Note on a Parameter Lumping in the Vibrations of Elastic Beams

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SOMMARIO. - *In questo articolo viene discussa una tecnica di discretizzazione per problemi agli autovalori che nascono nella vibrazione di travi ed aste elastiche. Formalmente, la discretizzazione corrisponde alla classica condensazione di deformazioni e masse ai nodi di un reticolo, ma piuttosto si pone l'attenzione sull'uso di tecniche di G-convergenza per generare problemi approssimanti in spazi di funzioni meno regolari di quanto venga richiesto nella classica teoria variazionale.*

SUMMARY. - *The paper discusses a discretization technique for the eigenvalue problems that arise in the vibration of elastic beams and rods. Formally, the discretization corresponds to the classical lumping of strain and masses at the nodes of a mesh, but the stress is rather put on the use of ideas from G-convergence for generating approximating problems set up in spaces of functions less regular than it is required in the ordinary variational framework.*

1. Introduction

In estimating the solution of differential problems by variational techniques, one often resorts to approximations that belong to the same

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Key words: Rod vibration, eigenfrequency approximation, non-smooth approximation, *G*-convergence.

function space of the solution by choosing spline functions that avoid certain discontinuities at the boundary of the elements. As a matter of fact discontinuities can be allowed, as is done in the hybrid methods [1], provided that they are penalized through the introduction in the objective functional of Lagrange multipliers to be determined in the approximation procedure. In this case the technique provides an approximation from the outside of the appropriate function space. One might adopt a different view and allow for singularities by relaxing the functional in a suitable manner, so as to obtain approximating problems whose solutions converge to the solution of the original problem in some weaker sense.

To do so I exploit an idea sketched by Kohn and Vogelius [2] in dealing with the minimization of the Dirichlet integral in a 2-dimensional conduction problem. Kohn and Vogelius propose to split the minimization in two steps: first, chosen a triangular mesh, to minimize the functional in the class of functions satisfying piecewise linear conditions at the boundary of each triangle, and then to minimize the outcome with respect to all possible choices of the boundary data. The pattern of the data along the sides of the mesh forms a kind of skeleton the approximating functions have to adhere to and the second step consists in a minimization of the functional in the class of the skeletons. The procedure is said to be particularly effective.

Kohn and Vogelius do not seem to have developed their idea in detail, nor it seems to contain any novelty when confined to the second order operators as they do. Yet, their idea can be combined with a relaxation of the objective functional to functions less regular than required by the original problem, obtaining advantages that are not only computational. For multidimensional problems, for instance, this relaxation avoids the need to guarantee the proper matching of the approximating functions across the elements to the mesh.

Here the previous idea is presented in the simpler context of the one-dimensional problems. For the sake of illustration I consider the eigenvalue problem relative to the axial vibration of an elastic rod and study piecewise constant approximations to the mode shapes. The method comes up with a sequence of lumped parameter systems that provide converging estimates of the modes and eigenfrequencies of the rod. While these systems are the usual finite difference models, the general setting is different because, with

the language of [3] [4], we are constructing a sequence of discrete problems that G-converge to that of the continuous beam. To be more precise it should be said "Γ-converge", since we are dealing with functionals, but being too specific about this point is not important, here, because we won't make use of the technical apparatus of Γ-convergence. In any case, the basic ingredients are the same and the technique is a straightforward application of ideas from G-convergence, with the distinguishing feature that the stress is on the construction of numerical approximations rather than on the characterization of the limit problem. Apparently, the researches in that area have privileged the latter aspect, so the application considered here offers a different perspective.

The method is not confined to the second order problems or to the eigenvalue problems. It is indeed for fourth order operators and in particular for the equilibrium problems of the plate theory that its advantages may become important. For the sake of illustration in Section 4, I consider the bending vibrations of a simply supported beam. The rate of convergence of the method is also briefly discussed at the end of the paper, so to allow us make a first comparison with the conform methods of the finite element theory. While the rate of convergence for the rod problem is lower than that obtained under standard conditions by using the simplest conform finite elements, it becomes the same for the bending problem subject to appropriate regularity properties of the modes. This is analyzed in more detail in another paper.

The present paper is a revised version of an unpublished note with the same title [5]. The discretization method has been applied by Davini et al. [6] [7] in the study of the eigenfrequencies of beams in some identification problems. The case of the bending vibrations under general boundary conditions is treated in [8]; the extension to the equilibrium of elastic plates has been considered by Pitacco [9] and Davini and Pitacco [10].

