THE GEGENBAUER TRANSFORMATION AND SINGULAR VALUE DECOMPOSITIONS FOR THE RADON TRANSFORMATION (*)

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SOMMARIO. - *La trasformazione di Radon è uno strumento largamente usato in problemi di ricostruzione d'immagine, cosicchè le tecniche d'inversione hanno raggiunto un interesse considerevole. Viene qui presentato un metodo basato sulla decomposizione dello spettro dell'operatore di Dirac.*
È cosa ben nota che la trasformazione di Radon è invariante per trasformazioni ortogonali. L'invarianza di rotazione della trasformazione di Radon viene illustrata tramite la trasformazione di Gegenbauer. In questo articolo viene trattata la relazione tra la trasformazione di Radon e l'operatore di Dirac, un operatore differenziale del primo ordine che è anche invariante per trasformazioni ortogonali. Entrambi soddisfano ad una certa proprietà d'interazione. Inoltre sia l'operatore di Dirac che la trasformazione di Gegenbauer sono strettamente collegati anche se in dimensioni differenti. In questo modo possono essere dimostrate due decomposizioni singolari della trasformazione di Radon: una nella palla unitaria ed un'altra nello spazio completo Euclideo. Esse appaiono come una diretta conseguenza della formula di Rodriguez per polinomi ortogonali in molte variabili come viene stabilito in [3].

SUMMARY. - *The Radon transformation is a widely used tool in image reconstruction problems, and thus inversion techniques have attracted considerable interest. Here a method is presented based on the spectral decomposition of the Dirac operator.*
It is well known that the Radon transformation is invariant under orthogonal transformations. The rotational invariance of the Radon transformation is illustrated by the Gegenbauer transformation. In this

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(*) Pervenuto in Redazione il 10 Novembre 1995.

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paper the relation between the Radon transformation and a first order differential operator which is also invariant under orthogonal transformations, the Dirac operator, is considered. The two satisfy a certain intertwining property. Moreover both the Dirac operator and the Gegenbauer transformation in different dimensions are closely linked. In this way two singular value decompositions for the Radon transformation can be proved: one in the unit ball and one in the complete Euclidean space. They appear as a straightforward consequence of the Rodrigues formulae for orthogonal polynomials in several variables as defined in [3].

Introduction.

In numerical inversion problems for the Radon transformation attention has been paid by several authors to the case where the transformation is defined between two (weighted) L_2-spaces. One of the tools used is a decomposition of functions into spherical harmonics, since this decomposition satisfies some orthogonality properties and since the Radon transforms of the components can be easily calculated using the Gegenbauer transformation. In this paper a relation between the Gegenbauer transformation for functions involving a spherical harmonic of degree greater than zero and the transform of spherical symmetric functions in a higher dimension is proved. Using this it is straightforward to see that the Rodrigues formula for orthogonal polynomials in several variables transforms to the classical formula for polynomials in one variable. This way two singular value decompositions are obtained. Similar decompositions were proved by Davison (see [4]) and later by Louis ([10]), using quite different methods. Here the relation between the Radon transformation and the Dirac operator is made clear. Using these decompositions it can be proved that the Radon transformation in both cases is not only continuous, but is also compact.

THE CLIFFORD ALGEBRA.

To express the Dirac operator we work in a Clifford algebra. We consider the 2^m-dimensional real vector space $\mathbb{R}_{0,m}$ given by the basis vectors $\{e_A = A \subset \{1, 2, \ldots m\}\}$ with the notation $e_0 = e_\emptyset$ and $e_A = e_{h_1 \ldots h_k}$ for $A = \{h_1, \ldots, h_k\}$ and $1 \leq h_1 < \ldots < h_k \leq m$.

On this vector space an associative product is defined by

$$e_{h_1}e_{h_2}\dots e_{h_k} = e_{h_1\dots h_k} \text{ for } 1 \le h_1 < \dots < h_k \le m$$

which is governed by the rules $e_i^2 = -1$ and $e_ie_j = -e_ie_j$ for $i \neq j$.

Hence $\mathbb{R}_{0,m}$ is the linear associative (but not commutative) algebra generated by the elements $e_1, e_2, \dots, e_m$. It is clear however that $\mathbb{R}_{0,m}$ is not commutative. Moreover for $m \ge 3$, $\mathbb{R}_{0,m}$ has zero divisors and hence is not a field. Since e_0 is the unit element for multiplication we can identify $\lambda \in \mathbb{R}$ with λe_0. An involution on $\mathbb{R}_{0,m}$ is defined by $\overline{e_i} = -e_i$ and $\overline{ab} = \overline{b}\overline{a}$. The Euclidean norm in $\mathbb{R}_{0,m}$ will be denoted by $|.|$. A vector $\vec{x}(x_1, \dots, x_m) \in \mathbb{R}^m$ can be identified with the Clifford number $\vec{x} = \sum_{i=1}^m e_ix_i$; a real number λ can be identified with λe_0. Thus the real part $[a]_0$ of a Clifford number a is defined as the coefficient of e_0 in the development of a in the basis $\{e_A\}$. For the product of a vector with itself we have $\vec{x}^2 = -|\vec{x}|^2$. A unit vector will be indicated in the sequel by a Greek letter, e.g. $\vec{\xi}$, $\vec{\theta}$, etc.

