

# On some Local Properties of Semigroups

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**SOMMARIO.** - *Vengono presentate dimostrazioni alternative di alcune proprietà locali per i semigruppì studiate da Hall [3]. Vengono inoltre studiate ulteriori proprietà inerenti il medesimo argomento e viene data una generalizzazione del risultato di Hall [3].*

**SUMMARY.** - *We provide alternative proofs of some local properties of semigroups studied by Hall [3]. In addition to that we contribute some more local properties of semigroups and generalization of a Hall's result [3].*

## 1. Introduction

McAlister [5] defines a semigroup  $S$  to have the property  $\mathcal{P}$ -locally, if the property  $\mathcal{P}$  is held by each of its local subsemigroups. Further, he defines a local subsemigroup of a semigroup  $S$  as the semigroup of the form  $eSe$  for an idempotent  $e \in S$ .

Nambooripad posed a problem in the Dekalb conference 1979: whether, for regular semigroups, the property of having idempotents,

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form a band, is inherited from the local subsemigroups by the larger subsemigroups  $eS$  and  $Se$ ? This is answered in affirmative by T. E. Hall [3]. That is, the class of regular semigroups is locally orthodox.

Apart from the property of regular semigroups being orthodox, there are many more local properties as discussed Theorems 1 and 5 of Hall [3].

Here we have divided our whole work in three Sections 4, 5, 6. In Section 4 we discuss alternative proofs of Hall's results [3]. In Section 5 we generalize some of the Hall's results [3], while in Section 6 we give some more local properties. In the end we add appendices as a comment on further generalization.

## 2. Preliminary

Though, we freely follow the notations of Clifford and Preston [1], and Hall [3], yet for convenience, we restate a few definitions and notations which are used every now and then in our text. However we omit the notations  $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$  for Green's relations as they are given in [1].

- (2.1)  $\text{Reg}(S)$  denotes the set of regular elements of  $S$ .
- (2.2)  $V(x)$  denotes the set of inverses of an element  $x$  of  $S$ .
- (2.3)  $(p, q, r)\text{-Reg}(S)$  denotes the set of  $(p, q, r)$ -regular elements of  $S$ .
- (2.4)  $(p, q, r)$ -Regular element: an element  $a$  of a semigroup  $S$  is called  $(p, q, r)$ -regular (where  $p, q, r$  are nonnegative integers) if  $a \in a^p S a^q S a^r$  (cf. [4]).
- (2.5) An element  $a$  of a semigroup  $S$  is said to be *right (left) divisible* by an element  $b$  if and only if  $a = xb, x \in S$  ( $a = bx_1; x_1 \in S$ ). An element  $a$  is said to be *divisible* by an element  $b$  if and only if it is both right and left divisible by  $b$ .
- (2.6) A semigroup  $S$  is said to be *right (left) archimedean* if and only if for any two elements  $a, b \in S$ , some power of each one of them is right (left) divisible by the other. A semigroup is said to be *archimedean* if and only if it is both right and left archimedean.

- (2.7) *Weakly commutative semigroup*: a semigroup  $S$  is called *weakly commutative* if for any  $a, b \in S$ , we have  $(ab)^k = xa = by$  for some  $x, y \in S$  and a positive integer  $k$  (cf. [6]).
- (2.8) *Bisimple*: A semigroup  $S$  is said to be *bisimple* if it consists of a single  $\mathcal{D}$ -class.

However the rest of the definitions are presumed well-known (cf. [1]).

### 3. Preparatory results

PROPOSITION 3.1. (Lemma 2.14), [1]. *Any idempotent element  $e$  of a semigroup  $S$  is a right identity element of  $L_e$ , a left identity element of  $R_e$ , and a two sided identity element of  $H_e$ .*

PROPOSITION 3.2. (Theorem 2.17, [1]). *If  $a$  and  $b$  are elements of a semigroup  $S$ , then  $ab \in R_a \cap L_b$  if and only if  $R_b \cap L_a$  contains an idempotent. If this is the case, then  $aH_b = H_a b = H_a H_b = H_{ab} = R_a \cap L_b$ .*

PROPOSITION 3.3. (Lemma 2.13-(i), [1]). *If  $a$  is a regular element of a semigroup  $S$ , then  $aS^1 = aS$  and  $S^1 a = Sa$ .*

PROPOSITION 3.4. (Theorem 2.16, [1]). *If  $a$ ,  $b$  and  $ab$  all belong to the same  $\mathcal{H}$ -class  $H$  of a semigroup  $S$ , then  $H$  is a subgroup of  $S$ . In particular, any  $H$ -class containing an idempotent is a subgroup of  $S$ .*

REMARK 3.5. In all the proofs of Sections 4, 5 and 6 we have used the result  $ex = x$ ;  $\forall x \in eS$  and  $ey = y$ ;  $\forall y \in eSe$  which is straight forward, to check.

