in \mathbf{R}$ where $L(t)$ is a symmetric matrix such that $c_1|q|^2 \le q\cdot L(t)q \le c_2|q|^2$ for any $(t,q) \in \mathbf{R}\times\mathbf{R}^N$ with c_1 and c_2 positive constants.

The condition (U2) implies that in the phase space the origin is a hyperbolic rest point for the system (HS). We look for homoclinic orbits to (HS) as critical points of the Lagrangian functional

$$\varphi(u) = \int_{\mathbf{R}} (\tfrac{1}{2}|\dot{u}|^2 - U(t,u))\, dt$$

defined on $X = H^1(\mathbf{R}, \mathbf{R}^N)$ and of class C^1, by (U1)–(U2).

This problem has been studied under further conditions when the potential U is autonomous (see [3, 4, 10, 11, 19, 20, 27, 31, 33]), periodic in time ([7, 8, 9, 12, 14, 15, 17, 26, 29, 30]), asymptotically periodic ([1, 25]) and, very recently, almost periodic ([6, 28]).

Here, as pointed out with the model case, we consider asymptotically periodic potentials. By this we mean that there is a function $U_+(t,q) = -\frac{1}{2}\, q \cdot L_+(t)\, q + V_+(t,q)$ satisfying (U1), (U2) and

(U3) there is $T_+ > 0$ such that $U_+(t,q) = U_+(t+T_+, q)$ for any $(t,q) \in \mathbf{R} \times \mathbf{R}^N$;

(U4) (i) there is $(t_+, q_+) \in \mathbf{R} \times \mathbf{R}^N$ such that $U_+(t_+, q_+) > 0$;

(ii) there are two constants $\beta_+ > 2$ and $\alpha_+ < \frac{\beta_+}{2} - 1$ such that:

$\beta_+ V_+(t,q) - V_+'(t,q) \cdot q \le \alpha_+ q \cdot L_+(t) q$ for all $(t,q) \in \mathbf{R} \times \mathbf{R}^N$;

(U5) $U'(t,q) - U_+'(t,q) \to 0$ as $t \to +\infty$ uniformly on the compact sets of $\mathbf{R}^N$.

As we have seen in remark 1.6, these assumptions are not sufficient in order that (HS) admits homoclinic solutions. In that example the potential U_+ is time independent and hence the corresponding functional

$$\varphi_+(u) = \int_{\mathbf{R}} (\tfrac{1}{2}|\dot{u}|^2 - U_+(t,u))\, dt$$

and $\|\varphi_+'(u)\|$ are invariant under the action of the translations group $\mathbf{R}$. In particular, if u is a non zero critical point of φ_+ (which always exists, by [3, 11, 27], for instance) then also $u(\cdot - t)$ is a critical point of φ_+ for any $t \in \mathbf{R}$. Therefore the set of critical points of φ_+ is uncountable.

To avoid this situation, we make an assumption on the cardinality of the critical set of φ_+.

As discussed in [12], the functional φ_+ satisfies the geometrical properties of the mountain pass lemma. If we denote with c_+ the mountain pass level of φ_+ and $K_+ = \{u \in X \,:\, u \neq 0,\ \varphi_+'(u) = 0\}$, we assume that

($*$) there exists $c_+^* > c_+$ such that the set $K_+ \cap \{u \in X \,:\, \varphi_+(u) \leq c_+^*\}$ is countable.

On one hand, as seen above, condition ($*$) excludes the class of asymptotically autonomous systems, because of the translational invariance under $\mathbf{R}$ of the functional φ_+.

On the other hand, ($*$) holds when the system at infinity exhibits countable intersection between the stable and unstable manifolds relative to the origin.

In fact, as noticed in [30], where ($*$) was first introduced, the hypothesis ($*$) is a weaker condition than the transversality one and also includes the case of countable, tangential intersection.

The condition ($*$) is the key to find a local mountain pass critical point for φ_+ and to develop a minimax argument as in [30] and [12]. The local character of such a procedure allows us to show existence of critical points also for the functional φ.

We can now state a first general result.

THEOREM 1.7. *Assume that U and U_+ satisfy (U1)-(U5) and* ($*$) *holds. Then (HS) admits infinitely many homoclinic solutions.*

Precisely there is $v_+ \in K_+$ with the following property: for any $r > 0$ there are $M, p \in \mathbf{N}$ such that for every $k \in \mathbf{N}$ and $(p_1, \ldots, p_k) \in \mathbf{Z}^k$ with $p_1 \geq p$ and $p_{j+1} - p_j \geq M$, for $j = 1, \ldots, k-1$, there exists a homoclinic solution v of (HS) which verifies:

$$|v(t) - v_+(t - p_j T_+)| < r \quad \text{and} \quad |\dot{v}(t) - \dot{v}_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_+, \frac{1}{2}(p_j + p_{j+1})T_+]$ and $j = 1, \ldots, k$, where $p_0 = -\infty$ and $p_{k+1} = +\infty$.

We notice that this theorem can be seen as a version of the shadowing lemma (see [21]).

Fixing $k = 1$, for any $r > 0$ the theorem assures the existence of an integer $p = p(r) \in \mathbf{N}$ and of a sequence v_j of homoclinic solutions of (HS) each of them belongs to a C^1–neighborhood of $v_+(\cdot - (p + j)T_+)$ of radius r. In general, unlike the periodic case, these solutions are geometrically distinct.

For a general $k \in \mathbf{N}$ the theorem provides a homoclinic orbit of (HS) having k bumps, whose positions are defined by the sequence $p_1, \ldots, p_k$. More precisely, for any $j = 1, \ldots, k$ there is an interval P_j centered on $p_j T_+$ where the k-bump solution v of (HS) is not farther from $v_+(\cdot - p_j T_+)$ than r. The value $\delta_j = p_{j+1} - p_j$ represents the distance between the corresponding bumps. Fixed r, we can find a solution of this kind for any choice of $k \in \mathbf{N}$ and of the sequence $p_1, \ldots, p_k$ provided that p_1 is sufficiently large, depending on r, and that the distances δ_j are greater than a certain value M which also depends only on r.

As noticed in [30], since the number M does not depend on k, one could consider the C^1_{loc}–closure of the set of the multibump homoclinic orbits, which contains solutions with possibly infinitely many bumps. Thus, by Ascoli Arzelà theorem, the previous result can be generalized in the following way.

THEOREM 1.8. *Under the same assumptions of theorem 1.7, it holds that for any $r > 0$ there are $M, p \in \mathbf{N}$ such that for every sequence $(p_j)_{j\in\mathbf{N}} \subset \mathbf{N}$ satisfying $p_1 \geq p$ and $p_{j+1} - p_j \geq M \quad (j \in \mathbf{N})$, and for every sequence $\sigma = (\sigma_j)_{j\in\mathbf{N}} \in \{0,1\}^{\mathbf{N}}$ there is a solution v_σ to (HS) such that*

$$|v_\sigma(t) - \sigma_j v_+(t - p_j T_+)| < r \quad \text{and} \quad |\dot{v}_\sigma(t) - \sigma_j \dot{v}_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_+, \frac{1}{2}(p_j + p_{j+1})T_+]$ and $j \in \mathbf{N}$, where $p_0 = -\infty$ and $v_+ \in K_+$ is the same of theorem 1.7. In addition any v_σ also satisfies $v_\sigma(t) \to 0$ and $\dot{v}_\sigma(t) \to 0$ as $t \to -\infty$ and it is actually a homoclinic orbit if $\sigma_j = 0$ definitively.

We see that theorem 1.3 easily follows from theorem 1.8 together with the classical results on equation (1.4) which imply that the intersection between the stable and unstable manifolds relative to 0 associated to system at infinity (1.4) is countable. This is true if $\epsilon \neq 0$ and small. In fact the Melnikov function of (1.4) (see [16, 23, 34]) is given by

$$M(s) = \sin(\omega_+ s) \int_{\mathbf{R}} \tfrac{\omega_+}{4} \cos(\omega_+ t) |q_0(t)|^4 \, dt = \sin(\omega_+ s)\, C(\omega_+)$$

where $q_0(t) = (2/a_+)^{\frac{1}{2}} (\cosh t)^{-1}$ is a homoclinic orbit of the unperturbed system $\ddot{q} = q - a_+ q^3$ and $C(\omega_+) \in (0, \infty)$ for any $\omega_+ \neq 0$.

Since the zeros of $\sin(\omega_+ s)$ are simple, for the Melnikov theorem [23], the stable and unstable manifolds of the perturbed system intersect transversally and so countably. Thus $(*)$ is verified.

The correspondence $\sigma \mapsto v_\sigma$ permits to define an approximate Bernoulli shift for the system (HS) (see [30]). The presence of this structure implies sensitive dependence on initial data.

We point out that in the previous theorems 1.7 and 1.8 no assumption is made on the behaviour of U as $t \to -\infty$, but the regularity and hyperbolicity hypotheses (U1) and (U2).

If the system (HS) is doubly asymptotic as $t \to \pm\infty$ to two, possibly different, periodic systems

$$\ddot{q} = -U'_\pm(t,q) \qquad \text{(HS)}_\pm$$

then, by theorem 1.8, we have two different sets of multibump solutions, that, at $\pm\infty$ are near to solutions of $\text{(HS)}_\pm$. Here and in the sequel, with (HS)_- we denote a system ruled by a potential $U_-(t,q) = -\frac{1}{2}\, q \cdot L_-(t)\, q + V_-(t,q)$ satisfying (U1)-(U4).

In fact, we prove that there are also multibump solutions of (HS) of mixed type, as said in the following theorem.

THEOREM 1.9. *Assume that U, U_+ and U_- satisfy (U1)-(U5) and that $(*)$ holds both for (HS)_+ and (HS)_-. Then there are v_+ and v_- homoclinic solutions respectively of (HS)_+ and (HS)_- having the following property: for any $r > 0$ there are $M, p \in \mathbf{N}$ such that for every sequence $(p_j)_{j\in\mathbf{Z}} \subset \mathbf{Z}$ satisfying $p_1 \geq p$, $p_{-1} \leq -p$, $p_{j+1} - p_j \geq M \quad (j \in \mathbf{Z})$ and for every sequence $\sigma = (\sigma_j)_{j\in\mathbf{Z}} \in \{0,1\}^{\mathbf{Z}}$ there is a solution v_σ to (HS) such that*

$$|v_\sigma(t) - \sigma_j v_+(t - p_j T_+)| < r \quad \textit{and} \quad |\dot{v}_\sigma(t) - \sigma_j \dot{v}_+(t - p_j T_+)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_+, \frac{1}{2}(p_j + p_{j+1})T_+]$, $j = 1, 2\ldots$ and

$$|v_\sigma(t) - \sigma_j v_-(t - p_j T_-)| < r \quad \textit{and} \quad |\dot{v}_\sigma(t) - \sigma_j \dot{v}_-(t - p_j T_-)| < r$$

for any $t \in [\frac{1}{2}(p_{j-1} + p_j)T_-, \frac{1}{2}(p_j + p_{j+1})T_-]$, $j = -1, -2\ldots$

In addition, if $\sigma_j = 0$ for all $j \geq j_0$ (respectively $j \leq j_0$) then the solution v_σ also satisfies $v_\sigma(t) \to 0$ and $\dot{v}_\sigma(t) \to 0$ as $t \to +\infty$ (respectively $t \to -\infty$).

