Single Non-Isolated Point Spaces

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Sommario. - *Si caratterizzano le proprietà di uno spazio con un solo punto non isolato utilizzando il resto della compattificazione di Čech-Stone dello spazio discreto corrispondente. Accanto a vari nuovi risultati, vengono pure presentate nuove dimostrazioni di alcuni fatti noti.*

Summary. - *We give characterization properties of spaces with a single non-isolated point in terms of Čech-Stone compactification's remainders of the corresponding discrete spaces. Together with various new results, we also present new proofs of several known facts.*

1. Introduction

The aim of this article is to characterize some properties of spaces with a single non-isolated point in terms of Čech-Stone compactification's remainders of the corresponding discrete spaces.

All spaces are assumed to be infinite.

Let βD be the Čech-Stone compactification of the discrete space D. For $A \subset D$ we will write $A^* = \overline{A} \setminus D$. So $D^* = \beta D \setminus D$ is the

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Čech-Stone remainder of D. If γ is a family of subsets of D then γ^* will denote the family $\{A^* : A \in \gamma\}$, conversely if γ^* is a family of clopen subsets of D^* then γ will be any family obtained by selecting for any $B \in \gamma^*$ some $A \subset D$ such that $A^* = B$.

As usual, points of βD are to be considered as ultrafilters on D. The whole space βD may be splitted into disjoint "pieces": D is the subset of all fixed ultrafilters, $D^*_{\aleph_0}$ is the subset of all free ultrafilters which have some countable subset and so on. For instance, if $|D| = \aleph_1$ then $\beta D = D \cup D^*_{\aleph_0} \cup D^*_{\aleph_1}$.

$Int(T)$ denotes the interior of T.

We will write D_F for the space $D \cup \{F\}$, where D is discrete and F is the single non-isolated point. The trace on D of the filter of all neighbourhoods of the point F, i.e. the family $\{A \setminus \{F\} = A \cap D : A$ is a neighbourhood of F in $D_F\}$ will be denoted by $\mathcal{F}$. On the other hand, if we have a filter $\mathcal{F}$ on D then we may add one point F to D and consider the space $D_F = D \cup \{F\}$ in which F is the only non-isolated point with neighbourhoods $\{A \cup \{F\} : A \in \mathcal{F}\}$.

For every filter $\mathcal{F}$ on D we may consider the subset $F = \cap\{\overline{A} : A \in \mathcal{F}\} \subset \beta D$. These two objects $\mathcal{F}$ and F correspond each other and we will not distinguish between them. It will be clear from the context what concrete object we mean. As we will consider only free filters, the set F will be always a subset of D^*.

FACT 1.1. *D^* is a Tychonoff zerodimensional compact space in which subsets of the form A^* are clopen and provide a base for D^*.*

FACT 1.2. *D^* is a F- space. This means that two disjoint open F_σ-sets have disjoint closures and hence are separated by two disjoint clopen sets. Equivalently, if $\mathcal{E}^*$ is a countable family of clopen subsets of D^* and $\cup\mathcal{E}^*$ is contained in some closed G_δ-set $T \subset D^*$ then $\overline{\cup\mathcal{E}^*} \subset Int(T)$.*

FACT 1.3. *For a space D_F the following things are true:*
a) If $A \subset D$ then $F \in \overline{A}$ iff $F \cap A^* \neq \emptyset$;
b) if $A \subset D$ is a countable set, then A converges as a sequence to F iff $A^* \subset F$;
c) let $D_{F'}$ be another space and consider the subspace $\Delta \cup \{F, F')\} = \{(x, x) : x \in D\} \cup \{(F, F')\}$ of the product $D_F \times D_{F'}$. Such subspace is a space with the single non-isolated point (F, F'). The trace of the neighbourhoods filter of (F, F') on the set Δ is the family $\mathcal{G} =$

$\{A \times B \cap \Delta : A \in \mathcal{F}, B \in \mathcal{F}'\}$. *By identifying* Δ *with* D, *the family* $\mathcal{G}$ *corresponds to the family* $\{A \cap B : A \in \mathcal{F}, B \in \mathcal{F}'\}$, *which in turn corresponds in* D^* *to the set* $F \cap F'$. *Thus, The space* $\Delta \cup \{(F, F')\}$ *is nothing else that* $D_{F \cap F'}$.