REMARK 3.6. For an element  $p \in S$  and  $p' \in V(p)$ , it can be easily checked that  $p \mathcal{L} p'p$  and  $p \mathcal{R} pp'$ .

### 4. Alternative proofs

In this section we discuss the alternative proofs of some of the Hall's results [3].

## THEOREM 4.1.

- (i) If  $\text{Reg}(eSe)$  is an idempotent-generated subsemigroup then  $\text{Reg}(eS)$  is also an idempotent-generated subsemigroup.
- (ii) If  $\text{Reg}(eSe)$  is a subsemigroup on which  $\mathcal{H}$  is a congruence then  $\text{Reg}(eS)$  is a subsemigroup on which  $\mathcal{H}$  is a congruence.
- (iii) If  $\text{Reg}(eSe)$  is a union of groups (not necessarily a subsemigroup) then  $\text{Reg}(eS)$  is also a union of groups.
- (iv) If  $eSe$  has at most one idempotent per  $L$ -class, then  $eS$  has at most one idempotent per  $L$ -class.

*Proof.* (i)–If  $\text{Reg}(eSe)$  is subsemigroup then so is  $\text{Reg}(eS)$  as proved by Hall [3]. Clearly  $\text{Reg}(eS) \subseteq \text{Reg}(eSe)E(eS)$  and  $\text{Reg}(eSe)$  is idempotent-generated hence  $\text{Reg}(eS)$  is also idempotent generated. (ii)–As  $\text{Reg}(eSe)$  is a subsemigroup, so  $\text{Reg}(eS)$  is also a subsemigroup as proved by Hall [3]. To show that  $\mathcal{H}$  is a congruence on  $\text{Reg}(eS)$ . Let  $p$  and  $q \in \text{Reg}(eS)$  such that  $p \mathcal{H} q$ . We show that  $pc \mathcal{H} qc$  and  $dp \mathcal{H} dq \forall c, d \in \text{Reg}(eS)$ .

$$p \mathcal{H} q \quad \text{viz.} \quad p \mathcal{L} q \text{ and } p \mathcal{R} q.$$

$$\text{But } \mathcal{L} \text{ is a right congruence and } p \mathcal{L} q, \text{ so } pc \mathcal{L} qc. \quad (4.1)$$

Since  $p \mathcal{L} q$  and  $p \mathcal{R} q$  therefore  $pe \mathcal{L} qe$  and also  $pe \mathcal{R} p \mathcal{R} q \mathcal{R} qe$ , (as it is easy to verify that  $pe \mathcal{R} p$  and  $qe \mathcal{R} q$ ). This yields  $pe \mathcal{R} qe$ . Thus finally  $pe \mathcal{H} qe$ . By hypothesis  $pce \mathcal{H} qce$ ; (for  $ce \in \text{Reg}(eSe)$ ), that is,  $pce \mathcal{L} qce$  and  $pce \mathcal{R} qce$ . Now,

$$pce \mathcal{R} pc \quad \text{and} \quad qce \mathcal{R} qc, \text{ hence } pc \mathcal{R} qc. \quad (4.2)$$

From (4.1) and (4.2) we have  $pc \mathcal{L} qc$ . Further,

$$p \mathcal{H} q \quad \text{viz.} \quad p \mathcal{L} q \text{ and } p \mathcal{R} q.$$

$$\text{Since } \mathcal{R} \text{ is a left congruence and } p \mathcal{R} q \text{ so } dp \mathcal{R} dq. \quad (4.3)$$

We now show that  $dp \mathcal{L} dq$ . We have  $\text{Reg}(eSe) = \text{Reg}(eS)e$ . Now,  $p \mathcal{L} q$  implies  $\text{Reg}(eS)p = \text{Reg}(eS)q$  which implies  $\text{Reg}(eSe)pe = \text{Reg}(eSe)qe$ . That is  $\text{Reg}(eSe)depe = \text{Reg}(eSe)deqe$  hence  $\text{Reg}(eS)$ .