Clearly, in the previous statement, when we say that U, U_+ and U_- satisfy (U5) we mean that $U'(t,q) - U'_+(t,q) \to 0$ as $t \to +\infty$

and $U'(t,q) - U'_-(t,q) \to 0$ as $t \to -\infty$ uniformly on the compact sets of $\mathbf{R}^N$.

Coming back to the model equation (1.1) with $a(t)$ bounded and strictly increasing we see that while for $\epsilon = 0$ the system has no homoclinic solutions different from the trivial one $q \equiv 0$, there exists $\epsilon_0 > 0$ such that $(*)$ is satisfied for $|\epsilon| < \epsilon_0$, $\epsilon \neq 0$ and so, by the theorem 1.7, the equation (1.1) has infinitely many homoclinic orbits. Geometrically this means that while the stable and unstable manifolds do not intersect when $\epsilon = 0$ (apart from in the origin), if $\epsilon \neq 0$ then they intersect in an infinite set. This suggests that the stable and unstable manifolds for (1.1) accumulate one on the other for $t \to +\infty$.

NOTATION.
Through this paper we denote:
$X = H^1(\mathbf{R}, \mathbf{R}^N)$.
$\langle u, v\rangle_A = \int_A (\dot u \cdot \dot v + u \cdot L(t)v)\, dt$ for $u, v \in X$ and A measurable subset of $\mathbf{R}$.
$\|u\|_A = \langle u, u\rangle_A^{1/2}$ for $u \in X$ and A as before.
In particular $\|u\| = \|u\|_{\mathbf{R}}$ is a norm on X equivalent to the standard one.
$\varphi(u) = \int_{\mathbf{R}} (\frac{1}{2}|\dot u|^2 - U(t,u))\, dt$ for $u \in X$.
$\{\varphi \le b\} = \{u \in X \,:\, \varphi(u) \le b\}$, $\{\varphi \ge a\} = \{u \in X \,:\, \varphi(u) \ge a\}$, $\{a \le \varphi \le b\} = \{\varphi \le b\} \cap \{\varphi \ge a\}$ where $a, b \in \mathbf{R}$.
$K = \{u \in X \setminus \{0\} \,:\, \varphi'(u) = 0\}$, $K^b = K \cap \{\varphi \le b\}$, $K(b) = K \cap \{\varphi = b\}$.
$B_r(S) = \{u \in X \,:\, \mathrm{dist}\,(u, S) < r\}$ where $S \subset X$, $S \neq \emptyset$ and $r > 0$.
$A_{r_1,r_2}(S) = \bigcup_{v\in S}\{u \in X \,:\, r_1 < \|u - v\| < r_2\}$ where $0 \le r_1 < r_2$.
The same notation for φ_+, φ_-, K_+, K_-, etc.
$\tau_n^+ u(t) = u(t - nT_+)$, $\tau_n^- u(t) = u(t - nT_-)$ for $u \in X$, $t \in \mathbf{R}$, $n \in \mathbf{Z}$.

2. A local compactness result

In this section we discuss some basic general facts which depend only on the hyperbolicity assumption and therefore are true for both the periodic and the asymptotically periodic problem. During this section we will always assume (U1)–(U2), without any hypothesis on the time dependence of the potential.

First of all we note that thanks to (U2) we have

$$\varphi(u) = \tfrac{1}{2}\|u\|^2 + o(\|u\|^2) \quad \text{and} \quad \varphi'(u) = \langle u, \cdot\rangle + o(\|u\|) \quad \text{as} \quad u \to 0\,. \tag{2.1}$$

Secondly we give some properties of the Palais Smale sequences of φ. In general (U1)–(U2) are not sufficient to guarantee the boundedness of these sequences. Anyhow we can state the following results, concerning the bounded Palais Smale sequences.

Lemma 2.2. *If $(u_n) \subset X$ is a Palais Smale sequence at the level b (namely $\varphi(u_n) \to b$ and $\|\varphi'(u_n)\| \to 0$) weakly convergent to some $u \in X$, then $\varphi'(u) = 0$ and $(u_n - u)$ is a Palais Smale sequence at the level $b - \varphi(u)$ weakly convergent to 0.*

Proof. We write $U(t,q) = -\frac{1}{2}q \cdot L(t)q + V(t,q)$ for any $(t,q) \in \mathbf{R} \times \mathbf{R}^N$.
If $u_n \to u$ weakly in X and so strongly in $L^\infty_{\text{loc}}(\mathbf{R}, \mathbf{R}^N)$, then, for any $w \in C^\infty_c(\mathbf{R}, \mathbf{R}^N)$ we have:

$$\begin{aligned}\varphi'(u)w &= \langle u, w\rangle - \textstyle\int_{\operatorname{supp} w} V'(t,u)\cdot w\,dt \\ &= \lim\langle u_n, w\rangle - \lim \textstyle\int_{\operatorname{supp} w} V'(t,u_n)\cdot w\,dt \\ &= \lim \varphi'(u_n)w.\end{aligned}$$

Therefore, since $\varphi'(u_n) \to 0$, $\varphi'(u) = 0$ follows.
To prove that $\|\varphi'(u_n - u)\| \to 0$, take any $w \in X$. It holds that for any $T > 0$:

$$\begin{aligned}&|\varphi'(u_n - u)w - \varphi'(u_n)w| \\ &\quad = |\textstyle\int_{\mathbf{R}} (V'(t,u_n-u) - V'(t,u_n) + V'(t,u))\cdot w\,dt| \\ &\quad \le |\textstyle\int_{|t|\le T} (V'(t,u_n-u) - V'(t,u_n) + V'(t,u))\cdot w\,dt| \\ &\qquad + \textstyle\int_{|t|>T} |V'(t,u_n-u) - V'(t,u_n)|\,|w|\,dt + \int_{|t|>T}|V'(t,u)|\,|w|\,dt \\ &\quad \le \delta_n(T)\,(\textstyle\int_{|t|\le T}|w|^2\,dt)^{\frac{1}{2}} + \int_{|t|>T} C_R|u|\,|w|\,dt \\ &\qquad + (\textstyle\int_{|t|>T}|V'(t,u)|^2\,dt)^{\frac{1}{2}}(\int_{|t|>T}|w|^2\,dt)^{\frac{1}{2}}\end{aligned}$$

where:

$$\delta_n(T) = (\textstyle\int_{|t|\le T}|V'(t,u_n-u) - V'(t,u_n) + V'(t,u)|^2\,dt)^{\frac{1}{2}}$$

$$C_R = \sup\{\,|V'(t,q) - V'(t,\bar q)|/|q - \bar q| \;:\; t \in \mathbf{R},\ |q|, |\bar q| \le R,\ q \ne \bar q\,\}$$

and $R > 0$ is such that $|u_n(t) - u(t)| \leq R$ and $|u_n(t)| \leq R$ for any $t \in \mathbf{R}$ and $n \in \mathbf{N}$. We note that $R < +\infty$ because (u_n) is bounded in X and so in $L^\infty(\mathbf{R}, \mathbf{R}^N)$. Then, by (U1), $C_R < \infty$ too. Hence we get:

$$\begin{aligned}
&|\varphi'(u_n - u)w - \varphi'(u_n)w| \\
&\quad \leq \delta_n(T)\,(\textstyle\int_{\mathbf{R}}|w|^2\,dt)^{\frac{1}{2}} + C_R(\int_{|t|>T}|u|^2 dt)^{\frac{1}{2}}(\int_{\mathbf{R}}|w|^2\,dt)^{\frac{1}{2}} + \\
&\qquad + (\textstyle\int_{|t|>T}|V'(t,u)|^2\,dt)^{\frac{1}{2}}(\int_{\mathbf{R}}|w|^2\,dt)^{\frac{1}{2}}
\end{aligned}$$

which implies:

$$\begin{aligned}
&\|\varphi'(u_n - u) - \varphi'(u_n)\| \\
&\quad \leq \delta_n(T) + C_R(\textstyle\int_{|t|>T}|u|^2 dt)^{\frac{1}{2}} + (\int_{|t|>T}|V'(t,u)|^2\,dt)^{\frac{1}{2}}
\end{aligned}$$

for every $T > 0$. Now, for any $\epsilon > 0$ we can choose $T > 0$ such that

$$C_R(\textstyle\int_{|t|>T}|u|^2 dt)^{\frac{1}{2}} + (\int_{|t|>T}|V'(t,u)|^2\,dt)^{\frac{1}{2}} < \epsilon.$$

By the dominated convergence theorem, $\delta_n(T) \to 0$ as $n \to \infty$. Therefore $\limsup \|\varphi'(u_n - u)\| \leq \epsilon$ and, for the arbitrariness of $\epsilon > 0$, we get that $\lim \|\varphi'(u_n - u)\| = 0$.
Finally we prove that if $b = \lim \varphi(u_n)$ then $\varphi(u_n - u) \to b - \varphi(u)$. Indeed, arguing as before, we have that:

$$\begin{aligned}
&|\varphi(u_n - u) - \varphi(u_n) + \varphi(u)| \\
&\quad \leq |\,\|u\|^2 - \langle u_n, u\rangle| \\
&\qquad + \textstyle\int_{|t|\leq T}|V(t, u_n - u) - V(t, u_n) + V(t,u)|\,dt \\
&\qquad + \textstyle\int_{|t|>T}|V(t, u_n - u) - V(t, u_n)|\,dt + \int_{|t|>T}|V(t,u)|\,dt.
\end{aligned}$$