2. Axial vibrations of rods: approximation of the eigenfrequencies

Let us consider first the axial vibrations of a rod of unit length and given homogeneous boundary conditions, and let the mass density $\rho(\cdot)$ and the axial stiffness $a(\cdot)$ be bounded and strictly positive.

Modes and natural frequencies are then the eigenpairs $\{p_j, u_j\}$ of the differential problem in the weak form:

$$u \in H^1_\Gamma(0,1)\,, \qquad -\left[a(x)\,u'\right]' = p^2\,\rho(x)\,u\,, \tag{2.1}$$

with H^1_Γ the space of functions satisfying the geometric boundary conditions, and they can be determined by studying the minima of the Rayleigh quotient

$$\mathcal{F}[u] \equiv \frac{\displaystyle\int_0^1 a(x)\,(u')^2\,dx}{\displaystyle\int_0^1 \rho(x)\,u^2\,dx} \tag{2.2}$$

in suitable subspaces of H^1_Γ.

Let $\{\mathcal{P}_N\}$ be a sequence of partitions of $(0,1)$ into sub-intervals $\Delta_i^N = (x_{i-1}^N, x_i^N)$, $x_0^N = 0$, $x_N^N = 1$, with lengths $|\Delta_i^N|$ uniformly converging to zero with N. In view of the minimizing of $\mathcal{F}$, we proceed to the minimization of the strain energy in each sub-interval for given values of the displacement at the extremes. Thus, for every $u \in H^1_\Gamma$, we have

$$\int_0^1 a(x)\,(u')^2\,dx \geq \sum \min \int_{\Delta_i^N} a(x)\,(w')^2\,dx\,, \tag{2.3}$$

where the minima are calculated over the functions w in $H^1(\Delta_i^N)$ that take on the values of u at the ends of the sub-interval:

$$\begin{aligned}\min \int_{\Delta_i^N} a(x)\,(w')^2\,dx \\ = \left\{ \min_{\substack{\psi \in H^1(\Delta_i^N) \\ [\![\psi]\!] = 1}} \int_{\Delta_i^N} a(x)\,(\psi')^2\,dx \right\} \delta u^N(i)^2\,,\end{aligned} \tag{2.4}$$

with $\delta u^N(i) \equiv u(x_i^N) - u(x_{i-1}^N)$.

From the minimization within the braces in (2.4) it turns out that the optimal ψ in each Δ_i^N corresponds to constant moment $a(x)\,\psi'$ (normal strength) and that the minimum is given by the harmonic mean of the stiffness

$$\min_{\substack{\psi \in H^1(\Delta_i^N) \\ [\![\psi]\!] = 1}} \int_{\Delta_i^N} a(x)\,(\psi')^2\,dx = \left(\int_{\Delta_i^N} \frac{1}{a(x)}\,dx \right)^{-1}. \tag{2.5}$$

Called $k^N(i)$ the harmonic mean of $a(x)$, it follows from (2.3) that

$$\int_0^1 a(x)\,(u')^2\,dx \geq \sum k^N(i)\delta u^N(i)^2 \qquad \forall\, u \in H^1_\Gamma\,. \tag{2.6}$$

Hence, from (2.3),

$$\mathcal{F}[u] \geq \frac{\sum k^N(i)\,\delta u^N(i)^2}{\int_0^1 \rho(x)u^2\,dx} \qquad \forall\, u \in H^1_\Gamma\,. \tag{2.7}$$

With the words of Kohn and Vogelius, each u fixes a skeleton of values at the nodes of $\mathcal{P}_N$. In the following an upper twiddle will indicate the function $\tilde{u}$ which fits with the skeleton of a given u in H^1_Γ and is optimal in each Δ_i^N in the sense of (2.4), and $\mathcal{O}^N_\Gamma \subset H^1_\Gamma$ will be the finite dimension subspace of such functions for all possible assignments of the skeleton. Note that for these functions (and only for them) the equality holds in (2.6).