In the sequel we will use these Facts without explicit reference.

2. Characterization properties of tightness type.

Let X be a topological space and $x \in X$. Recall that

a) x is a point of countable tightness if whenever $x \in \overline{A}$ there exists a countable set $B \subset A$ such that $x \in \overline{B}$;

In this section we are interested in various modifications of this classical notion.

b) x is a point of countable fan-tightness (see [A3]) if for any countable family $\{A_n : n \in \omega\}$ of subsets of X such that $x \in \overline{A}_n$ for every $n \in \omega$ it is possible to select finite sets $K_n \subset A_n$ in such a way that $x \in \overline{\cup\{K_n : n \in \omega\}}$;

c) x is a tight point (see [BM]) if for every family $\mathcal{E}$ of subsets of X that clusters at x there exists a countable family $\mathcal{S} \subset \mathcal{E}$ that clusters at x ($\mathcal{E}$ clusters at x if for every neighbourhood V of x there exists some $E \in \mathcal{E}$ such that $|E \cap V| \geq \aleph_0$).

PROPOSITION 2.1. *In the space* D_F *the point* F *is:*
a) a point of countable tightness iff $F = \overline{F \cap D^*_{\aleph_0}}$;
b) a point of countable fan-tightness iff for every closed G_δ*-set* $T \subset D^*$ $T \cap F \neq \emptyset$ *implies* $Int(T) \cap F \cap D^*_{\aleph_0} \neq \emptyset$;
b′) a point of countable fan-tightness iff it has countable tightness and for every closed G_δ*-set* $T \subset D^*$ $T \cap F \neq \emptyset$ *implies* $Int(T) \cap F \neq \emptyset$;
c) a tight point iff for every family $\mathcal{E}^*$ *of clopen subsets of* D^* *satisfying* $\overline{\cup\mathcal{E}^*} \cap F \neq \emptyset$ *there exists a countable family* $\mathcal{S}^* \subset \mathcal{E}^*$ *for which* $\overline{\cup\mathcal{S}^*} \cap F \neq \emptyset$.

Proof. a) the assertion "if $F \in \overline{A} \subset D_F$ then there exists some countable set $B \subset A$ for which $F \in \overline{B}$" is equivalent in D^* to "$A^* \cap F \neq \emptyset$ implies $B^* \cap F \neq \emptyset$ for some countable set $B \subset A$". The latter is in turn equivalent to "$A^* \cap F \neq \emptyset$ implies $A^* \cap F \cap D^*_{\aleph_0} \neq \emptyset$", that is just $F = \overline{F \cap D^*_{\aleph_0}}$.

b) let F be a point of countable fan-tightness in D_F and let T be a closed G_δ-subset of D^* such that $T \cap F \neq \emptyset$. We can assume $T = \cap\{A_n^* : n \in \omega\}$ and $A_{n+1}^* \subset A_n^*$ for every $n \in \omega$. Then according to the definition we can select finite sets $K_n \subset A_n$ in such a way that $F \in \overline{K}$, where $K = \cup\{K_n : n \in \omega\}$. So K is a countable set, $K^* \subset T$ and consequently $Int(T) \cap F \cap D_{\aleph_0}^* \neq \emptyset$. Conversely, let $\{A_n : n \in \omega\}$ be a family of subsets of D such that $A_{n+1} \subset A_n$ for each $n \in \omega$ and $F \in \overline{A_n}$ for each n and let $T = \cap\{A_n^* : n \in \omega\}$. Then since $T \cap F \neq \emptyset$, we have $Int(T) \cap F \cap D_{\aleph_0}^* \neq \emptyset$ and so there exists a set $K = \{x_n : n \in \omega\} \subset D$ such that $K^* \subset T$ and $K^* \cap F \neq \emptyset$. Letting $K_n = ((A_n \setminus A_{n+1}) \cap K) \cup \{x_k \in A_n : k \leq n\}$, we are done.