Taking $R > 0$ such that $|u_n(t) - u(t)| \leq R$ for any $t \in \mathbf{R}$ and $n \in \mathbf{N}$ and setting

$$C'_R = \sup\{\,|V'(t,q)|/|q| \;:\; t \in \mathbf{R},\ |q| \leq R,\ q \neq 0\,\}$$

from the mean value theorem we get that for any $t \in \mathbf{R}$:

$$\begin{aligned}
|V(t, u_n(t) - u(t)) - V(t, u_n(t))| &= |V'(t, u_n(t) - \theta u(t)) \cdot u(t)| \\
&\leq C'_R|u_n(t) - \theta u(t)|\,|u(t)| \\
&\leq C'_R|u_n(t)|\,|u(t)| + C'_R|u(t)|^2
\end{aligned}$$

where $\theta \in [0,1]$ and so

$$
\begin{aligned}
&|\varphi(u_n - u) - \varphi(u_n) + \varphi(u)| \\
&\quad \le |\,\|u\|^2 - \langle u_n, u\rangle| \\
&\qquad + \textstyle\int_{|t|\le T} |V(t, u_n - u) - V(t, u_n) + V(t, u)|\, dt \\
&\qquad + C'_R \textstyle\int_{|t|>T} |u_n|\,|u|\, dt + C'_R \int_{|t|>T} |u|^2 dt + \int_{|t|>T} |V(t,u)|\, dt.
\end{aligned}
$$

Taking now $\epsilon > 0$ we can find $T > 0$ independent from $n \in \mathbf{N}$ such that

$$
C'_R \textstyle\int_{|t|>T} |u_n|\,|u|\, dt + C'_R \int_{|t|>T} |u|^2 dt + \int_{|t|>T} |V(t,u)|\, dt < \epsilon.
$$

Since $\int_{|t|\le T} |V(t, u_n - u) - V(t, u_n) + V(t, u)|\, dt \to 0$ as $n \to \infty$ and $u_n \to u$ weakly, we infer that $\limsup |\varphi(u_n - u) - \varphi(u_n) + \varphi(u)| \le \epsilon$ which implies that $\lim \varphi(u_n - u) = b - \varphi(u)$. $\diamondsuit$

As next step we study the Palais Smale sequences which converge to 0 weakly in X.

LEMMA 2.3. *If $u_n \to 0$ weakly in X and $\varphi'(u_n) \to 0$ then $u_n \to 0$ strongly in $H^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^N)$ and the following alternative holds: either*

(i) $u_n \to 0$ strongly in X,

(ii) or $\exists\ |t_{n_k}| \to \infty$ *s.t.* $\inf_k |u_{n_k}(t_{n_k})| > 0$.

Proof. Let $(u_n) \subset X$ be a sequence such that $u_n \to 0$ weakly in X and $\varphi'(u_n) \to 0$. First we suppose that $u_n \to 0$ in $L^\infty(\mathbf{R}, \mathbf{R}^N)$. By (U2) there is $\delta > 0$ such that $|V'(t,q)\cdot q| \le \frac{1}{2} c_1 |q|^2$ for any $t \in \mathbf{R}$ and $|q| \le \delta$. Then we can find $\bar{n} \in \mathbf{N}$ such that $|u_n(t)| \le \delta$ for any $t \in \mathbf{R}$ and $n \ge \bar{n}$. Hence $\|u_n\|^2 = \varphi'(u_n)u_n + \int_{\mathbf{R}} V'(t,u_n)\cdot u_n dt \le \|\varphi'(u_n)\|\,\|u_n\| + \int_{\mathbf{R}} \frac{1}{2} c_1 |u_n|^2 dt$ and thus $\|u_n\|^2 \le C\,\|\varphi'(u_n)\|$ where $C = 2\sup \|u_n\|$. Therefore $\|u_n\| \to 0$ and the case *(i)* holds.
Let us now suppose that $u_n \not\to 0$ in $L^\infty(\mathbf{R}, \mathbf{R}^N)$. Then there are sequences $(n_k) \subseteq \mathbf{N}$ and $(t_{n_k}) \subset \mathbf{R}$ such that $n_k \to \infty$, $|t_{n_k}| \to \infty$, $u_{n_k} \to 0$ in $L^\infty_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^N)$ as $k \to \infty$ and $\inf_k |u_{n_k}(t_{n_k})| > 0$. So we are in the case *(ii)* and we only have to prove that $u_n \to 0$ strongly in $H^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^N)$, that is $\|u_n\|_{|t|\le T} \to 0$ for any $T > 0$. So, we fix a piecewise linear cut–off function $\chi : \mathbf{R} \to [0,1]$ such that $\chi(t) = 1$

for $|t| \leq T$ and $\chi(t) = 0$ for $|t| \geq T+1$. We point out that the mapping $u \mapsto \chi u$ is a bounded linear operator on X and

$$\begin{aligned} \|u_n\|^2_{|t|\leq T} &= \langle u_n, \chi u_n\rangle - \textstyle\int_{T\leq|t|\leq T+1}[\chi\,(|\dot u_n|^2 + |u_n|^2) + \dot\chi\,\dot u_n\cdot u_n]\,dt \\ &\leq \varphi'(u_n)\,\chi u_n + \textstyle\int_{\mathbf{R}} V'(t,u_n)\cdot\chi u_n\,dt + C_0|\textstyle\int_{T\leq|t|\leq T+1}\frac{d}{dt}\frac{1}{2}|u_n(t)|^2 dt| \\ &\leq C_1\|\varphi'(u_n)\| + \textstyle\int_{|t|\leq T+1} V'(t,u_n)\cdot\chi u_n\,dt + C_0\sup_{|t|\leq T+1}|u_n(t)|^2. \end{aligned}$$

This shows that $\|u_n\|_{|t|\leq T} \to 0$. $\diamondsuit$

Therefore if $(u_n) \subset X$ is a Palais Smale sequence which converges weakly but not strongly to some $u \in X$, then there exists a positive number r such that for any $T > 0$ we have $\limsup \|u_n\|_{|t|>T} > r$. As we will see in the next lemma, this value r can be taken independent from the sequence (u_n). Indeed, from (2.1) we easily get that

$$\exists\,\rho > 0 \text{ such that if } \limsup\|u_n\| \leq 2\rho\,,\ \varphi'(u_n) \to 0 \text{ then } u_n \to 0\,. \tag{2.4}$$

Then we have this first local compactness property of the functional φ.

LEMMA 2.5. *Let $u_n \to u$ weakly in X and $\varphi'(u_n) \to 0$. If there exists $T > 0$ for which* $\limsup\|u_n\|_{|t|>T} \leq \rho$ *(where ρ is given by (2.4)), then $u_n \to u$ strongly in X.*

Proof. Fix $R > 0$ such that $\|u\|_{|t|\geq R} \leq \rho$. Putting $M = \max\{R, T\}$, by lemma 2.3, we have that $\|u_n - u\|_{|t|\leq M} \to 0$. Therefore $\|u_n - u\|^2 = o(1) + \|u_n - u\|^2_{|t|>M} \leq o(1) + \rho^2 + 2\rho\,\|u_n\|_{|t|>M} + \|u_n\|^2_{|t|>M}$, from which we get $\limsup\|u_n - u\| \leq 2\rho$. Since $\varphi'(u_n - u) \to 0$ we derive from (2.4) that $u_n \to u$ strongly in X. $\diamondsuit$

From the previous lemma we deduce this second property.

LEMMA 2.6. *If* $\operatorname{diam}\{u_n\} < \rho$ *and $\varphi'(u_n) \to 0$ then (u_n) admits a strongly convergent subsequence.*

Proof. Let $\delta = \rho - \operatorname{diam}\{u_n\}$ and $T > 0$ such that $\|u_1\|_{|t|>T} \leq \delta$. Then $\|u_n\|_{|t|>T} \leq \|u_n - u_1\|_{|t|>T} + \delta \leq \rho$. Since the sequence (u_n) is bounded, there is a subsequence (u_{n_k}) which converges weakly to some $u \in X$. Hence, using lemma 2.5, $u_{n_k} \to u$ strongly in X. $\diamondsuit$

3. The periodic case

Here we first state some properties satisfied by the functional φ_+, by the periodicity and superquadraticity assumptions (U3) and (U4). Then, using the hypothesis $(*)$, we get further compactness properties which, together with lemma 2.6, give the existence of a local mountain pass-type critical point of φ_+. All these results were given in [12] to which we refer for the proofs.

First of all we note that the hypothesis (U4.ii) implies that

$$(\tfrac{1}{2} - \tfrac{1}{\beta_+} - \tfrac{\alpha_+}{\beta_+})\|u\|_+^2 - \tfrac{1}{\beta_+}\|\varphi_+'(u)\|\,\|u\|_+ \leq \varphi_+(u) \quad \forall\, u \in X \tag{3.1}$$

where $\|u\|_+^2 = \int_{\mathbf{R}}(|\dot{u}|^2 + u \cdot L_+(t)u)\,dt$. Therefore, if a sequence $(u_n) \subset X$ is such that $\varphi_+'(u_n) \to 0$ and $\limsup \varphi_+(u_n) < +\infty$, then (u_n) is bounded in X and $\liminf \varphi_+(u_n) \geq 0$.

So, as first result, we get that any Palais Smale sequence of φ_+ is a bounded sequence, at a non negative level.

Moreover, the hypothesis (U4) gives information about the behaviour of the potential at infinity with respect to q along the direction of q_+ in a neighborhood of t_+. In fact, from (U4), one can infer that

$$V_+(t, sq_+) \geq \delta s^{\beta_+} \quad \forall\, s \geq 1\,,\ \forall\, t \in [t_+ - \epsilon, t_+ + \epsilon] \tag{3.2}$$

where $\delta = 2[V_+(t_+, q_+) - \frac{\alpha_+}{\beta_+ - 2} q_+ \cdot L_+(t_+)q_+] > 0$ and $\epsilon > 0$ small enough. Hence, choosing $\rho \in C^\infty(\mathbf{R}, \mathbf{R}^+)$ with $\operatorname{supp}\rho = [t_+ - \epsilon, t_+ + \epsilon]$, and setting $u_0(t) = \rho(t)q_+$ we have that $\varphi_+(s\,u_0) \to -\infty$ as $s \to \infty$.

Together with (2.1), this says that the functional φ_+ verifies the geometrical hypotheses of the mountain pass theorem.

Then, if we define

$$\Gamma = \{\, \gamma \in C([0,1], X) \,:\, \gamma(0) = 0\,,\ \varphi_+(\gamma(1)) < 0 \,\}$$

and

$$c_+ = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \varphi_+(\gamma(s))$$

we infer that $c_+ > 0$ and there is a sequence $(u_n) \subset X$ such that $\varphi_+(u_n) \to c_+$ and $\|\varphi_+'(u_n)\| \to 0$. Using the periodicity hypothesis (U3), this sequence (u_n) can be chosen in such a way that $\sup_{t \in \mathbf{R}} |u_n(t)| = \sup_{t \in [0,T_+]} |u_n(t)| \geq \delta > 0$. Then, by (3.1), (u_n) is bounded and so, up to a subsequence, converges weakly to some

$u \in X$ which, by lemma 2.2, is a critical point of φ_+. Moreover $u \neq 0$ because $\sup_{t\in[0,T_+]}|u(t)| \geq \delta$.