The expression on the right hand side is in fact well defined in a space larger than H^1_Γ. Let $\mathcal{S}^N_\Gamma$ be the space of functions constant in the sub-intervals of $\mathcal{P}_N$ and satisfying the geometrical boundary conditions, and let $\hat{u} = s_N(u)$, with $u \in H^1_\Gamma$, denote the function in $\mathcal{S}^N_\Gamma$ such that $\hat{u}(x) = u(x_i^N)$ for $x \in \Delta_i^N$. In $\mathcal{S}^N_\Gamma$ the functional on the right hand side of (2.7) becomes

$$\mathcal{F}^N[\hat{u}] \equiv \frac{\sum k^N(i)\,\delta\hat{u}\,(i)^2}{\sum m^N(i)\,\hat{u}\,(i)^2}\,, \tag{2.8}$$

with

$$m^N(i) = \int_{\Delta_i^N} \rho(x)\,dx \tag{2.9}$$

and we have that

$$\mathcal{F}[u] \geq \mathcal{F}^N\,[\hat{u}]\,\frac{\sum m^N(i)\,\hat{u}\,(i)^2}{\int_0^1 \rho(x)\,u^2\,dx} \qquad \forall\, u \in H^1_\Gamma\,. \tag{2.10}$$

The functional $\mathcal{F}^N$ is the Rayleigh quotient of a chain of masses and springs lumped at the nodes of the mesh and satisfying the geometrical boundary conditions. Hereafter it is shown that the

eigenpairs of this system approximate arbitrarily well those of the continuous rod.

Let us consider the eigenpair $\{p_1, u_1\}$, with u_1 the first normalized mode of the rod. From (2.10) it follows that

$$p_1^2 \geq \mathcal{F}^N[\hat{u}_1] \sum m^N(i)\, \hat{u}_1(i)^2 . \tag{2.11}$$

Therefore, since the square of the first eigenfrequency p_1^N of the discrete system minimizes $\mathcal{F}^N$ in $\mathcal{S}_\Gamma^N$, it follows that

$$p_1^2 \geq \left(p_1^N\right)^2 \sum m^N(i)\, \hat{u}_1(i)^2 . \tag{2.12}$$

On the other hand, we also have

$$p_1^2 \leq \frac{\displaystyle\int_0^1 a(x)\,(u')^2\, dx}{\displaystyle\int_0^1 \rho(x)\, u^2\, dx} \qquad \forall\, u \in H_\Gamma^1 . \tag{2.13}$$

In particular,

$$p_1^2 \leq \mathcal{F}^N[\hat{u}]\, \frac{\displaystyle\sum m^N(i)\, \hat{u}\,(i)^2}{\displaystyle\int_0^1 \rho(x)\, u^2\, dx} \qquad \forall\, u \in \mathcal{O}_\Gamma^N , \tag{2.14}$$

with $\hat{u} = s_N(u)$, since the equality holds in (2.6). Therefore, if $\hat{u}_i^N$ is the first normalized mode of the discrete system and $\tilde{u}_1^N$ is the corresponding function in $\mathcal{O}_\Gamma^N$, it follows that

$$p_1^2 \leq \left(p_1^N\right)^2 \frac{1}{\displaystyle\int_0^1 \rho(x) \left(\tilde{u}_1^N\right)^2 dx} . \tag{2.15}$$

By collecting (2.12) and (2.15) we get

$$\begin{aligned} \left(p_1^N\right)^2 \sum m^N(i)\, \hat{u}_1(i)^2 \;&\leq\; p_1^2 \\ &\leq\; \left(p_1^N\right)^2 \frac{1}{\displaystyle\int_0^1 \rho(x) \left(\tilde{u}_1^N\right)^2 dx} . \end{aligned} \tag{2.16}$$

So, by comparison,

$$\lim_{N\to\infty} \left(p_1^N\right)^2 = p_1^2 \tag{2.17}$$

if we prove that

$$\lim_{N\to\infty} \sum m^N(i)\, \hat{u}_1(i)^2 = 1 = \lim_{N\to\infty} \int_0^1 \rho(x) \left(\tilde{u}_1^N\right)^2 dx\,. \tag{2.18}$$

The former of (2.18) follows from the equality

$$\sum m^N(i)\, \hat{u}_1(i)^2 = \int \rho(x)\, \hat{u}_1^2\, dx\,, \tag{2.19}$$

by taking into account that the integral is L^2 continuous in $\hat{u}_1$ and that $\hat{u}_1 \longrightarrow u_1$ in L^2. The latter requires a stronger property stated in the following lemma.

