REMARKS ON CONVERGENCE LINEAR SPACES (*)

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SOMMARIO. - *Vengono esaminate le topologie negli spazi lineari che generano convergenze $\mathcal{FLUSH}$ le quali soddisfano a certe condizioni di tipo diagonale. Vengono anche prese in esame convergenze massimali ("coarse"). In particolare, si dimostra che le convergenze "coarse" non sono normalizzabili. Vengono poi proposti alcuni problemi aperti.*

SUMMARY. - *Topologies in linear spaces that generate $\mathcal{FLUSH}$ convergences satisfying certain conditions of diagonal type are discussed. Maximal (coarse) convergences are also studied. In particular, it is shown that coarse convergences are not normable. Some open questions are posed.*

1. Assume that X is a linear space over the field $\mathcal{R}$ of real numbers. By a *convergence* $\mathcal{G}$ in X we mean an arbitrary subset of $X^{\mathcal{N}} \times X$. If $< (x_n), x > \in \mathcal{G}$ for $(x_n) \in X^{\mathcal{N}}$ and $x \in X$, then we say that the sequence (x_n) *is convergent to* x in $\mathcal{G}$ and write $x_n \to x$ $(\mathcal{G})$ or, simply, $x_n \to x$.

The following conditions are usually imposed on convergences $\mathcal{G}$ in linear spaces:

$\mathcal{F}$. *If $x_n \to x$, then $x_{p_n} \to x$ for every subsequence (x_{p_n}) of (x_n);*

$\mathcal{L}$. *If $x_n \to x$, $y_n \to y$ in $\mathcal{G}$ and $a_n \to a$, $b_n \to b$ in $\mathcal{R}$, then $a_n x_n + b_n y_n \to ax + by$ in $\mathcal{G}$;*

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$\mathcal{U}$. *If* $x_n \not\to x$, *then there is a subsequence* (x_{p_n}) *of* (x_n) *such that* $x_{q_n} \not\to x$ *for every subsequence* (x_{q_n}) *of* (x_{p_n})*;*

$\mathcal{S}$. *If* $x_n = x$ *for every* n, *then* $x_n \to x$*;*

$\mathcal{H}$. *If* $x_n \to x$ *and* $x_n \to y$, *then* $x = y$.

It is easy to check that the convergence in a linear topological space satisfies all the conditions $\mathcal{FLUSH}$. On the other side, there exist convergences satisfying conditions $\mathcal{FLUSH}$ which are non-topological, i. e. they are not generated by any linear topology. An example of non-topological convergence in a group was given by J. Novák in [8]. Then other examples of non-topological convergences satisfying the above and some additional sequential conditions in groups as well as in linear spaces were given by several authors (see e.g. [9, 3, 4, 5, 6, 10]).

The following question arises naturally: under what additional conditions every convergence satisfying conditions $\mathcal{FLUSH}$ is generated by a certain linear topology? The problem remains open.

In connection with the above problem, the following diagonal condition was introduced in [9] and [10]:

$\mathcal{P}$. *If* $x_{m,n} \to x_m$ *for each* $m \in \mathcal{N}$ *and* $x_{p_n,q_n} \to x$ *for every pair of increasing sequences* $(p_n), (q_n)$ *of positive integers, then* $x_m \to x$.

Since condition $\mathcal{P}$ is satisfied by every convergence generated by a T_3 topology, it has been used to show that some constructed convergences are not generated by any linear topology.

Moreover, convergences satisfying conditions $\mathcal{FLUSHP}$ have some properties that are satisfied in every topological linear space, but not in every $\mathcal{FLUSH}$ convergence linear space (see [3]).

The convergence given below satisfies conditions $\mathcal{FLUSHP}$ and is not generated by any linear topology (proofs can be found in [11]).

EXAMPLE. Let E be a linear space endowed with a convergence $\mathcal{G}_0$ satisfying conditions $\mathcal{FLUSH}$. Let X be the space of sequences with finite support whose coordinates belong to E, i.e. $x \in X$ if

$x = (t_1, t_2, \ldots), t_i \in E$ and $t_i = 0$ for almost all $i \in \mathcal{N}$. We define a convergence $\mathcal{G}$ on X as follows.

Let $x_n = (t_{1,n}, t_{2,n}, \ldots)$. Then we write $x_n \to 0$ $(\mathcal{G})$ if the following two conditions are satisfied:

1° $t_{i,n} \to 0$ $(\mathcal{G}_0)$ for each $i \in \mathcal{N}$;

2° there is a $j \in \mathcal{N}$ such that $\dim \operatorname{lin}\{t_{i,n}\colon\ i \geq j,\ n \in \mathcal{N}\} < \infty$.

Moreover, we define $x_n \to x$ $(\mathcal{G})$ if $x_n - x \to 0$ $(\mathcal{G})$.

Remark 1. The convergence $\mathcal{G}$ satisfies conditions $\mathcal{FLUSH}$.

A sequence (x_n) is called a K*-sequence* if every its subsequence contains a sumable subsequence.

Remark 2. If E contains a linearly independent K-sequence, then $\mathcal{G}$ is not generated by any linear topology.