REMARK 3.3. We point out that the assumption (U4.ii) permits to the potential U_+ to be negative on an unbounded region, as discussed in [12]. In the autonomous case, instead of (U4.ii), one can put milder conditions on the potential, to guarantee the existence of a homoclinic orbit (see [11, 20]).

To investigate the Palais Smale sequences, we introduce two sets of real numbers, already studied in [30]. Letting

$$\mathcal{S}^b_{\text{PS}}(\varphi_+) = \{\,(u_n) \subset X \,:\, \lim \varphi'_+(u_n) = 0\,,\ \limsup \varphi_+(u_n) \leq b\,\}$$

we define

$$\Phi^b_+ = \{\, l \in \mathbf{R} \,:\, \exists (u_n) \in \mathcal{S}^b_{\text{PS}}(\varphi_+)\ s.t.\ \varphi_+(u_n) \to l\,\}$$

the set of the asymptotic critical values lower than b and

$$D^b_+ = \{\, r \in \mathbf{R} \,:\, \exists\,(u_n), (\bar{u}_n) \in \mathcal{S}^b_{\text{PS}}(\varphi_+)\ s.t.\ \|u_n - \bar{u}_n\| \to r\,\}$$

the set of the asymptotic distances between two Palais Smale sequences under b.

As proved in [12] (see Lemma 3.7), Φ^b_+ and D^b_+ are closed subsets of $\mathbf{R}$. Thus, we have:

(3.4) given $b > 0$, for any $l \in (0,b)\setminus\Phi^b_+$ there exists $\delta > 0$ such that $[l-\delta, l+\delta] \subset (0,b)\setminus\Phi^b_+$ and there exists $\nu > 0$ such that $\|\varphi'_+(u)\| \geq \nu$ for any $u \in \{l-\delta \leq \varphi_+ \leq l+\delta\}$.

(3.5) given $b > 0$, for any $r \in \mathbf{R}^+\setminus D^b_+$ there exists $d_r > 0$ such that $[r-3d_r, r+3d_r] \subset \mathbf{R}^+\setminus D^b_+$ and there exists $\mu_r > 0$ such that $\|\varphi'_+(u)\| \geq \mu_r$ for any $u \in A_{r-3d_r,r+3d_r}(K^b_+) \cap \{\varphi_+ \leq b\}$;

Actually D^b_+ and Φ^b_+ can be described using the set K_+ of the critical points of φ_+. In fact, by the translational invariance of the functional φ_+, by concentration-compactness arguments [22], it is possible to prove the following result, already presented in [14, 15].

LEMMA 3.5. *Let $(u_n) \subset X$ be a Palais Smale sequence for φ_+ at the level b. Then there are $v_0 \in K_+ \cup \{0\}$, $v_1,\dots,v_k \in K_+$, a subsequence of (u_n), denoted again (u_n), and corresponding sequences $(t^1_n),\dots,(t^k_n) \in \mathbf{Z}$ such that, as $n \to \infty$:*

$$
\begin{aligned}
&\|u_n - (v_0 + \tau^+_{t^1_n} v_1 + \ldots + \tau^+_{t^k_n} v_k)\| \to 0 \\
&\varphi_+(v_0) + \cdots + \varphi_+(v_k) = b \\
&|t^j_n| \to +\infty \quad (j = 1, \ldots, k) \\
&t^{j+1}_n - t^j_n \to +\infty \quad (j = 1, \ldots, k-1).
\end{aligned}
$$

As proved in [12] (see lemma 3.10) this implies that

$$
\begin{aligned}
\Phi^b_+ &= \{\, \textstyle\sum_{j=1}^k \varphi_+(v_j) \,:\, k \in \mathbf{N}\,,\ v_j \in K_+\} \cap [0, b] \\
D^b_+ &= \{\, (\textstyle\sum_{j=1}^k \|v_j - \bar v_j\|^2)^{1/2} \,:\, k \in \mathbf{N}\,, v_j, \bar v_j \in K_+ \cup \{0\}\,, \\
&\qquad\qquad\qquad \textstyle\sum_{j=1}^k \varphi_+(v_j) \le b\,, \sum_{j=1}^k \varphi_+(\bar v_j) \le b\,\}\,.
\end{aligned}
$$

Now it is clear how the hypothesis $(*)$ enters in the argument. Indeed if

$(*)$ *there exists* $c^*_+ > c_+$ *such that* $K^{c^*_+}_+$ *is countable*

then both the sets $D^*_+ = D^{c^*_+}_+$ and $\Phi^*_+ = \Phi^{c^*_+}_+$ are countable too, and since they are closed, it holds that:

$$[0, c^*_+] \setminus \Phi^*_+ \text{ is open and dense in } [0, c^*_+] \tag{3.7}$$

$$\text{there is a sequence } (r_n) \subset \mathbf{R}^+ \setminus D^*_+ \text{ such that } r_n \to 0. \tag{3.8}$$

Therefore, by (3.5), near any level set $\{\varphi_+ = l\}$ at a critical value $l \in (0, c^*)$ there is a sequence of slices $\{l^1_n \le \varphi_+ \le l^2_n\}$ with $l^2_n - l^1_n$ smaller and smaller on which there are neither critical points or Palais Smale sequences. Analogously, by (3.5), around any critical point $u \in K^{c^*_+}_+$ there is a sequence of annuli of radii smaller and smaller (independently of u) on which, as above, there are neither critical points or Palais Smale sequences. From this last fact and from lemma 2.5 it is possible to show, as in [12], that the functional φ_+ admits a critical point of local mountain pass type, according to the following definition.

DEFINITION 3.9. Given a subset A of a Banach space X and two points $x_0, x_1 \in A$ we say that x_0 and x_1 are not connectible in A if there is no path $p \in C([0,1], X)$ joining x_0 and x_1, with $\operatorname{range} p \subseteq A$.

A critical point $\bar{x} \in X$ for a functional $f \in C^1(X, \mathbf{R})$ is called of *local mountain pass–type* if there is a neighborhood $\mathcal{N}_0$ of $\bar{x}$ such that for any neighborhood $\mathcal{N}$ of $\bar{x}$ contained in $\mathcal{N}_0$ the set $\{f < f(\bar{x})\} \cap \mathcal{N}$ contains two points not connectible in $\mathcal{N}_0 \cap \{f < f(\bar{x})\}$.

We refer to section 4 of [12] for the proof of the following lemma.

LEMMA 3.10. *If φ_+ verifies $(*)$ then it admits a non zero critical point of local mountain pass type. In particular there exist $\bar{c}_+ \in [c_+, c_+^*)$ and $\bar{r}_+ \in (0, \frac{\rho}{2})$ such that for any sequence $(r_n) \subset \mathbf{R}_+ \setminus D_+^*$ with $r_n \to 0$ there is a sequence $(v_n^+) \subset K_+(\bar{c}_+)$, $v_n^+ \to \bar{v}_+ \in K_+(\bar{c}_+)$ having this property: for any $n \in \mathbf{N}$ and for any $h > 0$ there is a path $\gamma_n^+ \in C([0,1], X)$ satisfying:*

(i) $\gamma_n^+(0), \gamma_n^+(1) \in \partial B_{r_n}(v_n^+)$;

(ii) $\gamma_n^+(0)$ *and* $\gamma_n^+(1)$ *are not connectible in* $B_{\bar{r}_+}(\bar{v}_+) \cap \{\varphi_+ < \bar{c}_+\}$;

(iii) $\operatorname{range} \gamma_n^+ \subseteq \bar{B}_{r_n}(v_n^+) \cap \{\varphi_+ \leq \bar{c}_+ + h\}$;

(iv) $\operatorname{range} \gamma_n^+ \cap A_{r_n - \frac{1}{2} d_{r_n}, r_n}(v_n^+) \subseteq \{\varphi_+ \leq \bar{c}_+ - h_n\}$;

(v) $\operatorname{supp} \gamma_n^+(s) \subset [-R_n, R_n]$ *for any* $s \in [0,1]$,

where $R_n > 0$ is independent of s, $h_n = \frac{1}{8} d_{r_n} \mu_{r_n}$ and d_{r_n} and μ_{r_n} are defined by (3.5).

REMARK 3.11. In [1, 15, 25, 29] a stronger condition than $(*)$ is considered. Precisely it is assumed that there exists $c_+^* > c_+$ such that $K_+^{c_+^*}/\mathbf{Z}$ is finite. In this case the property (3.8) becomes

$$\text{there exists } \bar{\epsilon} > 0 \text{ such that } D_+^* \cap (0, \bar{\epsilon}) = \emptyset. \qquad (3.8)'$$

This permits to get more information about the mountain pass structure described in lemma 3.10. Indeed, if $(3.8)'$ holds, in the statement of lemma 3.10 one can specify that $\bar{c}_+ = c_+$ and $v_n^+ = \bar{v}_+$ for all $n \in \mathbf{N}$.

4. Study of the Asymptotically Periodic System

In this section we tackle the problem of existence of homoclinic orbits for the Hamiltonian system (HS) in the two following cases:

1. (HS) is asymptotic, as $t \to +\infty$ to a given periodic system $(\mathrm{HS})_+$ with no assumption on the behaviour of U for $t \to -\infty$;

2. (HS) is asymptotic as $t \to \pm\infty$ to two, possibly different, periodic systems $(\mathrm{HS})_\pm$.

As shown in [12], if the functionals $\varphi_\pm$ satisfy the condition $(*)$, then each of them admits a class of homoclinic orbits obtained as multibump solutions.

To describe this situation in a precise way, we introduce some notation. For the sake of simplicity, for the moment, we consider only the problem $(\mathrm{HS})_+$. Given $M, k \in \mathbf{N}$ we set

$$\begin{aligned} P_k^+(M) &= \{p = (p_1, \dots, p_k) \in \mathbf{Z}^k : \\ &\qquad\qquad p_{j+1} - p_j \geq M \, \forall j = 1, \dots, k-1\} \\ P^+(M) &= \bigcup_{k \in \mathbf{N}} P_k^+(M). \end{aligned}$$

To any finite sequence $p = (p_1, \dots, p_k) \in P^+(M)$ we associate a partition of $\mathbf{R}$ into intervals $\{ P_1, \dots, P_k \}$ where, for any $j = 1, \dots, k$:

$$P_j = [\, \tfrac{1}{2}(p_j + p_{j-1})T_+, \tfrac{1}{2}(p_j + p_{j+1})T_+ \,]$$

with $p_0 = -\infty$ and $p_{k+1} = +\infty$.