$\diamondsuit$

Although it is very easy to find a space with countable tightness which has not countable fan-tightness [AB], we present here one more in the language of this paper. (It is the well known countable sequential fan.)

EXAMPLE 2.2. A point of countable tightness which has not countable fan-tightness.

Construction. Let $\{A_n^* : n \in \omega\}$ be a family of non-empty clopen disjoint subsets in ω^* and let $F = \overline{\cup\{A_n^* : n \in \omega\}}$, then $T = \omega^* \setminus \cup\{A_n^* : n \in \omega\}$ is a closed G_δ -set for which $T \cap F \neq \emptyset$ and $Int(T) \cap F = \emptyset$.

$\diamondsuit$

FACT 2.3. [AB] *If $F \in \omega^*$ then F has countable fan-tightness iff F is a P-point in ω^*.*

Proof. F is a P-point iff for any closed G_δ-set T in ω^* we have $F \notin T \setminus Int(T)$. Now apply condition b) (or b′)) in Proposition 2.1.

$\diamondsuit$

PROPOSITION 2.4. *If F is an infinite separable subset of D^* then F has not countable fan-tightness.*

Proof. Let $F = \overline{\{x_n : n \in \omega\}}$ Since F is infinite, we may find clopen subsets $A_n^*, n \in \omega$ such that $x_n \in A_n^*$ and no finite subfamily of $\{A_n^* : n \in \omega\}$ covers F. Letting $T = \cap\{D^* \setminus A_n^* : n \in \omega\}$, we have $T \cap F \neq \emptyset$, but $Int(T) \cap F = \emptyset$.

$\diamondsuit$

FACT 2.5. [BM] *If F is a tight point then it has countable fan-tightness.*

Proof. We show first that F has countable tightness. Let $A \subset D$ and $F \in \overline{A} \subset D_F$. The family $\mathcal{E}$ of all countable infinite subsets of A clearly clusters at F and so there exists a countable subfamily $\mathcal{S}$ of $\mathcal{E}$ which also clusters at F. Since $\cup\mathcal{S}$ is countable and $F \in \overline{\cup\mathcal{S}}$, we see that F has countable tightness. Now we can use b′) in Proposition 2.1. If F has not countable fan-tightness then there exists some closed G_δ-subset T such that $T \cap F \neq \emptyset$ but $Int(T) \cap F = \emptyset$. Let $\mathcal{E}^*$ be the family of all clopen sets lying in T. Clearly, we have $F \cap T = F \cap \overline{\cup\mathcal{E}^*} \neq \emptyset$, but no countable family $\mathcal{S}^* \subset \mathcal{E}^*$ can satisfy $F \cap \overline{\cup\mathcal{S}^*} \neq \emptyset$. Looking at 2.1 c), this means that F is not tight.

♢

FACT 2.6. [BM] *If $F \in \omega^*$ then F is not a tight point.*

Proof. By Fact 2.3 we need only to consider F a P-point. But it is trivial to observe that no P-point can be tight.

♢

EXAMPLE 2.7. A non tight point of countable fan-tightness.