HILBERT-MODULES OVER $\mathbb{R}_{0,m}$.

Let H be a right $\mathbb{R}_{0,m}$-module (i.e. there is a representation of $\mathbb{R}_{0,m}$ over H which can be written as right multiplication with Clifford numbers). A function $(.,.) : H \times H \to \mathbb{R}_{0,m}$ is called an inner product on H if for all f, g, $h \in H$ and $\lambda \in \mathbb{R}_{0,m}$

(i) $(f, g\lambda + h) = (f, g)\lambda + (f, h)$

(ii) $(f, g) = \overline{(g, f)}$

(iii) $[(f, f)]_0 \ge 0$ and $[(f, f)]_0 = 0 \Leftrightarrow f = 0$.

From this $\mathbb{R}_{0,m}$-valued inner product a real valued norm can be derived by $\|f\|^2 = [(f, f)]_0$. If H is complete for this norm we call it a Hilbert module.

An important example of Hilbert modules are the modules $L_2(\Omega, w)$ of $\mathbb{R}_{0,m}$-valued measurable functions over $\Omega \subset \mathbb{R}^m$, where the module structure is given by right multiplication, i.e. $f\lambda(\vec{x}) = f(\vec{x})\lambda$, and with the inner product

$$(f, g) = \int_\Omega \overline{f} g w \, d\vec{x}\,,$$

where w is a real valued weight function on Ω. Also we shall use the Hilbert module $L_2(S^{m-1})$ with the inner product

$$\langle f, g\rangle = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \overline{f} g dS$$

in which ω_{m-1} is the surface area of the unit sphere S^{m-1}.

THE RADON TRANSFORMATION.

Let f be a measurable, $\mathbb{R}_{0,m}$ valued function on $\mathbb{R}^m$ and let $\mathbf{P}^m$ the set of $m-1$-dimensional hyperplanes in $\mathbb{R}^m$. The Radon transform of f for a hyperplane σ (which can be characterized by a unit vector orthogonal to σ, $\vec{\theta}$, and the oriented distance p of σ to $\vec{o}$ such that $p\vec{\theta} \in \sigma$) is defined as

$$\mathcal{R}f(\vec{\theta}, p) = \mathcal{R}f(\sigma) = \int_\sigma f(\vec{x}) d\vec{x}$$

as far as this integral exists. It should be noted that σ can also be characterized by $(-\vec{\theta}, -p)$, and so $\mathcal{R}f$ is always an even function on $S^{m-1} \times \mathbb{R}$, where S^{m-1} is the unit sphere in m dimensions.

THE DIRAC OPERATOR.

The Dirac operator is given by

$$D = \sum_{i=1}^{m} e_i \partial_{x_i} .$$

If f is a C^1-function in a domain Ω then f is called monogenic if $Df = 0$ in Ω.

Since $D^2 = -\Delta$, where Δ is the Laplacian, each monogenic function is harmonic and hence analytic. The operator D can be separated in a radial and a spherical part:

$$D = \vec{\xi}(\partial_r + \frac{1}{r}\Gamma) .$$

The eigenmodules of the operator Γ are given by

$$\mathcal{P}_k = \left\{ f \in C^1(S^{m-1}) : \Gamma f = -kf \right\} ,$$

the module of *inner spherical monogenic functions* of degree k and

$$\mathcal{Q}_k = \left\{ f \in C^1(S^{m-1}) : \Gamma f = (k+m-1)f \right\} ,$$

the module of *outer spherical monogenic functions* of degree k. Clearly $\mathcal{P}_k \subset \mathcal{H}_k$ and $\mathcal{Q}_k \subset \mathcal{H}_{k+1}$, where $\mathcal{H}_k$ is the module of spherical harmonics of degree k. With P_k (Q_k) we shall denote an arbitrary element of $\mathcal{P}_k$ ($\mathcal{Q}_k$). Since Γ is selfadjoint all its eigenmodules are orthogonal The mapping, which maps P_k on $\vec{\xi} P_k$ is an isometry between $\mathcal{P}_k$ and $\mathcal{Q}_k$.

The module $\mathcal{P}_k$ has dimension $K(m,k) = \frac{(k+m-2)!}{k!(m-2)!}$ and has an orthogonal basis $\left\{ P_k^{(i)} : i = 1, \ldots, K(m,k) \right\}$. For details on this paragraph we refer to [1].