LEMMA *Let $\{v_j\}$ be a bounded sequence in H^1 and $\{s_N(v_j)\}$, $N = 1, 2, \ldots$ be the sequences of the corresponding simple functions defined above for the partitions $\mathcal{P}_N$. Then for any subsequence $\{v_{j_N}\}$*

$$\|s_N(v_{j_N}) - v_{j_N}\|_{L^2} \stackrel{N}{\longrightarrow} 0\,. \tag{2.20}$$

In fact, as $s_N(v_{j_N})$ is piecewise constant, Poincare' inequality reads

$$\int_{\Delta_i^N} \left[s_N(v_{j_N}) - v_{j_N}\right]^2 \le |\Delta_i^N| \int_{\Delta_i^N} \left(v_{j_N}{}'\right)^2 dx\,.$$

Then, by summing up over the Δ_i^N we get

$$\int_0^1 \left[s_N(v_{j_N}) - v_{j_N}\right]^2 dx \le |\Delta^N| \int_0^1 \left(v_{j_N}{}'\right)^2 dx \quad \text{with } |\Delta^N| = \sup_i |\Delta_i^N|\,,$$

which gives (2.20) since the sequence $\{v_j\}$ is bounded in H^1. Note that $s_N(v) \longrightarrow v$ in L^2 for every $v \in H^1$, a property that has been already used above.

Equalities $(2.18)_1$ and (2.12) imply that the energy integrals

$$\int_0^1 a(x) \left(\tilde{u}_1^{N\prime}\right)^2 dx\,,$$

which equal $(p_1^N)^2$ because the equality holds in (2.6), are equi-bounded. Therefore $\|\tilde{u}_1^N - \hat{u}_1^N\|_{L^2} \longrightarrow 0$ by the lemma, observed that $\hat{u}_1^N = s_N(\tilde{u}_1^N)$. This implies

$$\lim_{N\to\infty} \int_0^1 \rho(x) \left(\tilde{u}_1^N\right)^2 dx = \lim_{N\to\infty} \int_0^1 \rho(x) \left(\hat{u}_1^N\right)^2 dx\,. \tag{2.21}$$

On the other hand,

$$\int_0^1 \rho(x) \left(\hat{u}_1^N\right)^2 dx = \sum m^N(i)\, \hat{u}_1^N(i)^2 = 1 \tag{2.22}$$

since the discrete modes $\hat{u}_1^N$ are normalized, and this concludes the proof of $(2.18)_2$.

To see that $\lim_{N\to\infty} (p_j^N)^2 = {p_j}^2\,;\ j = 2, 3, \ldots$ requires a slight modification of the above argument.

Let us define the subspaces:

$$\begin{aligned}
\mathcal{A}_j &= \left\{ u \in H^1_\Gamma \,:\, \int_0^1 \rho(x)\, u\, u_k\, dx = 0\,; \quad k = 1, 2, \ldots, j-1 \right\}, \\
\mathcal{D}_j^N &= \left\{ \hat{u} \in \mathcal{S}_\Gamma^N \,:\, \sum m^N(i)\, \hat{u}\,(i)\, \hat{u}_k^N(i) = 0\,; \quad k = 1, 2, \ldots j-1 \right\}, \\
\mathcal{A}_j^N &= \left\{ u \in H^1_\Gamma \,:\, \hat{u} = s_N(u) \in \mathcal{D}_j^N \right\},
\end{aligned}$$

where u_k and $\hat{u}_k^N$ are the normalized modes of the beam and the discrete system, respectively.

From the min-max principle we have

$$p_j^2 = \min_{u\in\mathcal{A}_j} \mathcal{F}[u] \geq \min_{u\in\mathcal{A}_j^N} \mathcal{F}[u]\,. \tag{2.23}$$

Let a_j^N be the normalized minimizer of the Rayleigh quotient $\mathcal{F}$ in $\mathcal{A}_j^N$ and recall from (2.10) that

$$\mathcal{F}[a_j^N] \geq \mathcal{F}^N[(\hat{a}_j^N)] \sum m^N(i) \hat{a}_j^N(i)^2\,, \tag{2.24}$$

with $\hat{a}_j^N = s_N(a_j^N)$. Then, since $\hat{a}_j^N$ belongs to $\mathcal{D}_j^N$ and $\left(p_j^N\right)^2$ is the minimum of $\mathcal{F}^N$ in $\mathcal{D}_j^N$, (2.23) and (2.24) yield

$$p_j^2 \geq \left(p_j^N\right)^2 \sum m^N(i)\, \hat{a}_j^N(i)^2 \,. \tag{2.25}$$