$\cdot dpe = \text{Reg}(eS) dqe$ . So,  $\text{Reg}(eS) dp = \text{Reg}(eS) dp ep'p = \text{Reg}(eS) dq \cdot$   
 $\cdot ep'p = \text{Reg}(eS) dq p'p = \text{Reg}(eS) dq$  (because  $q \in \text{Reg}(eS) p$ ).

$$\text{Hence } dp \mathcal{L} dq \text{ in } \text{Reg}(eS). \quad (4.4)$$

From (4.3) and (4.4) we have  $dp \mathcal{H} dq$ .

Hence  $pc \mathcal{H} qc$  and  $dp \mathcal{H} dq$ ;  $\forall c, d \in \text{Reg}(eS)$ . Therefore,  $\mathcal{H}$  is a congruence on  $\text{Reg}(eS)$ .

(iii)–We want to show that  $\text{Reg}(eS)$  is a union of groups. Since the union of subgroups of  $\text{Reg}(eS)$  is contained in  $\text{Reg}(eS)$ , we need only show that  $\text{Reg}(eS)$  is contained in the union of subgroups of  $\text{Reg}(eS)$ . Let  $a \in \text{Reg}(eS)$ . To show  $H_a = R_{aa'} \cap L_{a'a}$  contains an idempotent. Now  $ae \in \text{Reg}(eSe)$ , therefore by hypothesis,  $H_{ae} = R_{aa'e} \cap L_{a'ae}$  contains an idempotent. So by Prop. 3.2  $a'a^2a'e \in R_{a'ae} \cap L_{aa'e}$ . Now,  $a'a^2a' \mathcal{R} a'a^2a'e \mathcal{R} a'ae \mathcal{R} a'a$ . And, since  $a'a^2a'e \mathcal{L} aa'e$ , so  $a'a^2 \cdot a' \mathcal{L} aa'$ . Therefore,  $a'a^2a' \in R_{a'a} \cap L_{aa'}$ . So, by Prop. 3.2,  $H_a = R_{aa'} \cap L_{a'a}$  contains an idempotent. So, by Prop. 3.4  $H_a$  is a subgroup. Hence  $a$  belongs to a subgroup.

(iv)–Let  $f, g \in E(eS)$ , such that  $f \mathcal{L} g$ . We wish to show that  $f = g$ . It is easy to check that  $f \mathcal{L} g$  implies  $fe \mathcal{L} ge$ . Since  $fe, ge \in E(eSe)$ , so  $fe = ge$ . Now,  $f = ff = fef = gef = gf$ . As  $f \mathcal{L} g$ , hence by Proposition 3.1, we have  $gf = g$ . Therefore  $f = g$ .  $\diamond$

REMARK 4.2. It appears that the statement of Theorem 4.1-(iv) can be generalized as “If  $eSe$  has at most one regular element per  $L$ -class then  $eS$  has at most one regular element per  $L$ -class”. However we note that every regular element  $x$  is  $\mathcal{L}$ -related to its corresponding idempotent  $x'x$  and every idempotent is a regular element, hence Theorem 4.1-(iv) the result of Hall and its so called generalization stated above are one and the same.

## 5. A generalization

In an attempt to generalize Hall [3] Theorem 1-(iv) to  $(p, q, r)$ -regularity, we make the following query:

QUERY 5.1. If  $(p, q, r)$ - $\text{Reg}(eSe)$  is a subsemigroup whether  $(p, q, r)$ - $\text{Reg}(eS)$  is also a subsemigroup.

The answer is partially true. According to the choices of  $p, q, r \geq 0$  we divide all 27 cases in three categories.