Construction. Recall that a Hausdorff-Luzin gap in ω^* (see, for example [HB, p. 125]) is a pair of transfinite increasing sequences $\mathcal{A}^* = \{A^*_\alpha : \alpha \in \omega_1\}$, $\mathcal{B}^* = \{B^*_\alpha : \alpha \in \omega_1\}$ of clopen subsets such that every A^*_α does not intersect every B^*_β, but the whole sequences $\mathcal{A}^*$ and $\mathcal{B}^*$ are not separated, i.e. there is no clopen subset E^* such that $\cup\mathcal{A}^* \subset E^*$ and $(\cup\mathcal{B}^*) \cap E^* = \emptyset$.
If $F = \overline{\cup\mathcal{A}^*}$ then this subset is the desired one. Indeed, let $T = \cap\{C^*_n : n \in \omega\}$ be a closed G_δ-subset of ω^* and assume $C^*_{n+1} \subset C^*_n$. If $T \cap F \neq \emptyset$ then $C^*_n \cap F \neq \emptyset$ for every $n \in \omega$. Now, for every $n \in \omega$ there exists some $\alpha_n \in \omega_1$ such that $C^*_n \cap A^*_{\alpha_n} \neq \emptyset$. Letting $\epsilon = Sup\{\alpha_n : n \in \omega\} + 1$, we have $C^*_n \cap A^*_\epsilon \neq \emptyset$ for every $n \in \omega$ and so $T \cap A^*_\epsilon \neq \emptyset$. This implies $Int(T) \cap A^*_\epsilon \neq \emptyset$ and a fortiori $Int(T) \cap F \neq \emptyset$. Thus F has countable fan-tightness. The fact that F is not a tight point is evident.

♢

FACT 2.8. [BM] *The product $\omega_F \times S_\mathfrak{C}$ has countable tightness iff F is a tight point ($S_\mathfrak{C}$ is the sequential fan of size $\mathfrak{C}$).*

Since the product of a sequential space and a countably compact regular space with countable tightness has as well countable tightness, we have the following:

COROLLARY 2.9. *If the space ω_F can be embedded into a countably compact regular space with countable tightness then F is a tight point.*

3. Characterization properties of Fréchet-Urysohn type.

Let X be a topological space and $x \in X$. Recall that

a) x is a Fréchet-Urysohn (FU) point provided that if $x \in \overline{A}$ then there exists a sequence lying in A and converging to x;

b) x is a strongly Fréchet-Urysohn (SFU) point if for every decreasing family $\{A_n : n \in \omega\}$ of subsets of X such that $x \in \overline{A_n}$ for every $n \in \omega$ then there exist $x_n \in A_n$ such that the sequence $\{x_n : n \in \omega\}$ converges to x;

c) x is a bisequential point provided that if x belongs to the adherence of some filter ξ then there exists a filter ν with a countable base which converges to x and is synchronous with ξ (synchronous means that for each $A \in \xi$ and each $B \in \nu$ the intersection $A \cap B$ is not empty).

PROPOSITION 3.1. *In a space D_F the point F is:*
a) a FU point iff $F = \overline{Int(F)}$, i.e. F is a regular closed subset in D^*;
b) a SFU point iff, for every closed G_δ-set $T \subset D^*$, $T \cap F \neq \emptyset$ implies $T \cap Int(F) \neq \emptyset$ (or equivalently $Int(T) \cap Int(F) \neq \emptyset$;
c) a bisequential point provided that F is a union of closed G_δ -subsets of D^*.

Proof. The simple proofs can be obtained as in Proposition 2.1. Observe that trivially 3.1a implies 2.1a, because $D^*_{\aleph_0}$ is dense in D^*. ♢

The second named author [M2] has already given analogous characterizations for the countable case.

We continue by presenting other results on FU-points. Some of these have been previously considered by other topologists, but some other are new.

EXAMPLE 3.2. [M3] *There are two countable SFU spaces D_{F_A} and D_{F_B} whose product is not a FU space. Moreover, both points F_A and F_B are not tight points and hence not bisequential.*

Construction. We consider a Hausdorff-Luzin gap as in Example 2.7. Let us define $F_{\mathcal{A}} = \overline{\cup \mathcal{A}^*}$ and $F_{\mathcal{B}} = \overline{\cup \mathcal{B}^*}$. These two subsets of ω^* have the required properties. Let us check only that the product $\omega_{F_{\mathcal{A}}} \times \omega_{F_{\mathcal{B}}}$ is not a FU space. Take the subspace $Y = \{(n,n) : n \in \omega\} \cup \{(F_{\mathcal{A}}, F_{\mathcal{B}})\}$. As in 1.3 c), we see that the space Y is homeomorphic to ω_F, where F corresponds in the space ω^* to the set $F_{\mathcal{A}} \cap F_{\mathcal{B}}$ which satisfies $Int(F) = \emptyset$.