Remark 3. If the space E is a linear topological space and $\mathcal{G}_0$ is the convergence generated by its topology, then the convergence $\mathcal{G}$ satisfies condition $\mathcal{P}$.

Corollary. *There is a $\mathcal{FLUSHP}$ convergence in a linear space that is not generated by any linear topology.*

It is worth pointing out that in the proof of non-topologicality of the convergence $\mathcal{G}$ the following diagonal condition plays an important part.

$\mathcal{A}$. *If $x_{m,n} \to 0$ for each $n \in \mathcal{N}$, $x_{m,n} \to 0$ for each $m \in \mathcal{N}$ and for every subsequence (p_m) of (m) there is a subsequence (q_m) of (p_m) and a sequence (x_m) such that $\sum_{n=1}^{\infty} x_{p_m,q_n} = x_m$ for $m \in \mathcal{N}$ and $x_m \to 0$, then $x_{n,n} \to 0$.*

Condition $\mathcal{A}$ was introduced by P. Antosik. It was used in proofs of many classical theorems of functional analysis and measure theory (see [1, 2]).

2. Now, consider the following two conditions of diagonal type:

$\mathcal{D}$. *If $x_{m,n} \to 0$ for each m, then there is a subsequence (p_m) of natural numbers such that $x_{m,p_m} \to 0$.*

$\mathcal{D}_0$. *If $x_{m,n} \to 0$ for each m, then there are two subsequences (p_n) and (q_n) of natural numbers such that $x_{p_n,q_n} \to 0$.*

Note that the convergence in the constructed example does not satisfy condition $\mathcal{D}_0$.

Evidently, $\mathcal{D}$ implies $\mathcal{D}_0$ and $\mathcal{D}_0$ implies $\mathcal{P}$. On the other hand $\mathcal{P}$ does not imply $\mathcal{D}_0$. Moreover, $\mathcal{D}_0$ does not imply $\mathcal{D}$ if we assume only conditions $\mathcal{FUSH}$ (without $\mathcal{L}$) (see [7]).

The following questions are unanswered:

PROBLEM 1. Does condition $\mathcal{D}_0$ imply $\mathcal{D}$ among convergences satisfying conditions $\mathcal{FLUSH}$?

PROBLEM 2. Is every $\mathcal{FLUSHD}_0$ ($\mathcal{D}$) convergence generated by some linear topology?

Now, consider the family of all $\mathcal{FLUSH}$ convergences on a linear space X. Suppose that $\{\mathcal{G}_\alpha : \alpha \in \Gamma\}$ is a chain of convergences with respect to inclusion. Then the convergence $\bigcup_{\alpha\in\Gamma} \mathcal{G}_\alpha$ satisfies conditions $\mathcal{FLSH}$ and, consequently, its Urysohn extention satisfies all the conditions $\mathcal{FLUSH}$. By the Kuratowski-Zorn Lemma there is a maximal (coarse) $\mathcal{FLUSH}$ convergence on X. In case X is a group, coarse convergences were considered in many papers (see e.g. [5, 6]).

The following question is unanswered:

PROBLEM 3. Does there exist a coarse convergence (in infinitely dimensional linear space) that is generated by some linear topology?

We prove the following partial result:

CLAIM. *Every coarse convergence in infinitely dimensional space is not normable.*

Given a $\mathcal{FLUSH}$ convergence $\mathcal{G}$, consider a sequence $(\bar{x}_n)$ satisfying the following conditions:

(a) $(\bar{x}_n) \not\to 0$ $(\mathcal{G})$;

(b) every sequence of the form $\sum_{i=1}^{k} \alpha_{i,n}\bar{x}_{p_{i,n}}$ is not convergent to x for any $x \neq 0$,

where $k \in \mathcal{N}$, $(\bar{x}_{p_{i,n}})$ are subsequences of $(\bar{x}_n)$ $(i = 1, 2, \ldots, k)$ and the set of scalars $\{\alpha_{i,n} : \ 1 \leq i \leq k, \ n \in \mathcal{N}\}$ is bounded.

The proof of Claim is based on the following Lemma:

LEMMA. *Let $\mathcal{G}$ be a $\mathcal{FLUSH}$ convergence and $(\bar{x}_n)$ a sequence satisfying (a) and (b). Then there exists a convergence $\tilde{\mathcal{G}}$ satisfying the following conditions:*

1° $\bar{x}_n \to 0$ $(\tilde{\mathcal{G}})$;

2° $\mathcal{G} \subset \tilde{\mathcal{G}}$;

3° *$\tilde{\mathcal{G}}$ is a $\mathcal{FLUSH}$ convergence.*

Proof. We first define a convergence $\bar{\mathcal{G}}$ assuming that

$$x_n \to 0 \ (\bar{\mathcal{G}}), \tag{1}$$

if there is a natural k such that

$$x_n = \sum_{i=1}^{k} \alpha_{i,n}\bar{x}_{p_{i,n}} + y_n, \tag{2}$$

where $y_n \to 0$ $(\mathcal{G})$, $(\bar{x}_{p_{i,n}})$ are subsequences of $(\bar{x}_n)$ for $i = 1, 2, \ldots, k$, $y_n \to 0$ $(\mathcal{G})$ and the set $\{\alpha_{i,n} : \ 1 \leq i \leq k, n \in \mathcal{N}\}$ is bounded. Evidently, we adopt $x_n \to x$ $(\bar{\mathcal{G}})$ if $x_n - x \to 0$ $(\bar{\mathcal{G}})$.