Furthermore, by operating as in (2.14) and (2.15) it follows that

$$p_j^2 \leq \mathcal{F}[u] = \mathcal{F}^N[s_N(u)] \frac{\sum m^N(i)\, s_N(u)(i)^2}{\displaystyle\int_0^1 \rho(x)\, u^2\, dx} \quad \forall\, u \in \mathcal{A}_j \cap \mathcal{O}_\Gamma^N \tag{2.26}$$

for p_j^2 is the minimum of $\mathcal{F}$ in $\mathcal{A}_j$. Note that $\mathcal{A}_j \cap \mathcal{O}_\Gamma^N$ contains elements different from the null one, as $\mathcal{O}_\Gamma^N$ has dimension N and $\mathcal{A}_j$ has deficiency $\leq N-1$. In particular,

$$p_j^2 \;\leq\; \mathcal{F}^N\left[s_N\left(\tilde{a}_j^N\right)\right] \frac{1}{\displaystyle\int_0^1 \rho(x) \left(\tilde{a}_j^N\right)^2 dx} \,, \tag{2.27}$$

if $\tilde{a}_j^N$ is the normalized minimizer of $\mathcal{F}^N\left[s_N(u)\right]$ in $\mathcal{A}_j \cap \mathcal{O}_\Gamma^N$. By recalling that

$$\min_{u \in \mathcal{A}_j \cap \mathcal{O}_\Gamma^N} \mathcal{F}^N[s_N(u)] \leq \min_{\hat{u} \in \mathcal{D}_j^N} \mathcal{F}^N[\hat{u}] = \left(p_j^N\right)^2 \tag{2.28}$$

from the min-max principle, we finally obtain

$$p_j^2 \leq \left(p_j^N\right)^2 \frac{1}{\displaystyle\int_0^1 \rho(x) \left(\tilde{a}_j^N\right)^2 dx} \,. \tag{2.29}$$

Inequalities (2.25) and (2.29), together, read

$$\begin{aligned} \left(p_j^N\right)^2 \sum m^N(i) \hat{a}_j^N(i)^2 \;&\leq\; p_j^2 \\ &\leq\; \left(p_j^N\right)^2 \frac{1}{\displaystyle\int_0^1 \rho(x) \left(\tilde{a}_j^N\right)^2 dx} \,. \end{aligned} \tag{2.30}$$

Hence, the repetition of the argument (2.18)-(2.22) yields that

$$\lim_{N \to \infty} \left(p_j^N\right)^2 = p_j^2 \,. \tag{2.31}$$

3. Approximation of the axial modes

The modes of the lumped parameter systems also provide an approximation to the modes of the rod. To see this let us proceed by induction.

Assume that the normalized discrete modes $\hat{u}_k^N$; $k = 1, 2, \ldots, j-1$ converge in L^2 to the first $j-1$ modes of the rod

$$\hat{u}_k^N \xrightarrow{N} u_k \text{ in } L^2 \quad \text{for } k = 1, 2, \ldots j-1 \,. \tag{3.1}$$

For each N, the j-th discrete mode satisfies the conditions

$$\sum m^N(i)\, \hat{u}_j^N(i)^2 = 1 \,, \tag{3.2}$$

$$\sum m^N(i) \hat{u}_j^N(i)\, \hat{u}_k^N(i) = 0 \,, \qquad k = 1, 2, \ldots j-1 \,. \tag{3.3}$$

In addition, if $\tilde{u}_j^N$ is the corresponding function in $\mathcal{O}_\Gamma^N$, we have that

$$\int_0^1 a(x) \left(\tilde{u}_j^{N\prime}\right)^2 dx = \left(p_j^N\right)^2 \,. \tag{3.4}$$

Thus, from (2.31), $\left\{\tilde{u}_j^N\right\}$ is bounded in H^1. Hence, by possibly passing to a subsequence, there is some v such that

$$\tilde{u}_j^N \longrightarrow v \quad \text{weakly in } H^1 \,. \tag{3.5}$$

It follows from the lemma that $\hat{u}_j^N \longrightarrow v$ in L^2 and this implies that

$$\int_0^1 \rho(x)\, v^2\, dx = 1$$

$$\text{and } \int_0^1 \rho(x)\, v\, u_k\, dx = 0 \,, \qquad k = 1, 2, \ldots j-1 \tag{3.6}$$

by continuity. Moreover, by semicontinuity,

$$\int_0^1 a(x)\, (v')^2\, dx \le \liminf_{N\to\infty} \int_0^1 a(x) \left(\tilde{u}_j^{N\prime}\right)^2 dx = p_j^2 \,, \tag{3.7}$$

where (3.4) and (2.31) have been taken into account. Then, as $(3.6)_2$ states that $v \in \mathcal{A}_j$ and p_j^2 is the minimum of the energy under the

constraint $(3.6)_1$, the equality must hold in (3.7). It follows that v coincides with the normalized mode u_j of the rod. So, we conclude from (3.5) and the lemma that

$$\hat{u}_j^N \longrightarrow u_j \quad \text{in } L^2\,. \tag{3.8}$$

As the above argument straightforward applies to the case $j = 1$, this ends the proof. It is only worth noticing that a standard argument states that any subsequence of $\left\{\tilde{u}_j^N\right\}$ and then of $\left\{\hat{u}_j^N\right\}$ is convergent to the same limit, because the minimizer in $\mathcal{A}_j$ is unique.