Generalized Gegenbauer and Hermite polynomials.

In [3] we proved that Gegenbauer and Hermite polynomials admit a straightforward generalization to several dimensions. The defining Rodrigues formulae read

$$C^\alpha_{n,m,k}(\vec{x})P_k(\vec{x}) = \left(1-r^2\right)^{-\alpha} (D)^n \left((1-r^2)^{n+\alpha} P_k(\vec{x})\right)$$

$$H_{n,m,k}(\vec{x})P_k(\vec{x}) = e^{r^2/2}\vec{x}^2(-D)^n \left(e^{-r^2/2} P_k(\vec{x})\right)$$

These functions constitute an orthogonal basis for the Hilbert modules $L_2(B(1),\, (1-r^2)^\alpha)$ and $L_2(\mathbb{R}^m, e^{-r^2/2})$ respectively.

1. The Gegenbauer transformation.

The Gegenbauer transformation was introduced by Ludwig [11] and Deans [6]. We give it here in a slightly modified form and indicate precisely under which conditions it can be used. The proof is similar to the one given in [6] and will be omitted. Notice that $|F|$ is defined by $|F|(\vec{x}) = |F(\vec{x})|$, where $|\cdot|$ is the norm on the Clifford algebra.

Theorem 1.1. (*Gegenbauer transformation*) *If F is of the form $S_k(\vec{x})f(r)$, where S_k is a spherical harmonic function of degree k,* *and moreover $|F|$ has a Radon transform in $(\vec{\theta}, p)$, $p > 0$, then*

$$\mathcal{R}F(\vec{\theta}, p) = S_k(\vec{\theta})g(p)$$

with

$$g(p) = \frac{2\pi^{\frac{m-1}{2}}\Gamma(2\nu)}{\Gamma(\nu+\frac{1}{2})\Gamma(2\nu+k)} \int_p^\infty r^{2\nu+k} f(r) C_k^\nu\left(\frac{p}{r}\right)\left(1-\frac{p^2}{r^2}\right)^{\nu-\frac{1}{2}} dr \tag{1}$$

where $\nu = \frac{m-2}{2}$.

In the sequel we shall denote the coefficient before the integral as $b(k, m)$ i.e. (using the doubling formula for the Γ function)

$$b(l, t) = \frac{(4\pi)^{\frac{t-2}{2}}\Gamma(l+1)\Gamma(\frac{t-2}{2})}{\Gamma(l+t-2)}.$$

There is a close relation between the Radon transform of a function of the form $S_k(\vec{x})f(r)$ in m dimensions and the Radon transform of $f(|\vec{b}|)$ in $m + 2k$ dimensions (in the sequel we shall denote by $\vec{b} = \rho.\vec{\beta}$, $\rho = |\vec{b}|$, a variable in $m + 2k$ dimensions). To obtain the relations between the different Radon transforms we introduce the following notations: let f be a real valued function defined on $\mathbb{R}^+$. Let $S_k(\vec{x}) = r^k S_k(\vec{\xi})$ be a spherical harmonic function of degree k. We shall indicate by $\mathcal{R}_m(\vec{\theta}, p)$ and $\mathcal{R}_{m+2k}(\vec{\omega}, t)$ the Radon transformation in m and $m + 2k$ dimensions respectively. We have

$$\mathcal{R}_m(S_k(\vec{x})f(r))(\vec{\theta}, p) = S_k(\vec{\theta})G(p)$$

$$\mathcal{R}_{m+2k}(f(\rho))(\vec{\theta}, t) = G^*(t)$$

as far as these exist. We now show

THEOREM 1.2. *If $f \in L_1\left(\mathbb{R}^+, r^{m-2+2k}\right)$ and $\mathcal{R}_m(|f|)$ exists everywhere then*

$$G(p) = \left(\frac{-1}{2\pi}\frac{d}{dp}\right)^k G^*(p).$$

Proof. From the formula for the Gegenbauer transformation we have that

$$G^*(p) = b(0, m+2k) \int_p^\infty f(r) \left[\left(1 - \frac{p^2}{r^2}\right)^{m-3+2k} \right] r^{m-2+2k} dr \, .$$

This integral exists and converges absolutely since the term between brackets is bounded. Under the conditions for f derivation under the integral sign is allowed and it is easy to see, by a limit procedure (the integrand is not continuous, so we cannot use Leibniz' rule directly) that the boundary term vanishes. Repeating this k times gives

$$\begin{aligned} \left(\frac{d}{dp}\right)^k G^*(p) &= b(0, m+2k) \int_p^\infty f(r) \times \\ &\times \left[\left(\frac{\partial}{\partial p}\right)^k \left(1 - \frac{p^2}{r^2}\right)^{\frac{m-3}{2}+k} \right] r^{m-2+2k} dr \, . \end{aligned}$$