$\diamondsuit$

EXAMPLE 3.3. (CH) [M2]. *There are two countable SFU spaces D_{F_1} and D_{F_2} whose product contains a single ultrafilter P-point subspace.*

Construction. It is know that under CH the subspace $\omega^* \setminus \{p\}$, where p is a P-point, is homeomorphic to the space $D^*_{\aleph_0}$ - the subspace of all free "countable" ultrafilters on a discrete space D of cardinality $\aleph_1$. Let us divide D in two disjoint uncountable mutually complemented parts A and B, then let $A^*_{\aleph_0}$ and $B^*_{\aleph_0}$ be the corresponding subspaces of $D^*_{\aleph_0}$ consisting of all free "countable" ultrafilters on the sets A and B. These two subspaces are homeomorphic to two disjoint mutually complemented subspaces of the space $\omega^* \setminus \{p\}$. Next, adding to these sets the P-point p, we get the closed sets F_1 and F_2 we are looking for. It is important to observe that $F_1 \cap F_2 = \{p\}$.

$\diamondsuit$

In [M2] it is shown that there is a bisequential non first countable point.

For a stronger example see 3.8 below.

Now we consider another classification of Fréchet-Urysohn properties.

In some sense, before we were looking at a filter F "from outside" and now we are going to look at F "from inside".

We need to remind the properties ($\alpha 1 - \alpha 4$), introduced by Arhangel'skiĭ[A2].

According with the current usage, the properties that we describe below differ slightly from those of Arhangel'skiĭ.

For a point x let us fix a countable family of non trivial sequences $\{l_n : n \in \omega\}$ converging to x. Such a family is called a fan at the point x.

The point $x \in X$ has property $(\alpha i), i \in \{1,2,3,4\}$ if, for every fan of sequences $\{l_n : n \in \omega\}$ converging to x, there exists a nontrivial converging sequence l, such that:
$(\alpha 1)$: $|l_n \setminus l| < \aleph_0$ for every $n \in \omega$;
$(\alpha 2)$: $l_n \cap l \neq \emptyset$ for every $n \in \omega$ ($\alpha 5$ by Arhangel'skiĭ), this is equivalent to $|l_n \cap l| = \aleph_0$ for every $n \in \omega$;
$(\alpha 3)$: $|l_n \cap l| = \aleph_0$ for infinitely many $n \in \omega$;
$(\alpha 4)$: $l_n \cap l \neq \emptyset$ for infinitely many $n \in \omega$.

Now a point (and also a space) is called iFU if it is FU and has (in each of its points) the property αi.

PROPOSITION 3.4. *In a space D_F the FU point F is a iFU point iff for every countable family $\mathcal{E}^*$ of clopen subsets lying in F*
1) $\overline{\cup \mathcal{E}^*} \subset Int(F)$;
2) there exists a clopen set $V^* \subset F$ such that $V^* \cap E^* \neq \emptyset$ for every $E^* \in \mathcal{E}^*$;
3) there exists a clopen set $V^* \subset F$ such that $V^* \cap E^* \neq \emptyset$ for infinitely many $E^* \in \mathcal{E}^*$;
4) $(((\cup \mathcal{E})^*) \setminus \cup(\mathcal{E}^*)) \cap Int(F) \neq \emptyset$.