4. Bending vibrations of beams: approximation of the eigenpairs

For the approximation of the bending eigenpairs we can proceed similarly. There only are a difference in the notion of skeleton (which involves the derivatives) and minor changes in the argument when one wants to account for general boundary conditions. For the sake of illustration let us consider the case of a beam of unit length simply supported at the ends.

In the framework of Euler Bernoulli theory the eigenvalue problem for the transverse vibrations in the weak form reads

$$u \in H_0^1 \cap H^2 \;:\; (a(x)\,u'')'' = p^2\,\rho(x)\,u \tag{4.1}$$

and the associated Rayleigh quotient is

$$\mathcal{F}[u] \equiv \frac{\displaystyle\int_0^1 a(x)(u'')^2\,dx}{\displaystyle\int_0^1 \rho(x)\,u^2\,dx}\,. \tag{4.2}$$

We assume that stiffness $a(x)$ and mass density $\rho(x)$ are positive and bounded away from 0.

Let $\mathcal{P}_N$ be defined as in Section 2 and indicate by $\overline{\mathcal{P}}_N$ the partition of $(0,1)$ into sub-intervals $\overline{\Delta}_i^N = (y_i^N,\, y_{i+1}^N)$ obtained by taking, say, the middle points y_i^N of each Δ_i^N, with $y_0^N = 0$ and $y_{N+1}^N = 1$. As in the previous sections, for every $u \in H_1^0 \cap H^2$,we minimize the strain energy in each sub-interval of $\overline{\mathcal{P}}_N$ under the condition that the variation of the derivative in the intermediate sub-intervals matches

with that of u. We leave on the contrary unrestrained the minimum problem in $\overline{\Delta}_0^N$ and $\overline{\Delta}_N^N$. Accordingly, we write

$$\int_0^1 a(x)(u'')^2\,dx \geq \sum \min \int_{\overline{\Delta}_i^N} a(x)(w'')^2\,dx\,, \tag{4.3}$$

where the minima in each $\overline{\Delta}_i^N$ are relative to the boundary conditions specified above.

As the optimal w are linear in $\overline{\Delta}_0^N$ and $\overline{\Delta}_N^N$, it turns out that (4.3) takes the form

$$\int_0^1 a(x)\,(u'')^2\,dx \;\geq\; \sum_1^{N-1} k^N(i)\,\delta u'(i)^2 \qquad \forall\, u \in H_0^1 \cap H^2\,, \tag{4.4}$$

where $\delta u'(i) \equiv u'(y_{i+1}^N) - u'(y_i^N)$ and the $k^N(i)$ are given by

$$k^N(i) = \left(\int_{\overline{\Delta}_1^N} \frac{1}{a(x)}\,dx\right)^{-1} \qquad i = 1, \ldots, N-1\,. \tag{4.5}$$

The term on the right hand side of (4.4) represents the energy of a system of rigid bars connected to one another by elastic hinges of stiffness $k^N(i)$ simply supported at the ends and presenting rotation jumps (generalized curvature) $\delta u'(i)$ at the intermediate nodes x_i. Let $\mathcal{S}_0^N \subset H_0^1$ be the space of the piecewise linear functions in the sub-intervals of $\mathcal{P}_N$ and let

$$\mathcal{F}^N[\hat{u}] \equiv \frac{\sum k^N(i)\,\delta\hat{u}'(i)^2}{\sum m^N(i,j)\,\hat{u}(i)\,\hat{u}(j)} \qquad \hat{u} \in \mathcal{S}_0^N\,, \tag{4.6}$$

be the Rayleigh quotient of the discrete system, with

$$\sum m^N(i,j)\,\hat{u}(i)\,\hat{u}(j) \equiv \int_0^1 \rho(x)\,\hat{u}^2\,dx\,. \tag{4.7}$$

For each configuration u of the beam we can associate the configuration $\hat{u} = s_N(u)$ of the discrete system that shows the skeleton of variations $\delta u'(i)$ at the intermediate nodes x_i. To do so, it is enough to add a suitable linear correction (rigid rotation) to the piecewise

function that vanishes at $x = 0$ and whose derivatives coincide with those of u at y_i^N ; $i = 1, 2, \ldots, N-1$. It follows then from (4.4) that

$$\mathcal{F}[u] \geq \mathcal{F}^N[\hat{u}] \frac{\sum m^N(i,j)\, \hat{u}\,(i) \hat{u}\,(j)}{\int_0^1 \rho(x)\, u^2\, dx} \qquad \forall\, u \in H_0^1 \cap H^2\,, \quad (4.8)$$

with $\hat{u} = s_N(u)$.