Category I: (i)  $(p, q, r) = (0, 0, 0)$

Category II:  $p \geq 1, q, r \geq 0$

- |                              |                                |
|------------------------------|--------------------------------|
| (ii) $(1, 0, 0)$             | (iii) $(1, 0, 1)$              |
| (iv) $(1, 1, 0)$             | (v) $(1, 0, r) r > 1$          |
| (vi) $(1, 1, 1)$             | (vii) $(1, 1, r) r > 1$        |
| (viii) $(1, q, 0) q > 1$     | (ix) $(1, q, 1) q > 1$         |
| (x) $(1, q, r) q, r > 1$     | (xi) $(p, 0, 0) p > 1$         |
| (xii) $(p, 0, 1) p > 1$      | (xiii) $(p, 0, r) p, r > 1$    |
| (xiv) $(p, 1, 0) p > 1$      | (xv) $(p, 1, r) p, r > 1$      |
| (xvi) $(p, q, 0) p, q > 1$   | (xvii) $(p, 1, 1) p > 1$       |
| (xviii) $(p, q, 1) p, q > 1$ | (xix) $(p, q, r) p, q, r > 1.$ |

Category III:  $p = 0, q, r \geq 0$

- |                          |                               |
|--------------------------|-------------------------------|
| (xx) $(0, 0, 1)$         | (xxi) $(0, 0, r) r > 1$       |
| (xxii) $(0, 1, 0)$       | (xxiii) $(0, q, 0) q > 1$     |
| (xxiv) $(0, 1, 1)$       | (xxv) $(0, 1, r) r > 1$       |
| (xxvi) $(0, q, 1) q > 1$ | (xxvii) $(0, q, r) q, r > 0.$ |

For the case of Category I the query's answer is affirmative.

**THEOREM 5.2.** *If  $(0, 0, 0)$ -Reg( $eSe$ ) is a subsemigroup then so is  $(0, 0, 0)$ -Reg( $eS$ ).*

*Proof.* Suppose  $a, b \in (0, 0, 0)$ -Reg( $eS$ ) then

$$\begin{aligned} a &= x_1 x_2 && \text{for } x_1, x_2 \in eS \text{ and} \\ b &= y_1 y_2 && \text{for } y_1, y_2 \in eS. \end{aligned}$$

Now

$$\begin{aligned} ab &= (x_1 x_2) (y_1 y_2) \\ &= (ab)^0 (x_1 x_2) (ab)^0 (y_1 y_2) (ab)^0 \in (ab)^0 (eS) (ab)^0 (eS) (ab)^0 \end{aligned}$$

So  $ab \in (0, 0, 0)$ -Reg( $eS$ ). The query's answer is again affirmative for all the cases of Category II.

**THEOREM 5.3.** *If  $(p, q, r)$ -Reg( $eSe$ ) is a subsemigroup then so is  $(p, q, r)$ -Reg( $eS$ ) for  $p \geq 1$ , and  $q, r \geq 0$ .*

*Proof.* Let  $a, b \in (p, q, r)$ -Reg( $eS$ ). If  $p \geq 1$  then  $b \in b(eS)$  and  $b = bec$  for some  $c \in S$ . We have  $ae, be \in (p, q, r)$ -Reg( $eSe$ ) so  $abe = (ae)(be) \in (p, q, r)$ -Reg( $eSe$ ). Hence  $abe \in Te$  where  $T = (ab)^p(eS)(ab)^q(eS)(ab)^r$ . Therefore  $ab \in abec \in Tec \subseteq T$   $\diamond$

**REMARK 5.4.** All the cases of Category III are believed to lead the answer of the query in negative. The counter examples can be constructed. We give below such an example for the case  $(p, q, r) = (0, 1, 0)$ .

**EXAMPLE 5.5.** Let  $F$  be a monoid over an alphabet  $\{0, 1\}$  with respect to concatenation and with the empty word  $\delta$ . For  $x, y \in F$  we say  $xpy$  iff  $x = u010vy = u1v$  for some  $u, v \in F$  and  $x\sigma y$  iff there exists a sequence  $x = x_1, x_2, \dots, x_n = y$  ( $n \geq 1$ ) where  $x_i\rho x_{i+1}$  or  $x_{i+1}\rho x_i$  for  $i = 1, 2, \dots, n-1$  ( $n \geq 2$ ). It is easy to check that  $\sigma$  is a congruence on  $F$ .