Then, for $r > 0$, $p = (p_1, \dots, p_k) \in P^+(M)$ and $v \in X$, we set

$$B_r^+(v; p) = \{ \, u \in X \; : \; \|u - \tau_{p_j}^+ v\|_{P_j} < r \quad \forall \, j = 1, \dots, k \, \}.$$

The elements of $B_r^+(v; p)$ are $k-$bump functions associated to v according to the sequence p.

In [12] the following result is proved.

Theorem 4.1. *If U_+ satisfies (U1)–(U4) and if $(*)$ holds, then for any $r > 0$ there exists $M \in \mathbf{N}$ such that $B_r^+(\bar{v}_+; p) \cap K_+ \neq \emptyset$ for every $p \in P^+(M)$, where $\bar{v}_+ \in K_+$ is given by Lemma 3.10.*

Clearly the same holds for the system $(HS)_-$. In this case we modify the notation in the following way. Given $M, k \in \mathbf{N}$ we write

$$
\begin{aligned}
P_k^-(M) &= \{p = (p_{-k}, \ldots, p_{-1}) \in \mathbf{Z}^k : \\
&\qquad\qquad p_{j+1} - p_j \geq M \,\forall j = -k, \ldots, -2\} \\
P^-(M) &= \bigcup_{k \in \mathbf{N}} P_k^-(M).
\end{aligned}
$$

and, for $r > 0$, $p = (p_{-k}, \ldots, p_{-1}) \in P^-(M)$ and $v \in X$, we set

$$
B_r^-(v; p) = \{\, u \in X \;:\; \|u - \tau_{p_j}^- v\|_{P_j} < r \quad \forall\, j = -k, \ldots, -1 \,\}
$$

where $P_j = [\,\frac{1}{2}(p_j + p_{j-1})T_-, \frac{1}{2}(p_j + p_{j+1})T_-\,]$.

Now we study the functional φ corresponding to the problem (HS) assuming the periodically asymptotic behaviour of U only at $+\infty$.

First of all we point out that for (U5), the operator $\varphi'(u)$ is close to $\varphi'_+(u)$ for those elements $u \in X$ with support "at $+\infty$", as stated in the next lemma.

LEMMA 4.2. *For any $\epsilon > 0$ and for any $C > 0$ there exists $T \in \mathbf{R}$ such that*

$$
\|\varphi'(u) - \varphi'_+(u)\| \leq \epsilon
$$

for any $u \in X$ with $\|u\| \leq C$ and $\operatorname{supp} u \subseteq [T, +\infty)$.

Proof. For any $u, w \in X$ and $\delta > 0$ it holds that:

$$
\begin{aligned}
&|(\varphi'(u) - \varphi'_+(u))w| \\
&\quad = |\textstyle\int_{\mathbf{R}} (u \cdot (L(t) - L_+(t))w - (V'(t,u) - V'_+(t,u)) \cdot w)\, dt| \\
&\quad \leq \textstyle\int_{\mathbf{R}} |L(t) - L_+(t)|\, |u|\, |w|\, dt + \int_{|u(t)| > \delta} |V'(t,u) - V'_+(t,u)|\, |w|\, dt \\
&\qquad\quad + \textstyle\int_{|u(t)| \leq \delta} |V'(t,u)|\, |w|\, dt + \int_{|u(t)| \leq \delta} |V'_+(t,u)|\, |w|\, dt \\
&\quad \leq c_1^{-\frac{1}{2}} \sup_{t \in \operatorname{supp} u} |L(t) - L_+(t)|\, \|u\|\, \|w\| \\
&\qquad\quad + (\textstyle\int_{|u(t)| > \delta} |V'(t,u) - V'_+(t,u)|^2 dt)^{\frac{1}{2}} \|w\| \\
&\qquad\quad + (\textstyle\int_{|u(t)| \leq \delta} |V'(t,u)|^2 dt)^{\frac{1}{2}} \|w\| + (\int_{|u(t)| \leq \delta} |V'_+(t,u)|^2 dt)^{\frac{1}{2}} \|w\|.
\end{aligned}
$$

Then, taking $u \in X$ with $\|u\| \le C$ and fixing $\epsilon > 0$, by (U2) we can find $\delta > 0$ such that $|V'(t,q)| \le \frac{\epsilon}{4C}|q|$ and $|V_+'(t,q)| \le \frac{\epsilon}{4C}|q|$ for any $t \in \mathbf{R}$ and $|q| \le \delta$. Therefore

$$(\textstyle\int_{|u(t)|\le\delta}|V'(t,u)|^2dt)^{\frac{1}{2}} + (\textstyle\int_{|u(t)|\le\delta}|V_+'(t,u)|^2dt)^{\frac{1}{2}} \le \frac{\epsilon}{2}.$$

Moreover we have that:

$$\begin{aligned}&(\textstyle\int_{|u(t)|>\delta}|V'(t,u) - V_+'(t,u)|^2dt)\\ &\quad\le \frac{1}{\delta^2}\sup_{t\in \operatorname{supp} u}|V'(t,u) - V_+'(t,u)|^2\textstyle\int_{\mathbf{R}}|u|^2dt\\ &\quad\le \frac{C^2}{\delta^2}\sup_{t\in\operatorname{supp} u,\ |q|\le R}|V'(t,q) - V_+'(t,q)|^2\end{aligned}$$

for a suitable $R > 0$. Finally, by (U2) and (U5) we can take $T \in \mathbf{R}$ so large that, if $\operatorname{supp} u \subseteq [T, +\infty)$, then

$$\begin{aligned}&c_1^{-\frac{1}{2}}C\sup_{t\in\operatorname{supp} u}|L(t) - L_+(t)| + \\ &\quad + \frac{C}{\delta}(\sup_{t\in\operatorname{supp} u,\ |q|\le R}|V'(t,q) - V_+'(t,q)|^2)^{\frac{1}{2}} \le \frac{\epsilon}{2}.\end{aligned}$$

Then $\|\varphi'(u) - \varphi_+'(u)\| \le \epsilon$. $\diamondsuit$

Now, since $B_r^+(\bar{v}_+;p) \cap K_+ \ne \emptyset$ for every $p \in P^+(M)$, provided that $M \in \mathbf{N}$ is large, we expect that also $B_r^+(\bar{v}_+;p) \cap K \ne \emptyset$ for those sequences $p \in P^+(M)$ with p_1 so large that lemma 4.2 can be applied. In other words, also the system (HS), as well as $(\mathrm{HS})_+$, admits a family of homoclinic orbits obtained as multibump solutions.

We define, for $M, p_0 \in \mathbf{N}$

$$P^+(M,p_0) = \{\, p \in P^+(M) \ :\ p_1 \ge p_0 + M \,\}$$

and analogously

$$P^-(M,p_0) = \{\, p \in P^-(M) \ :\ p_{-1} \le -p_0 - M \,\}.$$

We now state the result concerning the case of asymptotic periodicity of (HS) only for $t \to +\infty$. We omit the proof which can be obtained by simple modification of the proof of theorem 4.5 below.

THEOREM 4.3. *If U and U_+ satisfy (U1)–(U5) and if the condition $(*)$ holds for the functional φ_+, then for any $r > 0$ there are $M, p_0 \in \mathbf{N}$ such that $B_r(\bar{v}_+; p) \cap K \neq \emptyset$ for every $p \in P^+(M, p_0)$, where $\bar{v}_+ \in K_+$ is given by lemma 3.10.*

REMARK 4.4. The multibump homoclinic solutions of (HS) found with the previous theorem are near to $\tau^+_{p_j}\bar{v}_+$ on the interval P_j in the H^1–norm and so in the sup norm. Since they verify (HS), we infer that they are actually near to $\tau^+_{p_j}\bar{v}_+$ on P_j in the C^1–norm, too, as stated in theorem 1.7.

When U is doubly asymptotic to $U_\pm$ for $t \to \pm\infty$, we can find critical points of φ among doubly multibump functions, according to the following procedure.

Given $M, p_0 \in \mathbf{N}$ we put:

$$P(M, p_0) = (P^-(M, p_0) \times P^+(M, p_0)) \cup P^-(M, p_0) \cup P^+(M, p_0)$$

For $p = (p_{-h}, \ldots, p_{-1}, p_1, \ldots, p_k) \in P(M, p_0)$ we define the family $\{P_{-h}, \ldots, P_k\}$ by setting

$$\begin{aligned} &P_j = [\tfrac{1}{2}(p_j + p_{j-1})T_\pm, \tfrac{1}{2}(p_j + p_{j+1})T_\pm] \text{ for } -h \leq j \leq k,\ j \neq 0, -1 \\ &P_{-1} = [\,\tfrac{1}{2}(p_{-1} + p_{-2})T_-, \tfrac{1}{2}(p_{-1} - p_0)T_-\,] \\ &P_0 = [\,\tfrac{1}{2}(p_{-1} - p_0)T_-, \tfrac{1}{2}(p_0 + p_1)T_+\,] \end{aligned}$$

where $p_{-h-1} = -\infty$, $p_{k+1} = +\infty$, $T_\pm = T_-$ for $j < 0$ and $T_\pm = T_+$ for $j > 0$.

Finally, for $v^-, v^+ \in X$ and $r > 0$ we set

$$B_r(v^-, v^+; p) = B_r^-(v^-; p^-) \cap B_r^0 \cap B_r^+(v^+; p^+)$$

where $p^- = (p_{-h}, \ldots, p_{-1})$, $p^+ = (p_1, \ldots, p_k)$ and $B_r^0 = \{\, u \in X : \|u\|_{P_0} < r \,\}$.

With this notation the theorem concerning the doubly asymptotic case can be stated in this form.

THEOREM 4.5. *If U, U_+ and U_- verify (U1)–(U5) and if the condition $(*)$ holds for the functionals φ_+ and φ_-, then for any $r > 0$ there are $M, p_0 \in \mathbf{N}$ such that $B_r(\bar{v}_-, \bar{v}_+; p) \cap K \neq \emptyset$ for every*

$p \in P(M, p_0)$, where $\bar{v}_\pm$ are critical points of $\varphi_\pm$ given by Lemma 3.10.