PROPOSITION 3.5. [A2] *Every bisequential point is 3FU.*

Proof. Let us check that for a bisequential point F the condition 3 of Proposition 3.4 is fulfilled. Let $\mathcal{E}^*$ be a countable infinite family of non empty clopen subsets lying in F. Let x be any limit point for this family, i.e. every neighbourhood of $x \in F$ intersects infinitely many members of $\mathcal{E}^*$. As F is a bisequential point, there is a closed G_δ subset T such that $x \in T \subset F$. Let $T = \cap\{A_n^* : n \in \omega\}$ and $A_{n+1}^* \subset A_n^*$. Choose an infinite family $\{E_n^* : n \in \omega\} \subset \mathcal{E}^*$ such that $A_n^* \cap E_n^* \neq \emptyset$ and let $K = \cup\{A_n \cap E_n : n \in \omega\}$. Obviously, K intersects infinitely many members of $\mathcal{E}$. It remains to verify that $K^* \subset F$. For every $n \in \omega$ we have $K^* \subset (F \cup A_n^*)$, hence $K^* \subset (F \cup (\cap\{A_n^* : n \in \omega\})$. Since the last intersection lies in F, all is done.

$\diamondsuit$

FACT 3.6. [A2] *SFU points coincide with 4FU points.*

Proof. Let us check that for a 4FU point F condition b) of Proposition 3.1 is fulfilled. Let T be a closed G_δ-subset and $T \cap F \neq \emptyset$. Let $T = \cap\{A_n^* : n \in \omega\}$. We may assume that $A_{n+1}^* \subset A_n^*$. For any n select a non empty clopen set $E_n^* \subset A_n^* \cap F$ and observe that for any n we have $(\cup\{E_n : n \in \omega\})^* = E_1^* \cup \cdots \cup E_n^* \cup (\cup\{E_i > n\})^* \subset E_1^* \cup \cdots \cup E_n^* \cup A_{n+1}^*$. Therefore, we have $(\cup\{E_n : n \in \omega\})^* \setminus \cup\{E_n^* : n \in \omega\} \subset T$ and, using the fact that F is 4FU, we get $T \cap Int(F) \neq \emptyset$. For the converse, let F be a SFU point. Let $\{E_n^* : n \in \omega\}$ be a countable family of clopen subsets lying in F and let $K_n = \cup\{E_i : i \in \omega \setminus n\}$. It is evident that $T = \cap\{K_n^* : n \in \omega\}$ is a closed G_δ-subset which intersects F. Since $T \cap Int(F) \neq \emptyset$, the requirement for F to be 4FU is satisfied.

$\diamondsuit$

EXAMPLE 3.7. *There is a bisequential non 2FU compact space.*

Construction. Let T be the well known "two arrows" space. T is a separable first countable non metrizable compact space. So its diagonal $\Delta \subset T^2$ is a separable non G_δ -subset of T^2. By identifying Δ to a single point, we get a bisequential non 2FU compact space. Only that Δ is not a 2FU point needs to be verified.
To this end, recall that the space "two arrows" T is the union (but not the topological sum!) of two spaces "one arrow" (or Sorgenfrey's line). One of these Sorgenfrey's lines is a "right arrow" $R = [0, 1)$, basic open sets in which are semiintervals $[a, b)$. The other one is the symmetric "left arrow", $L = (0, 1]$. Let us fix a countable dense subset $S = \{(s_n, s_n) : n \in \omega\} \subset \Delta \subset T^2$ lying on the "right arrow" $R \subset \Delta$. For any n choose a sequence $L_n \subset T^2 \setminus \Delta$ converging to (s_n, s_n) and lying on some "horizontal" line, i.e. $L_n = \{(x_i^n, s_n) : n \in \omega\}$. Furthermore, all these sequences will go "from right to left", i.e. $x_{i+1}^n < x_i^n$, all x_i^n belong to "horizontal" rightarrows. If we suppose that Δ is a 2FU point then we must have some sequence L which converges to Δ and intersects every sequence L_n in infinitely many elements. By replacing L_n with $L_n \cap L$, this is equivalent to the condition that $\cup\{L_n : n \in \omega\} \setminus \mathcal{V}$ is finite for every neighbourhood $\mathcal{V}$ of Δ in T^2. We will show that it is impossible to find some sequences like L_n such that the last condition is fulfilled.
Now let us note that a base of neighbourhoods of Δ in T^2 is formed by neighbourhoods of the type $\mathcal{V} = V_1^2 \cup \cdots \cup V_n^2$, where $\{V_1, \ldots, V_n\}$ is some finite disjoint cover of T. So, $(x, y) \in \mathcal{V}$ holds iff both points