We shall show that the convergence $\bar{\mathcal{G}}$ defined above is a $\mathcal{FLSH}$ convergence satisfying 1° and 2°.

To prove 1° it suffices to observe that $x_n = \bar{x}_n$ if we put in (2) $k := 1, \alpha_{1,n} := 1, \bar{x}_{p_{1,n}} := \bar{x}_n$ and $y_n := 0$ for all $n \in \mathcal{N}$. Setting $\alpha_{i,n} := 0$ for $1 \le i \le k, n \in \mathcal{N}$ we get $x_n = y_n$ which implies 2°.

It is easy to check that the convergence $\bar{\mathcal{G}}$ satisfies conditions $\mathcal{FLS}$. Suppose (1), and the same time, $x_n \to x$ $(\bar{\mathcal{G}})$ for some $x \ne 0$, i.e. (2) holds and

$$x_n - x = \sum_{i=1}^{l} \beta_{i,n} \bar{x}_{q_{i,n}} + z_n$$

for some $l \in \mathcal{N}$ and sequences $(\beta_{i,n})$, $(\bar{x}_{q_{i,n}}), (z_n)$ satisfy the conditions of the definition of $\bar{\mathcal{G}}$.

Hence

$$x = \sum_{i=1}^{k} \alpha_{i,n} \bar{x}_{p_{i,n}} - \sum_{i=1}^{l} \beta_{i,n} \bar{x}_{q_{i,n}} + y_n - z_n$$

and since $y_n - z_n \to 0$ $(\mathcal{G})$, this contradicts (b) and proves that the convergence $\bar{\mathcal{G}}$ satisfies also condition $\mathcal{H}$.

It is easy to see that the Urysohn modification $\tilde{\bar{\mathcal{G}}}$ of $\bar{\mathcal{G}}$ possesses all required properties. ♢

Proof of Claim. Suppose that X is a normed space of infinite dimension and $\mathcal{G}$ the norm convergence. By Riesz's Lemma [11], there is a sequence $(\bar{x}_n)$ such that, for each n,

$$\| \bar{x}_n \| = 1; \tag{3}$$

$$\operatorname{dist}(\bar{x}_n, Y_n) > \frac{1}{2}, \tag{4}$$

where $Y_n := \operatorname{lin}\{\bar{x}_i : \ 1 \le i < n\}$.

We shall show that the sequence $(\bar{x}_n)$ satisfies assumptions (a) and (b) of Lemma.

Condition (a) is obvious. To prove (b) consider the sum

$$\sum_{i=1}^{k} \alpha_i \bar{x}_{p_i} \qquad (p_i > p_j \text{ for } i > j)$$

and note that, by (3) and (4),

$$\| \sum_{i=1}^{k} \alpha_i \bar{x}_{p_i} \| \leq \sum_{i=1}^{k} | \alpha_i |; \tag{5}$$

$$\| \sum_{i=1}^{k} \alpha_i \bar{x}_{p_i} \| \geq \frac{1}{2} | \alpha_j | - \sum_{p_j < p_i} \alpha_i; \qquad 1 \leq j \leq k. \tag{6}$$

Let (x_n) be a a sequence such that

$$x_n = \sum_{i=1}^{k} \alpha_{i,n} \bar{x}_{p_{i,n}},$$

where $(\bar{x}_{p_{i,n}})$ for $i = 1, 2, \ldots, k$ and $(\alpha_{i,n})$ are described in condition (b) of Lemma. If

$$\lim_{n \to \infty} \sum_{i=1}^{n} | \alpha_{i,n} | = 0,$$

then $x_n \to 0$ $(\mathcal{G})$, in view of (5). In the opposite case we may assume, without loss of generality, that

$$\max_i \; p_{i,n} < \min_i \; p_{i,n+1}$$

and

$$| \alpha_{j,n} | > 4\varepsilon; \quad | \alpha_{i,n} | < \frac{\varepsilon}{k}$$

for some j and $p_{i,n} > p_{j,n}$.

Hence

$$\begin{aligned} \| x_{n+m} - x_n \| &= \| \sum_{i=1}^{k} \alpha_{i,n+m} \bar{x}_{p_{i,n+m}} - \sum_{i=1}^{k} \alpha_{i,n} \bar{x}_{p_{i,n}} \| \geq \\ &\geq \frac{\varepsilon}{2} | \alpha_{i_o,n+m} | - \sum_{p_i > p_{i_o}} | \alpha_{i,m+n} | > \varepsilon, \end{aligned}$$

by (6). Consequently, the sequence (x_n) does not satisfy the Cauchy condition and thus is not convergent in $\mathcal{G}$.

Due to Lemma, there is a $\mathcal{FLUSH}$ extension of the convergence $\mathcal{G}$. This completes the proof. $\diamondsuit$

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