Now let  $T = F/\sigma$  be the factor monoid with identity  $\Delta = \{\delta\}$ . For  $X, Y \in T$  we have  $XY = \Delta \implies X = \Delta = Y$ . Suppose  $B, C \in T$  where  $1 \in B$  and  $0 \in C$ . It can be proved that  $B = CBC$  and  $B^2 = XB^2Y \implies X = \Delta = Y$ . We denote by  $E$  the right zero semigroup containing two elements  $\alpha, \beta$ . Consider the semigroup  $EXT$  with coordinatewise usual operation.  $EXT$  satisfies the condition:

$$(u, X)(u, Y) = (\alpha, \Delta) \implies (u, Y) = (\alpha, \Delta).$$

Put  $S = (EXT) \setminus \{(\alpha, \Delta)\}$  and  $e = (\beta, \Delta)$ ,  $b = (\alpha, B)$ ,  $c = (\alpha, C)$ . We have  $e^2 = e$ ,  $eb = b$  and  $ec = c$ . It is easy to show that the semigroup  $S$  is generated by  $e, b, c$  and so  $eS = S$ ,  $eSe = Se$ . Now for any element  $x$  of  $Se$  we have  $x = exe \in SexSe$  and so  $(0, 1, 0)$ -Reg( $eSe$ ) =  $Se$  is a semigroup of  $S$ . It can easily be checked that  $b = cbc$  and so  $b \in (0, 1, 0)$ -Reg( $eS$ ). On the contrary suppose, that  $b^2 \in (0, 1, 0)$ -Reg( $eS$ ). Then  $b^2 \in Sb^2S$  and so  $b^2 = xb^2y$  for some  $x$  and  $y \in S$ . Hence we have  $(\alpha, B^2) = (u, X)(\alpha, B^2)(v, Y)$  where  $x = (u, X)$  and  $y = (v, Y)$ . Therefore  $v = \alpha$  and  $B^2 = XB^2Y$ . This implies that  $Y = \Delta$  and so  $y = (\alpha, \Delta) \notin S$ , a contradiction, consequently  $b^2 \notin Sb^2S$  and  $b^2 \notin (0, 1, 0)$ -Reg( $eS$ ). This shows that  $(0, 1, 0)$ -Reg( $eS$ ) is no subsemigroup of  $S$ .

## 6. Additional properties

This section contains a few additional local properties.

**THEOREM 6.1.** (i)–If  $eSe$  is a bisimple subsemigroup, then  $eS$  is also a bisimple subsemigroup.

(ii)–If  $\text{Reg}(eSe)$  is a right zero subsemigroup, then so is  $\text{Reg}(eS)$ .

*Proof.* (i)–Let  $D_a$  and  $D_b$  be two  $\mathcal{D}$ -classes of  $eS$ , for  $a, b \in eS$ . Then we need to show that  $D_a = D_b$  for proving the assertion.

Let  $p \in D_a$ , i.e.  $p\mathcal{D}a$ , namely,  $p\mathcal{L} \circ \mathcal{R}a$ , viz.  $p\mathcal{L}z$  and  $z\mathcal{R}a$  for some  $z \in eS$ . This yields  $pe\mathcal{L}ze$  and  $ze\mathcal{R}ae$  (as  $z\mathcal{R}ze$  and  $a\mathcal{R}ae$ ). Therefore  $pe\mathcal{D}ae$ . But by hypothesis  $D_{ae} = D_{be}$ . So  $pe\mathcal{D}be$ , i.e.  $pe\mathcal{L}k$  and  $k\mathcal{R}be$  (for some  $k \in eSe$ ). Also,  $be\mathcal{R}b$ . Therefore  $pe\mathcal{L}k$  and  $k\mathcal{R}b$ , yields  $pe\mathcal{D}b$ . Further,  $pe\mathcal{R}p$  implies  $pe\mathcal{D}p$ , which further yields  $p\mathcal{D}pe\mathcal{D}b$ , so that  $p\mathcal{D}b$ , i.e.  $p \in D_b$ . Therefore,  $D_a \subseteq D_b$ . Similarly it can be shown that  $D_b \subseteq D_a$ . Hence  $D_a = D_b$ .

(ii)–As  $\text{Reg}(eSe)$  is a subsemigroup, so by Hall's result [3], Theorem 1-(iv)  $\text{Reg}(eS)$  is also a subsemigroup. Let  $p, q \in \text{Reg}(eS)$ . So  $pq = pqq'q = pqq'q = qe'q'q = qe'q'q$  (by hypothesis)  $= qq'q = q$ . Therefore  $pq = q$ ;  $\forall p, q \in \text{Reg}(eS)$ .  $\diamond$

**THEOREM 6.2.** In a weakly commutative semigroup  $S$ , if  $eSe$  is right (left) archimedean then  $eS$  is also right (left) archimedean.