Proof. We start by giving an idea of the proof. Fix a sequence $(r_n) \subset \mathbf{R}^+ \setminus (D_+^* \cup D_-^*)$ such that $r_n \to 0$. Let $v_n^-, \bar{v}_- \in K_-$ and $v_n^+, \bar{v}_+ \in K_+$ be given by lemma 3.10. Arguing by contradiction, suppose that the conclusion of the theorem is false. Then there exists $r_0 > 0$ such that for any $M, p_0 \in \mathbf{N}$ there is a finite sequence $p = (p_{-h}, \ldots, p_{-1}, p_1, \ldots, p_k) \in P(M, p_0)$ for which $B_{r_0}(\bar{v}_-, \bar{v}_+; p) \cap K = \emptyset$. Fixing a suitable $h > 0$, lemma 3.10 assigns two sequences of paths γ_n^-, γ_n^+ such that $\gamma_n^-(0)$ and $\gamma_n^-(1)$ belong to two different components of $B_{\bar{r}_-}(\bar{v}_-) \cap \{\varphi_- < \bar{c}_-\}$ and analogously for $\gamma_n^+(0)$ and $\gamma_n^+(1)$. To reach a contradiction, we will construct a path $\bar{\gamma}$ joining $\gamma_n^-(0)$ and $\gamma_n^-(1)$ (or $\gamma_n^+(0)$ and $\gamma_n^+(1)$) inside $B_{\bar{r}_-}(\bar{v}_-) \cap \{\varphi_- < \bar{c}_-\}$ (respectively, inside $B_{\bar{r}_+}(\bar{v}_+) \cap \{\varphi_+ < \bar{c}_+\}$). This path $\bar{\gamma}$ is built in the following way. We consider the surface $G : [0,1]^{h+k} \to X$ defined by

$$G(\theta_{-h}, \ldots, \theta_{-1}, \theta_1, \ldots, \theta_k) = \sum_{-h \le j \le -1} \tau_{p_j}^- \gamma_n^-(\theta_j) + \sum_{1 \le j \le k} \tau_{p_j}^+ \gamma_n^+(\theta_j). \tag{4.6}$$

For the properties of $\gamma_n^\pm$ listed in lemma 3.10, we have that

$$\varphi_j(G(\theta)) \le \bar{c}_\pm + h \quad \text{for any } j \text{ and for any } \theta \in [0,1]^{h+k}$$

where

$$\varphi_j(u) = \int_{P_j} \left(\tfrac{1}{2}|\dot{u}|^2 - U_\pm(t, u)\right) dt$$

with $U_\pm = U_+$ if $j > 0$ and $U_\pm = U_-$ if $j < 0$. Since $v_n^- \to \bar{v}_-$, $v_n^+ \to \bar{v}_+$ and $r_n \to 0$, we can choose $n \in \mathbf{N}$ so large that $B_{r_n}(v_n^-, v_n^+; p) \cap K = \emptyset$. This allows us to construct a deformation η of X such that the surface $\eta \circ G$ has the property that:

(4.7) for any $\theta \in [0,1]^{h+k}$ there is an index j such that $\varphi_j(\eta \circ G(\theta)) < \bar{c}_\pm$.

Using (4.7), by a Miranda fixed point theorem ([24]), on the surface $\eta \circ G$ we can select a path g joining two opposite faces $\eta \circ G(\{\theta_j = 0\})$ and $\eta \circ G(\{\theta_j = 1\})$ such that range $g \subset \{\varphi_j < \bar{c}_\pm\}$. Finally, let $\bar{\gamma}$ be the path obtained by multiplying g by a suitable cut-off function χ on P_j and by translating by $p_j T_\pm$. It turns out that $\bar{\gamma}$ is the required path which gives the contradiction.

The deformation η is obtained as a solution of a Cauchy problem

$$\begin{cases} \dfrac{d\eta}{ds} = -\mathcal{V}(\eta) \\ \eta(0,u) = u \end{cases}$$

ruled by a pseudogradient vector field $\mathcal{V} : X \to X$ for φ which acts in this way. First of all $\mathcal{V}$ is a bounded locally Lipschitz continuous function on X which does not move the points of X outside the set $B = B_{r_n - \frac{1}{3}d_{r_n}}(v_n^-, v_n^+; p)$ and such that the functional φ decreases along its flow lines. This holds asking that:

($\mathcal{V}1$) $\varphi'(u)\mathcal{V}(u) \geq 0 \quad \forall u \in X$, $\|\mathcal{V}(u)\| \leq 1 \quad \forall u \in X$, $\mathcal{V}(u) = 0 \quad \forall u \in X \setminus B$.

To get the property (4.7), we want to use the following argument:

- we can choose $b_\pm > \bar{c}_\pm$ near as we want to $\bar{c}_\pm$ such that starting from a point $u \in \{\varphi_j \leq b_\pm\}$, along the positive flow line $\{\eta(s,u) : s \geq 0\}$, one always remains inside $\{\varphi_j \leq b_\pm\}$.

- if $u \in \bigcap_j \{\varphi_j \leq b_\pm\}$ and the trajectory $\{\eta(s,u) : s \geq 0\}$ crosses an annular region of the type $\mathcal{A}_i = \{u \in B : r_n - \frac{1}{2}d_{r_n} \leq \|u - \tau_{p_i}^\pm v_n^\pm\| \leq r_n - \frac{5}{12}d_{r_n}\} \cap \bigcap_j \{\varphi_j \leq b_\pm\}$ then the functional φ_i decreases of a positive uniform amount $\Delta\varphi_i$ independent of the sequence $(p_{-h}, \ldots, p_k)$.

- we can choose $a_\pm < \bar{c}_\pm$ near as we want to $\bar{c}_\pm$ such that also the sets $\{\varphi_j \leq a_\pm\}$ are positively invariant with respect to the flow η.

Thus, taking $a_\pm$ and $b_\pm$ such that $b_\pm - a_\pm \leq \Delta\varphi_i$, if the trajectory $\{\eta(s,u) : s \geq 0\}$ crosses some $\mathcal{A}_i$ starting from $\{\varphi_i \leq b_\pm\}$ then it reaches the sublevel $\{\varphi_i \leq a_\pm\}$.

These properties are obtained requiring that:

($\mathcal{V}2$) $\varphi_j'(u)\mathcal{V}(u) \geq \nu \quad \forall u \in \mathcal{A}_j$

for some $\nu > 0$ independent of $(p_{-h}, \ldots, p_k)$ and:

($\mathcal{V}3$) $\varphi_j'(u)\mathcal{V}(u) \geq 0 \quad \forall u \in \{a_\pm \leq \varphi_j \leq a_\pm + \delta\} \cup \{b_\pm \leq \varphi_j \leq b_\pm + \delta\}$

for some $\delta > 0$.

Thanks to lemma 2.5 and to the contradiction assumption, for which $B \cap K = \emptyset$, it is possible to construct $\mathcal{V}$ in such a way:

$(\mathcal{V}5)$ $\varphi'(u)\mathcal{V}(u) \geq \nu' \quad \forall u \in B_{r_n - \frac{5}{12}d_{r_n}}(v_n^-, v_n^+; p)$

for some $\nu' > 0$.

This implies that any flow line starting from a point $u \in B_{r_n - \frac{1}{2}d_{r_n}}(v_n^-, v_n^+; p)$ crosses some $\mathcal{A}_i$ for an index i depending on u.

Finally we need a property of $\mathcal{V}$ which permits us to control the error $|\varphi_j(g) - \varphi_j(\chi g)|$ produced by the cut–off procedure. As we will see this can be realized if:

$(\mathcal{V}4)$ $\langle u, \mathcal{V}(u)\rangle_{Q_j} \geq 0 \quad \forall u \in X \setminus Y_\epsilon$ and $\forall j = -h, \ldots, k$

where we assume $M \geq 2m^2 + 3m$ for some $m \in \mathbf{N}$,
$Q_j = [p_j T_+ + m(m+1)T_+, p_{j+1}T_+ - m(m+1)T_+] \quad (1 \leq j \leq k)$,
$Q_j = [p_{j-1}T_- + m(m+1)T_-, p_j T_- - m(m+1)T_-] \quad (-h \leq j \leq -1)$,
$Q_0 = [p_{-1}T_- + m(m+1)T_-, p_1 T_+ - m(m+1)T_+]$ and
$Y_\epsilon = \{\, u \in X \ : \ \|u\|^2_{Q_j} \leq \epsilon \quad \forall j \,\}$
with $\epsilon > 0$ small enough.
The vector field $\mathcal{V}$ as well as the positive constant h chosen at the beginning is assigned by the following lemma, whose proof can be found in [12, 25].

Lemma 4.8. *For any r_n sufficiently small there is $\nu = \nu(r_n) > 0$ such that for any $a_-, a_+, b_-, b_+ \in \mathbf{R}$ and $\delta > 0$ with*

$$\begin{array}{ll} [a_- - \delta, a_- + 2\delta] \subset (0, \bar{c}_-) \setminus \Phi^*_- & [b_- - \delta, b_- + 2\delta] \subset (\bar{c}_-, c^*_-) \setminus \Phi^*_- \\ [a_+ - \delta, a_+ + 2\delta] \subset (0, \bar{c}_+) \setminus \Phi^*_+ & [b_+ - \delta, b_+ + 2\delta] \subset (\bar{c}_+, c^*_+) \setminus \Phi^*_+ \end{array} \tag{4.9}$$

*there exist $p_0 \in \mathbf{N}$ and $\epsilon_1 > 0$ for which the following holds:
for any $\epsilon \in (0, \epsilon_1)$ there is $m \in \mathbf{N}$ such that for each $p \in P(2m^2 + 3m, p_0)$ there exists a locally Lipschitz continuous vector field $\mathcal{V}$: $X \to X$ satisfying $(\mathcal{V}1)$-$(\mathcal{V}4)$.
Moreover, if $B \cap K = \emptyset$ then there is $\nu' > 0$ such that $(\mathcal{V}5)$ holds.*

So, we follow this scheme: we first fix $n \in \mathbf{N}$ such that $\|v_n^\pm - \bar{v}_\pm\| < \frac{\rho}{2}$, $r_n < \min\{\frac{\rho}{2}, r_0\}$ and $B_{2r_n}(v_n^\pm) \subset B_{\bar{r}_\pm}(\bar{v}_\pm)$. In particular we have that $B_{r_n}(v_n^-, v_n^+; p) \subset B_\rho(\bar{v}_-, \bar{v}_+; p)$ for all $p \in P(M, p_0)$ and for all $M, p_0 \in \mathbf{N}$. In correspondence of the value $r_n > 0$ above fixed, lemma