x, y belong to some V_j. By "projecting" to the "horizontal" factor we can reduce our consideration as follows.
We have a "right arrow" space R, its dense subset $\{s_n : n \in \omega\}$ and for any $n \in \omega$ there is a sequence L_n lying on the right of s_n and converging to it and let points of this sequence go "from right to left". Our goal will be achieved by finding a binary disjoint cover γ of R in such a way that infinitely many sequences L_n have points lying "outside" of γ in the sense that these points belong to different members of γ. Consequently, the corresponding "hung up" set of all these points will be outside the neighbourhood $\mathcal{V} = \cup\{V^2 : V \in \gamma\}$ of Δ.
Let d be an usual distance on R (understood as a set of reals).
Let $r_n = diam(L_n) = \sup\{d(a, b) : a, b \in L_n\}$. If the sequence $\{r_n : n \in \omega\}$ does not converge to 0 then, in order to find the binary cover, select an infinite set $K \subset \omega$, a real number $r > 0$ and points $x^i_{n_i} \in L_i$ for every $i \in K$ such that $d(s_i, x^i_{n_i}) \geq r$. Fix an infinite set $K' \subset K$ and a real number z' in such a way that $\{x^j_{n_j} : j \in K'\}$ is a sequence converging to z' with respect to the usual topology of the real line. Next, fix another infinite set $K'' \subset K'$ and a real number z'' in such a way that $\{s_j : j \in K''\}$ is a sequence converging to z'' again with respect to the real line. Certainly is $z'' < z'$ and we see that, for any $c \in (z'', z')$, the sequences $L_j, j \in K''$ have points in both $[0, c)$ and $[c, 1)$. So the cover $\{[0, c), [c, 1)\}$ has the required property. Assume now that $\{r_n : n \in \omega\}$ converges to 0. Let us consider the sequence $L_0 = \{x^0_i : i \in \omega\}$. We can find some sequence L_{i_1} such that $x^0_1 < \inf(L_{i_1}) < \max(L_{i_1}) < x^0_0$. Next, we can find some sequence L_{i_2} such that $x^1_1 < \inf(L_{i_2}) < \max(L_{i_2}) < x^1_0$ and so on. If c is any limit point (in the sense of real line) for the sequence $\{x^k_0 : k \in \omega\}$ then infinitely many points of the sequences L_{i_k} are "outside" of the cover $\{[0, c), [c, 1)\}$. So all is done.

$\diamondsuit$

EXAMPLE 3.8. *There is a bisequential non 2FU point on a countable space.*

Construction. The space T^2 in 3.7 has a countable dense set $D \subset T^2 \setminus \Delta$. By declaring all points of D to be isolated, the construction in 3.7 shows that the space D_Δ is the required one.

$\diamondsuit$

Observe that, in the last two examples, both spaces are 3FU according to Proposition 3.5.

EXAMPLE 3.9. *The two SFU spaces of Proposition 3.3 are 2FU.*

Proof. The proof is similar to that of Example 2.7.

♢

There are not too many results in ZFC concerning ω^* and its subspaces. The elegant result of P. Simon [S] about two compact FU spaces whose product is not FU is one of them. Later we will extract something from the Simon's construction.