The equality in (4.4), and hence in (4.8), holds if and only if u is optimal in all the $\overline{\Delta}_1^N$ (in particular, linear in $\overline{\Delta}_0^N$ and $\overline{\Delta}_N^N$). This allows us to write the two side bounds (2.16) (2.30). Moreover, the previous lemma applied to the derivative of the functions and Poincare' inequality imply that

$$\| s_N(v_{j_N}) - v_{j_N} \|_{H^1} \xrightarrow{N} 0 \quad (4.9)$$

for every subsequences of any family of functions $\{v_j\}$ bounded in H^2. All this suffices to prove that

$$\lim_{N \to \infty} \left(p_j^N\right)^2 = p_j^2 \quad (4.10)$$

and

$$\hat{u}_j^N \longrightarrow u_j \quad \text{in } H^1\,, \quad (4.11)$$

for $j = 1, 2, \ldots$, by repeating step by step the proof given in Sections 3 and 4.

5. Error estimate for the eigenfrequencies

The two side bounds (2.16) (2.30), or the equivalent ones for the beam in bending, provide a direct evaluation of the approximation error. Again, the treatment of the axial and bending vibration is similar. So, let us consider the former and in particular the approximation of the first eigenfrequency.

Put $e \equiv \left[p_1^2 - \left(p_1^N\right)^2\right]$. From (2.16) it follows that

$$\left(p_1^N\right)^2 \left[\sum m^N(i)\, \hat{u}\,(i)^2 - 1\right] \leq e \leq \left(p_1^N\right)^2 \left(\frac{1}{\int_0^1 \rho(x) \left(\tilde{u}_1^N\right)^2 dx} - 1 \right) \quad (5.1)$$

Then, if we put

$$a \equiv \left(\sum m^N(i)\, \hat{u}_1(i)^2 - 1 \right) , \quad b \equiv \left(\frac{1}{\displaystyle\int_0^1 \rho(x) \left(\tilde{u}_1^N \right)^2 dx} - 1 \right) \tag{5.2}$$

and

$$\ell \equiv \max \{ |a|, |b| \} , \tag{5.3}$$

the absolute value of the error is bounded above by

$$|e| \leq \left(p_1^N \right)^2 \ell . \tag{5.4}$$

If we take into account that $\hat{u}_1^N$ is constant in each interval of $\mathcal{P}_N$ and that $\tilde{u}_1^N$ coincides with $\hat{u}_1^N$ at one end of it, Poincare' inequality implies

$$\int_{\Delta_i^N} \rho(x) \left(\tilde{u}_1^N - \hat{u}_1^N \right)^2 dx \leq \frac{\rho_M}{a_m} |\Delta_i^N|^2 \int_{\Delta_i^N} a(x) \left(\tilde{u}_1^{N\prime} \right)^2 dx \tag{5.5}$$

with $\rho_M \equiv \sup \rho(x)$, $a_m \equiv \inf a(x)$. So, summing up over the Δ_i^N, from (3.4) we obtain

$$\int_0^1 \rho x \left(\tilde{u}_1^N - \hat{u}_1^N \right)^2 dx \leq \frac{\rho_M}{a_m} |\Delta^N|^2 \left(p_1^N \right)^2 \tag{5.6}$$

which implies

$$\left(\int_0^1 \rho(x) \left(\tilde{u}_1^N \right)^2 dx \right)^{\frac{1}{2}} \geq 1 - \left(\frac{\rho_M}{a_m} \right)^{\frac{1}{2}} |\Delta^N|\, p_1^N \tag{5.7}$$

for $\hat{u}_1^N$ is normalized. The term on the right hand side is positive for $|\Delta^N|$ small enough. Hence,

$$\frac{1}{\displaystyle\int_0^1 \rho(x) \left(\tilde{u}_1^N \right)^2 dx} \leq \left(1 - \left(\frac{\rho_M}{a_m} \right)^{\frac{1}{2}} |\Delta^N|\, p_1^N \right)^{-2} \tag{5.8}$$

and $|b|$ is bounded above by

$$|b| \leq \left(1 - \left(\frac{\rho_M}{a_m} \right)^{\frac{1}{2}} |\Delta^N|\, p_1^N \right)^{-2} - 1 . \tag{5.9}$$