*Proof.* Let  $eSe$  be right archimedean then we show that  $eS$  is also right archimedean. Let  $a, b \in S$ , then  $ae, be \in eSe$ . But  $eSe$  is right archimedean, hence  $(ae)^n = x(be)$  and  $(be)^m = y(ae)$  for some positive integers  $n$  and  $m$  and  $x, y \in eSe$ . Now,  $a^{n+1} = a^n a = (ae)^n a$  (by Remark 3.5)  $= (xbe)a = xba$ . Further  $(xba)^p = uxb$  for some positive integer  $p$  and  $u \in eSe$ . So,  $a^{(n+1)p} = (xba)^p = uxb$ . Analogously, we can have  $b^{(m+1)q} = vya$  for some positive integer  $q$  and  $v \in eSe$ . Hence  $eS$  is right archimedean. The theorem for left archimedean can be proved on similar lines.  $\diamond$

## 7. Appendix I

We have used Definition 2.6 of archimedean property in the Theorem 6.2. This property is of two types, left archimedean and right



archimedean. We furnish here some examples of various combinations of these two possibilities. Let

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in Z_2 \right\}$$

where  $Z_2$  denotes integers modulo 2.

EXAMPLE 7.1. Suppose

$$A = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

Then  $A$  which is a subsemigroup of  $S$ , is left archimedean but not right archimedean.

EXAMPLE 7.2. Let

$$B = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

Then  $B$  which is a subsemigroup of  $S$ , is right archimedean but not left archimedean.

EXAMPLE 7.3. Suppose

$$C = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Then  $C$  which is a subsemigroup of  $S$  is neither left nor right archimedean.

EXAMPLE 7.4. Suppose

$$D = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Then  $D$  which is a subsemigroup of  $S$  is both, right as well as left archimedean.

## 8. Appendix II

There arises a natural question whether the concept of local property can further be generalized. Namely, if we replace idempotent  $e$  in Hall's literature by a regular element  $x \in S$  and take subsemigroups  $xSx$  and  $xS$ , whether the local properties are carried from  $xSx$  over to  $xS$ . The answer is in negative. This is because Hall's [3] results in this perspective need consideration of the fact that  $x' \in xSx$ , which is false. We furnish below the two counter examples-one suggested by T. E. Hall and other given by the referee of a Journal of American Mathematical Society.

(i) Let  $U = U^1$  be any semigroup with an identity element 1. Put  $S = \mathcal{M}^0(U; 2, 2; \Delta)$ , the  $2 \times 2$  Rees semigroup over  $U$  with sandwich matrix  $\Delta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Then

$$S = \{1, 2\} \times U \times \{1, 2\} \cup \{0\} ,$$

with multiplication given by

$$(i, u, j) (k, v, 1) = \begin{cases} (i, uv, 1) & \text{if } \Delta_{jk} \neq 0 \\ 0 & \text{if } \Delta_{jk} = 0 . \end{cases}$$

Put  $x = (1, 1, 2)$ . The unique inverse of  $x$  in  $S$  is  $x' = (2, 1, 1)$ . Now,  $xSx \subseteq \{1\} \times U \times \{2\} \cup \{0\}$ , so  $x' \notin xSx$ .

(ii) Consider the inverse semigroup  $S$ .

	$a$	$a^{-1}$	$e$	$f$	$0$
$a$	$0$	$e$	$0$	$a$	$0$
$a^{-1}$	$f$	$0$	$a^{-1}$	$0$	$0$
$e$	$a$	$0$	$e$	$0$	$0$
$f$	$0$	$a^{-1}$	$0$	$f$	$0$
$0$	$0$	$0$	$0$	$0$	$0$

This is the Brandt semigroup  $\mathcal{M}^0(\{1\}; 2, 2; \Delta) = B_2$ . Put  $x = a$ . Then  $x' = a^{-1}$ , the unique inverse of  $a$  in  $S$ . Checking shows that  $aSa = \{0, a\}$ , so  $a^{-1} \notin aSa$ , that is  $x' \notin xSx$ .

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