4.8 gives a suitable positive constant ν. Thanks to (3.7), we can choose $a_\pm > \bar{c}_\pm - \min\{h_n, \frac{1}{24}\nu d_{r_n}\}$ and $b_\pm < \min\{c^*_\pm, \bar{c}_\pm + \frac{1}{24}\nu d_{r_n}\}$ and $\delta > 0$ satisfying (4.9). Then lemma 4.8 assigns two values $p_0 \in \mathbf{N}$ and $\epsilon_1 > 0$. Now we take $\epsilon_2 > 0$ such that for any Borel set $A \subseteq \mathbf{R}$ with $|A| \geq 1$ and for any $u \in X$ with $\|u\|^2_A \leq \epsilon_2$ it holds that $\int_A |V_\pm(t,u)|\, dt \leq \|u\|^2_A$. This is possible because $U_\pm$ satisfy (U2). Then we fix $\epsilon \in (0, \min\{\epsilon_1, \frac{1}{2}\epsilon_2, \frac{1}{4}(\bar{c}_+ - a_+), \frac{1}{4}(\bar{c}_- - a_-), \frac{1}{3}r_n^2, \frac{1}{2}d^2_{r_n}\})$. By lemma 4.8 there exists $m_0 \in \mathbf{N}$ such that for any $p \in P(2m_0^2 + 3m_0, p_0)$ there is a vector field $\mathcal{V}_p : X \to X$ satisfying ($\mathcal{V}$1)-($\mathcal{V}$5). Now we apply lemma 3.10 fixing $h = \min\{b_- - \bar{c}_-, b_+ - \bar{c}_+\}$ and finding two paths $\gamma_n^\pm$ with $\operatorname{supp}\gamma_n^\pm(s) \subset [-R, R]$ for any $s \in [0,1]$, where $R > 0$ depends only on n. Moreover we can always assume that $\|v_n^\pm\|^2_{|t|\geq R} < \epsilon$. Then we choose $m > \max\{m_0, R, T_-^{-1}, T_+^{-1}\}$ and we use the contradiction assumption, for which there is $p = (p_{-h}, \dots, p_{-1}, p_1, \dots, p_k) \in P(2m^2 + 3m, p_0)$ such that $B_{r_n}(v_n^-, v_n^+; p) \cap K = \emptyset$. Consequently, there is a vector field $\mathcal{V}_p = \mathcal{V} : X \to X$ that satisfies ($\mathcal{V}$1)-($\mathcal{V}$5).
Finally, for any $s \geq 0$ we consider the continuous function $G_s : [0,1]^{h+k} \to X$ given by

$$G_s(\theta) = \eta(s, G(\theta)) \qquad (\theta \in [0,1]^{h+k})$$

where $G(\theta)$ is defined by (4.6) and η is the flow generated by $-\mathcal{V}$.

LEMMA 4.10. *(i) For any $s \geq 0$ $G_s = G$ on the boundary of* $[0,1]^{h+k}$.

(ii) For any $s \geq 0$ $\operatorname{range} G_s \subseteq Y_\epsilon$.

(iii) There exists $\bar{s} > 0$ such that $\operatorname{range} G_{\bar{s}} \subseteq \bigcup_j \{\varphi_j \leq a_\pm\}$.

Before proving lemma 4.10 we continue the proof of the theorem showing that:

(4.11) there is an index $j \in \{-h, \dots, -1, 1, \dots, k\}$ and a path $\xi \in C([0,1],[0,1]^{h+k})$ such that $\xi(0) \in \{\theta_j = 0\}$, $\xi(1) \in \{\theta_j = 1\}$ and $\varphi_j(G_{\bar{s}}(\theta)) < a_\pm + \epsilon$ for any $\theta \in \operatorname{range}\xi$.

Indeed, if (4.11) were false, for any $i \in \{-h, \dots, -1, 1, \dots, k\}$ the set $D_i = \{\theta \in [0,1]^{h+k} : \varphi_i(G_{\bar{s}}(\theta)) \geq a_\pm + \epsilon\}$ should separate the faces $\{\theta_i = 0\}$ and $\{\theta_i = 1\}$. Then, from a Miranda fixed point theorem

([24]), it follows that $\bigcap_i D_i \neq \emptyset$, that is there exists $\theta \in [0,1]^{h+k}$ such that $\varphi_i(G_{\bar{s}}(\theta)) \geq a_\pm + \epsilon$ for any i, in contrast with the point *(iii)* of lemma 4.10.

From now on, let j be the index for which (4.11) holds. Let us assume that $j > 0$. Clearly the same argument works if $j < 0$. Set $Q = \bigcup_{i=-h}^{k} Q_i$. Let $\chi : \mathbf{R} \to [0,1]$ be a piecewise linear, cut–off function such that $\chi(r) = 1$ if $r \in P_j \setminus Q$ and $\chi(r) = 0$ if $r \in \mathbf{R} \setminus P_j$. Notice that, since $m \geq 2$, for any $u \in X$

$$\|\chi\, u\|^2_{P_j \cap Q} \leq 2\|u\|^2_{P_j \cap Q} \quad \text{and} \quad \|(1-\chi)\, u\|^2_{P_j \cap Q} \leq 2\|u\|^2_{P_j \cap Q} \tag{4.12}$$

and for any $s \in [0,1]$

$$\operatorname{supp} \tau^+_{p_j} \gamma^+_n(s) \subseteq [p_j - R, p_j + R] \subseteq P_j \setminus Q. \tag{4.13}$$

Then we define a path $\gamma : [0,1] \to X$ by setting

$$\gamma(s) = \tau^+_{-p_j} \chi G_{\bar{s}}(\xi(s)) \qquad (s \in [0,1]).$$

By lemma 4.10, part *(i)*, and from (4.13), we have that

$$\gamma(0) = \gamma^+_n(0) \;\text{ and }\; \gamma(1) = \gamma^+_n(1). \tag{4.14}$$

Now we will prove that

$$\operatorname{range} \gamma \subset B_{\bar{r}_+}(\bar{v}_+). \tag{4.15}$$

Indeed, if we set $u = G_{\bar{s}}(\xi(s))$ we have that

$$\begin{aligned} &|\gamma(s) - v^+_n|^2 \\ &\quad = |\chi u - \tau^+_{p_j} v^+_n|^2 + |\tau^+_{p_j} v^+_n|^2_{\mathbf{R} \setminus P_j} \\ &\qquad + |u - \tau^+_{p_j} v^+_n|^2_{P_j \setminus Q} + |\chi u - \tau^+_{p_j} v^+_n|^2_{P_j \cap Q}. \end{aligned} \tag{4.16}$$

By (4.12) and (4.13) it holds that $\|\tau^+_{p_j} v^+_n\|^2_{\mathbf{R} \setminus P_j} \leq \|v^+_n\|^2_{|t| \geq R} \leq \epsilon$ and analogously we also get $\|(1-\chi)\, \tau^+_{p_j} v^+_n\|^2_{P_j \cap Q} \leq 2\|v^+_n\|^2_{|t| \geq R} \leq 2\epsilon$. Consequently from (4.16) we infer that

$$\|\gamma(s) - v^+_n\|^2 \leq 3\epsilon + 3\|u - \tau^+_{p_j} v^+_n\|^2_{P_j}. \tag{4.17}$$

Since, by $(\mathcal{V}1)$, B is η-invariant and, from lemma 3.10, $\operatorname{range} \gamma^+_n \subseteq \overline{B}_{r_n}(v^+_n)$, we deduce that $\|u - \tau^+_{p_j} v^+_n\|_{P_j} \leq r_n$. Thus, from (4.17), we

get that $\|\gamma(s)-v_n^+\|^2<4r_n^2$, because $\epsilon<\frac{1}{3}r_n^2$ and, since $B_{2r_n}(v_n^+)\subset B_{\overline{r}_+}(\bar{v}_+)$, (4.15) follows.
Now, we show that for any $s\in[0,1]$

$$\varphi_+(\gamma(s))<\bar{c}_+. \tag{4.18}$$

As before, we set $u=G_{\bar{s}}(\xi(s))$. It holds that $\varphi_+(\gamma(s))=\varphi_+(\chi\, u)=\varphi_j(\chi\, u)=\varphi_j(u)+\frac{1}{2}\|\chi u\|^2_{P_j\cap Q}-\frac{1}{2}\|u\|^2_{P_j\cap Q}+\int_{P_j\cap Q}[V_+(t,u)-V_+(t,\chi u)]\times dt$. From lemma 4.10.iii, we know that $\varphi_j(u)\le a_+$. Using again lemma 4.10.iii, and (4.11) we estimate $\frac{1}{2}\|\chi\, u\|^2_{P_j\cap Q}\le\|u\|^2_{P_j\cap Q}\le\epsilon$ and, for $\epsilon<\frac{1}{2}\epsilon_2$, $\int_{P_j\cap Q}|V_+(t,u)|\,dt\le\|u\|^2_{P_j\cap Q}$. Hence $\varphi_+(\gamma(s))\le a_++4\epsilon$ and (4.18) follows, because $\epsilon<\frac{1}{4}(\bar{c}_+-a_+)$.
In conclusion, from (4.14), (4.15) and (4.18), γ is a path joining $\gamma_n^+(0)$ with $\gamma_n^+(1)$ inside $B_{\overline{r}_+}(\bar{v}_+)\cap\{\varphi_+<\bar{c}_+\}$ and this gives the contradiction and concludes the proof of the theorem. ♢