Analogously, it can be proved that

$$1 - \left(\frac{\rho_M}{a_m}\right)^{\frac{1}{2}} |\Delta^N|\, p_1 \;\leq\; \left(\sum m^N(i)\; \hat{u}_1(i)^2\right)^{\frac{1}{2}}$$
$$\leq\; 1 + \left(\frac{\rho_M}{a_m}\right)^{\frac{1}{2}} |\Delta^N|\, p_1\,. \qquad (5.10)$$

Thus,

$$(1 - \left(\frac{\rho_M}{a_m}\right)^{2} |\Delta^N| p_1)^2 - 1$$
$$\leq a \leq \left(1 + \left(\frac{\rho_M}{a_m}\right)^{\frac{1}{2}} |\Delta^N|\, p_1\right)^2 - 1\,, \qquad (5.11)$$

which gives

$$|a| \leq \left(1 + \left(\frac{\rho_M}{a_m}\right)^{\frac{1}{2}} |\Delta^N|\, p_1\right)^2 - 1\,, \qquad (5.12)$$

where p_1 can be replaced by the upper bound

$$p_1 \leq p_1^N \left(1 - \left(\frac{\rho_M}{a_m}\right)^{\frac{1}{2}} |\Delta^N|\, p_1\right)^{-1} \qquad (5.13)$$

deduced from $(2.16)_2$ and (5.8).

The inequalities (5.4) (5.9) (5.12) (5.13) provide an estimate for the approximation of the technique. To within a term $O(|\Delta^N|^2)$, the error is bounded above by

$$|e| \leq 2\,(p_1^N)^3 \left(\frac{\rho_M}{a_m}\right)^{\frac{1}{2}} |\Delta^N|\,. \qquad (5.14)$$

It is known that the rate of convergence for the eigenfrequencies in the finite element approximation is the same as that of the energies for the eigenfunctions. Then, (5.14) can be compared with the basic error estimate in the theory of the conform finite elements of polynomial type, see e.g. Strang and Fix [11], Theorems 3.7 and 6.1. With obvious adjustments of notation that estimate reads

$$|e| \leq C\, |\Delta^N|^{2(k-m)}\, |u_1|_k^2 \qquad (5.15)$$

where $2m$ is the order of the differential problem, $k-1$ the degree of the finite element space and $|u_1|_k$ in L^2 norm of the k-th derivative of u_1. The simplest choice for (5.15) to make sense is $k = m+1$, which gives

$$|e| \leq C\,|\Delta^N|^2\,|u_1|^2_{m+1} \tag{5.16}$$

In our case, $m = 1$ for the axial vibrations and $m = 2$ for the beam in bending, which correspond to linear and quadratic splines, respectively. In both cases the rate of the convergence of the present technique is lower than that of the ordinary conform finite element methods, provided that the mode is smooth enough. Note however that (5.14) prescinds from the results of the regularity theory, whereas (5.16) requires that $u_1 \in H^{m+1}$. A property that does not hold if we only assume that $a(\cdot)$ and $\rho(\cdot)$ are bounded.

Formula (5.1) relates the rate of the convergence to the eigenfrequency p_1 to that of the approximation to the mode u_1 in L^2 by means of functions in the finite element spaces $\mathcal{O}^N_\Gamma$ and $\mathcal{S}^N_\Gamma$. There is then an important difference between rods and beams for the rate of convergence increases with the order of the problem due to the use of more regular functions in $\mathcal{S}^N_\Gamma$ (piecewise constant and piecewise linear, respectively). For the beam problem it is proved in [8] that:

$$|e| = \mathrm{O}\left(\omega(\Delta^N)\,|\Delta^N| + |\Delta^N|^2\right)\,, \tag{5.17}$$

with

$$\omega\left(|\Delta^N|\right) \equiv \sup_{|x-y|<\Delta^N} |u_1''(x) - u_1''(y)|\,,$$

if $u_1'' \in C^0[0,1]$; and that:

$$|e| = \mathrm{O}\left(|\Delta^N|^{1+\alpha}\right)\,, \tag{5.18}$$

if $u_1'' \in C^{0,\alpha}[0,1]$ with $0 < \alpha \leq 1$.

When u_1'' is a Lipschitz function, in particular, the rate of convergence coincides with that of the conform methods, but under slightly different regularity requirements.

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Pervenuto in Redazione l'8 Maggio 1996.