PROPOSITION 3.10. *If Σ^* is a disjoint family of clopen subsets of ω^* such that $F = \omega^* \setminus \cup\Sigma^*$ is regular closed then F is a 4FU (=SFU) point.*

Proof. We could refer to Arhangel'skiĭresult from [A2]: Each subspace of a countably compact FU space is 4FU. However, our direct proof is not too heavy. Let us assume that Σ^* is infinite, the other case being trivial. Use condition b) of Proposition 3.1 for SFU(=4FU). Let T be a closed G_δ-subset and $T \cap F \neq \emptyset$. We have to prove that $Int(T) \cap F \neq \emptyset$. Let us assume the contrary then $Int(T) \subset \cup\Sigma^*$. Since $Int(T)$ is a open non clopen subset, we can fix a countable infinite family $\Sigma'^* \subset \Sigma^*$ whose members intersect $Int(T)$. Let σ^* be the family obtained by choosing a clopen set in $E^* \cap Int(T)$ for any $E^* \in \Sigma'^*$. We have $\overline{\cup\sigma^*} \subset Int(T)$ and $\emptyset \neq \overline{\cup\sigma^*} \setminus \cup\sigma^* \subset F$ - a contradiction.

♢

Let us remind Simon's construction. He takes an arbitrary maximal disjoint family Σ^* of cardinality $2^{\aleph_0}$ of non empty clopen subsets of ω^* and splits it into two parts Σ_1^*, Σ_2^* having the following special property:

if A^* is a clopen subset of ω^* then A^* intersects infinitely many members of Σ_1^* iff A^* intersects infinitely many members of Σ_2^*.

It is clear that $F = \omega^* \setminus \cup\Sigma_1^*$ is regular closed and we immediately get that F is a 4FU point by Proposition 3.10. But F is not a 3FU point, because if σ^* is any infinite subfamily of Σ_1^* then every A^* which intersects infinitely many members of σ^* must intersects

infinitely many members of Σ_2^* as well and therefore this A^* can not lie in F.

This proves the following:

PROPOSITION 3.11. *There are 4FU, not 3FU points.*

PROPOSITION 3.12. *There is an uncountable 1FU space D_F which is not first countable.*

Construction. Probably Arhangel'skiĭ [A2] was the first to point out that every point of a Σ-product of an uncountable family of first countable non single point spaces is 1FU, but non first countable. Let D_F be such a Σ-product, where all points, except one called F, are declared isolated. This space D_F is the desired one.

$\Diamond$

A. Dow and J. Steprans [DS] constructed a model of ZFC in which every countable 1FU space is first countable.

So, we can not give in ZFC an example of countable 1FU space D_F which is not first countable. However, there are models where such examples can be constructed.

EXAMPLE 3.13. The two 2FU spaces of Proposition 3.2 are not 1FU in Dow-Steprans' model (they are countable not first countable spaces).

EXAMPLE 3.14. The two 2FU spaces of Proposition 3.2 are 1FU under (MA+$\neg$ CH).

Proof. It is enough to prove that $Int(F_{\mathcal{A}}) = \cup\mathcal{A}^*$ -see Example 3.2. Assume the contrary and let V^* be some clopen subset such that $V^* \subset F_{\mathcal{A}^*}$, but $V^* \not\subset \cup\mathcal{A}^*$. In this case $V^* \setminus \cup\mathcal{A}^*$ should have empty interior and this is impossible under (MA+$\neg$ CH).

$\Diamond$

Therefore, we may formulate the following

PROPOSITION 3.15. *The property to be 1FU is undetermined for the 2FU spaces of Example 3.2.*

PROPOSITION 3.16. *The two 2FU spaces constructed under CH in Example 3.3 are 1FU.*

PROPOSITION 3.17. *The two 2FU spaces of Example 3.2 assuming (MA+¬ CH) are examples of 1FU non bisequential spaces, as well as the two 2FU spaces constructed under CH in Example 3.3.*

The second named author proved in [M3] that, in any model, obtained by adding more than $\aleph_1$ new Cohen reals to a model in which CH is true, every countable 2FU space has character not greater than $\aleph_1$. From this result we extract the following

PROPOSITION 3.18. *The character of the 2FU spaces in Example 3.2 is equal to* $\mathfrak{c}$ *assuming MA and to* $\aleph_1 < \mathfrak{c}$ *in the Cohen's model.*

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