Proof of Lemma 4.10. (i) If θ belongs to the boundary of $[0,1]^{h+k}$ then $\theta_i=0$ or $\theta_i=1$ for some $i\in\{-h,\dots,-1,1,\dots,k\}$. Let us suppose for instance that $i>0$ and $\theta_i=0$. From (4.13) and lemma 3.10 *(i)*, we deduce that $\|G(\theta)-\tau^+_{p_i}v_n^+\|^2_{P_i}=\|\gamma_n^+(0)-v_n^+\|^2-\|\tau^+_{p_i}v_n^+\|^2_{\mathbf{R}\setminus P_i}\ge r_n^2-\epsilon\ge\overline{r}^2$ because $\epsilon<\frac{1}{3}r_n^2$. Then $G(\theta)\in X\setminus B_{\overline{r}}$ and consequently, by $(\mathcal{V}1)$, $\eta(s,G(\theta))=G(\theta)$.
(ii) By (4.13), we have that $\|G(\theta)\|_{Q_j}=0$ for any j and so $G(\theta)\in Y_\epsilon$. But $(\mathcal{V}4)$ gives that Y_ϵ is positively η–invariant. Hence $\eta(s,G(\theta))\in Y_\epsilon$ for all $\theta\in[0,1]^{h+k}$ and for all $s\ge0$.
(iii) First of all, $(\mathcal{V}1)$ and $(\mathcal{V}3)$ imply that the sets $\{\varphi_i\le a_\pm\}$ and $\{\varphi_i\le b_\pm\}$ are positively η–invariant sets.
Fix now $\theta\in[0,1]^{h+k}$. If $G(\theta)\notin B_{r_n-\frac{1}{2}d_{r_n}}(v_n^-,v_n^+;p)$ then there is an index i, for example positive, for which $\|G(\theta)-\tau^+_{p_i}v_n^+\|_{P_i}\ge r_n-\frac{1}{2}d_{r_n}$. But, using (4.13) and lemma 3.10 *(iii)*, we have also $\|G(\theta)-\tau^+_{p_i}v_n^+\|_{P_i}\le\|\gamma_n^+(\theta_i)-v_n^+\|\le r_n$. Therefore $\gamma_n^+(\theta_i)\in A_{r_n-\frac{1}{2}d_{r_n},r_n}(v_n^+)$ and, by lemma 3.10 *(iv)* $\varphi_+(\gamma_n^+(\theta_i))\le\bar{c}_+-h_n$. Thus, since $a_+\ge\bar{c}_+-h_n$ we have that $G(\theta)\in\{\varphi_i\le a_+\}$, and, for the positive η–invariance of $\{\varphi_i\le a_+\}$, also $G_{\bar{s}}(\theta)\in\{\varphi_i\le a_+\}$.
Suppose now that $G(\theta)\in B_{r_n-\frac{1}{2}d_{r_n}}$. First, we notice that, from (4.13), lemma 3.10 *(iii)* and by the definition of φ_i and h, $G(\theta)\in\bigcap_i\{\varphi_i\le b_\pm\}$. Hence, on one hand, for the positive η–invariance of each $\{\varphi_i\le b_\pm\}$, all the positive trajectory $s\mapsto G_s(\theta)$ remains in $\bigcap_i\{\varphi_i\le b_\pm\}$. On the other hand, we claim that:

(4.19) as $s \geq 0$ increases, the curve $s \mapsto G_s(\theta)$ must go out from $B_{r_n - \frac{5}{12} d_{r_n}}$ in a finite time $\bar{s} \geq 0$ independent of θ.

During this amount of time $\bar{s}$, $G_s(\theta)$ crosses the annular region $\mathcal{A}_i$. In fact, there exists an index i, let us say positive, such that $\|G_{s^1_\theta}(\theta) - \tau^+_{p_i} v^+_n\|_{P_i} = r_n - \frac{1}{2} d_{r_n}$, $\|G_{s^2_\theta}(\theta) - \tau^+_{p_i} v^+_n\|_{P_i} = r_n - \frac{5}{12} d_{r_n}$ and $r_n - \frac{1}{2} d_{r_n} \leq \|G_s(\theta) - \tau^+_{p_i} v^+_n\|_{P_i} \leq r_n - \frac{5}{12} d_{r_n}$ for $s \in (s^1_\theta, s^2_\theta)$. Then, by $(\mathcal{V}2)$, $\varphi'_i \mathcal{V} \geq \nu$ along the curve described by $G_s(\theta)$ as s goes from s^1_θ to s^2_θ and consequently, $\varphi_i(G_s(\theta))$ decreases. Precisely $\varphi_i(G_{s^2_\theta}(\theta)) \leq \varphi_i(G_{s^1_\theta}(\theta)) - \nu\,(s^2_\theta - s^1_\theta)$. But, using $(\mathcal{V}1)$ it holds that $\frac{1}{12} d_{r_n} \leq \|G_{s^2_\theta}(\theta) - G_{s^1_\theta}(\theta)\| \leq \int_{s^1_\theta}^{s^2_\theta} \|\mathcal{V}(\eta(s, G(\theta)))\|_{P_i} ds \leq s^2_\theta - s^1_\theta$ and so $\varphi_i(G_{s^2_\theta}(\theta)) \leq b_+ - \frac{1}{12} d_{r_n} \nu \leq \bar{c}_+ - \frac{1}{24} d_{r_n} \nu \leq a_+$. Then the positive η–invariance of $\{\varphi_i \leq a_+\}$ implies that $\varphi_i(G_{\bar{s}}(\theta)) \leq a_+$.
Now, it remains to prove the claim (4.19). Arguing by contradiction, if (4.19) is false, then there are sequences $(s_n) \subset \mathbf{R}_+$ and $(\theta_n) \subset [0,1]^{h+k}$ such that $s_n \to +\infty$, $\theta_n \to \theta$ and, for any $n \in \mathbf{N}$, $G_s(\theta_n) \in B_{r_n - \frac{5}{12} d_{r_n}}(v^-_n, v^+_n; p)$ for $s \in [0, s_n]$. Then, from $(\mathcal{V}5)$ $\varphi(G_{s_n}(\theta_n)) \leq \varphi(G(\theta_n)) - s_n \nu'$ and so $\varphi(G_{s_n}(\theta_n)) \to -\infty$. This is in contrast with the fact that $\varphi(B^{b\pm}_r)$ is bounded. $\diamondsuit$

References

[1] Alama S. and Li Y. Y., *On "Multibump" Bound States for Certain Semilinear Elliptic Equations*, J. Diff. Eq. **96** (1992), 89-115.

[2] Angenent S., *The Shadowing Lemma for Elliptic PDE*, Dynamics of Infinite Dimensional Systems, (S.N. Chow and J.K. Hale eds.) **F37**, 1987.

[3] Ambrosetti A. and Bertotti M. L., *Homoclinics for second order conservative systems*, In *Partial Differential Equations and Related Subjects*, ed. M. Miranda, Pitman Research Notes in Math. Ser. (London, Pitman Press), 1992.

[4] Ambrosetti A. and Coti Zelati V., *Multiple Homoclinic Orbits for a Class of Conservative Systems*, Rend. Sem. Mat. Univ. Padova, **89** (1993), 177-194.

[5] Benci V. and Giannoni F., *Homoclinic orbits on compact manifolds*, J. Math. Anal. Appl. **157** (1991), 568-576.

[6] Bertotti M. L. and Bolotin S., *Homoclinic Solutions of Quasiperiodic Lagrangian Systems*, Differential and Integral Equations **8** (1995), 1733-1760.

[7] Bessi U., *A Variational Proof of a Sitnikov-like Theorem*, Nonlin. Anal. T. M. A. **20** (1993), 1303-1318.

[8] Bessi U., *Global Homoclinic Bifurcation for Damped Systems*, Math. Z. **218** (1995), 387-415.

[9] Bessi U., *Homoclinic and Period-doubling Bifurcations for Damped Systems*, Ann. Inst. H. Poincaré, Anal. NonLinéaire **12** (1995), 1-25.

[10] Bolotin S. V., *Existence of homoclinic motions*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **6** (1980), 98-103.

[11] Caldiroli P., *Existence and multiplicity of homoclinic orbits for potentials on unbounded domains*, Proc.Roy.Soc. Edinburgh **124A** (1994), 317-339.

[12] Caldiroli P. and Montecchiari P., *Homoclinic orbits for second order Hamiltonian systems with potential changing sign*, Comm. Applied Nonlinear Analysis, **1** (1994), 97-129.

[13] Cieliebak K. and Séré E., *Pseudo-holomorphic curves and multiplicity of homoclinic orbits*, Duke Math. J. **77** (1995), 483-518.

[14] Coti Zelati V., Ekeland I. and Séré E., *A Variational approach to homoclinic orbits in Hamiltonian systems*, Math. Ann. **288** (1990), 133-160.

[15] Coti Zelati V. and Rabinowitz P. H., *Homoclinic orbits for second order hamiltonian systems possessing superquadratic potentials*, J. Amer. Math. Soc. **4** (1991), 693-727.

[16] Guckeneimer J. and Holmes P., *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Springer–Verlag, 1983.

[17] Giannoni F., Jeanjean L. and Tanaka K., *Homoclinic orbits on non-compact Riemannian manifolds for second order Hamiltonian systems*, Rend. Sem. Mat. Univ. Padova, **93** (1995), 153-176.

[18] Girardi M. and Yanheng D., *Periodic and homoclinic solutions to a class of Hamiltonian systems with the potential changing sign*, Dyn. Syst. and Appl. **2** (1993), 131-145.

[19] Hofer H. and Wysocki K., *First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems*, Math. Ann. **288** (1990), 483-503.

[20] Jeanjean L., *Existence of connecting orbits in a potential well*, Dyn. Sys. Appl. **3** (1994), 537-561.

[21] Kirchgraber U. and Stoffer D., *Chaotic behavior in simple dynamical systems*, SIAM Review **32** (1990), 424-452.

[22] Lions P. L., *The concentration–compactness principle in the calculus of variations*, Rev. Mat. Iberoamericana **1** (1985), 145-201.

[23] MELNIKOV V. K., *On the stability of the center for periodic perturbations*, Trans. Moscow Math. Soc. **12** (1963), 1-57.

[24] MIRANDA C., *Un'osservazione su un teorema di Brouwer*, Boll. Unione Mat. Ital. **3** (1940), 5-7.

[25] MONTECCHIARI P., *Existence and multiplicity of homoclinic solutions for a class of asymptotically periodic second order Hamiltonian systems*, Ann. Mat. Pura ed App. (4) **CLXVIII** (1995), 317-354. See also: Multiplicity of homoclinic solutions for a class of asymptotically periodic second order Hamiltonian systems, *Rend. Mat. Acc. Lincei s. 9* **4**, 265-271 (1993).

[26] RABINOWITZ P. H., *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Roy. Soc. Edinburgh **114 A** (1990), 33-38.

[27] RABINOWITZ P. H. and TANAKA K., *Some results on connecting orbits for a class of Hamiltonian systems*, Math. Z. **206** (1991), 473-479.

[28] SERRA E, TARALLO M. and TERRACINI S., *On the existence of homoclinic solutions for almost periodic second order systems*, Preprint, 1994.

[29] SÉRÉ E., *Existence of infinitely many homoclinic orbits in Hamiltonian systems*, Math. Z. **209** (1992), 27-42.

[30] SÉRÉ E., *Looking for the Bernoulli shift*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **10** (1993), 561-590.

[31] SÉRÉ E., *Homoclinic orbits on compact hypersurfaces in* $\mathbf{R}^{2N}$*, of restricted contact type*, Comm. Math. Phys. **172** (1995), 293-316.

[32] TANAKA K., *Homoclinic orbits in a first order superquadratic Hamiltonian system: Convergence of subharmonic orbits*, J. Diff. Eq. **94** (1991), 315-339.

[33] TANAKA K., *A note on the existence of multiple homoclinic orbits for a perturbed radial potential*, Nonlinear Diff. Eq. Appl. **1** (1994), 149-162.

[34] WIGGINS S., *Global bifurcations and chaos*, Applied Mathematical Sciences, Springer–Verlag, **73**, 1988.

Pervenuto in Redazione l'8 Luglio 1996.