From the definition of the Gegenbauer polynomials

$$C_k^{\frac{m-3}{2}}(u) \left(1 - u^2\right)^{\frac{m-3}{2}} = a \left(\frac{d}{du}\right)^k \left(1 - u^2\right)^{\frac{m-3}{2}+k}$$

where a is a constant given by

$$a = \frac{(-1)^k \Gamma(m-2+k) \Gamma(\frac{m-1}{2})}{2^k k! \Gamma(m-2) \Gamma(\frac{m-1}{2}+k)}$$

it follows that the term between square brackets is equal to

$$\frac{1}{a} \frac{1}{r^k} C_k^{\frac{m-3}{2}} \left(\frac{p}{r}\right) \left(1 - \frac{p^2}{r^2}\right)^{\frac{m-3}{2}} . \tag{2}$$

Inserting this gives, after simplifying the necessary constant, the desired result. ♢

2. Singular value decompositions for the Radon transformation.

2.1. Singular value decompositions.

A much used technique for inversion problems for operators between Hilbert spaces is the singular value decomposition, which we describe shortly first.

Let $A : H_0 \to H_1$ be a closed densely defined operator where H_0 and H_1 are separable Hilbert spaces. Then one tries to find sequences $\{g_j\}$, $\{f_i\}$ and $\{h_i\}$ of vectors such that

(1) $\{g_j\} \cup \{f_i\}$ is an orthonormal basis of H_0.

(2) $$\begin{aligned} Ag_j &= 0, && \text{for all } j \\ Af_i &= a_i h_i \quad a_i \in \mathbb{R}_0^+, && \text{for all } i. \end{aligned}$$

(3) $\{h_i\}$ is an orthonormal sequence.

Here the sequence $\{g_j\}$ can be finite or infinite, possibly empty, the sequence $\{f_i\}$ can also be finite or infinite. The scalars a_i are the singular values of the system. We shall limit ourselves to the infinite case. The singular value decomposition is used to describe explicitly the Moore-Penrose inverse $A^\dagger$ of A. Given the image $Af = g$ the least square approximation of f is the vector $A^\dagger g$, which is the vector with the smallest possible norm satisfying $AA^\dagger g = g$. In terms of the singular values $A^\dagger g$ is given by

$$A^\dagger g = \sum f_i \langle g, h_i \rangle \frac{1}{a_i}\,.$$

It is classical to consider singular value decompositions only for compact operators. However there is a more general criterion for decomposability.

THEOREM 2.1. *If $A : H_0 \to H_1$ is a closed densely defined operator where H_0 and H_1 are separable then A has a singular value decomposition if and only if A^*A has pure point spectrum*[1].

1. We use the definition of operator with pure point spectrum as being an operator such that an orthogonal basis of eigenvectors exists.

Proof. If A has a singular value decomposition then we have for all i, j that

$$A^* A g_j = 0, \text{ and } A^* A f_i = a_i^2 f_i\,.$$

Hence H_0 has an orthogonal basis of eigenvectors of A^*A and A^*A has pure point spectrum.

Conversely, assume that A^*A has pure point spectrum. Then an orthonormal basis $\{g_j\}$ of $\ker(A^*A)$ and an orthonormal basis $\{f_i\}$ of $\ker(A^*A)^\perp$, existing of eigenvectors of A^*A can be taken. The elementary operator (partial isometry) R associated with A has $\ker(R) = \ker(A) = \ker(A^*A)$ and is an isometry of $\ker(A^*A)^\perp \to$ Range(R). Hence $\{Rf_i\}$ is an orthonormal sequence in H_1. If moreover $A^*Af_i = \lambda_i f_i$ then $\lambda_i > 0$, since A^*A is non-negative and $f_i \notin \ker(A^*A)$. Hence $Af_i = \sqrt{\lambda_i} R f_i$, taking into account the polar decomposition $A = R\sqrt{A^*A}$. As a consequence $\{g_j\}$, $\{f_i\}$, $\{Rf_i\}$ is a singular value decomposition of A. $\diamondsuit$

REMARK. The singular value decomposition can be used to prove compactness of an operator: indeed, A is compact if and only if $\lim_{i\to+\infty} a_i = 0$.

2.2. The Radon transformation on $L_2\left(B(1), (1+\vec{x}^2)^{-\alpha}\right)$.

In [2] it was proved that the Radon transformation is continuous considered as an operator from $H_{-\alpha}$ to $h_{-\alpha}$, $\alpha > -1$, where $H_{-\alpha}$ is the Hilbert module $L_2\left(B\,(1)\,, (1+\vec{x}^2)^{-\alpha}\right)$ and $h_{-\alpha}$ is the image module

$$\begin{aligned} h_{-\alpha} \;\; = \;\; & \left\{ f \in L_2\left(S^{m-1} \times [-1,1], \left(1-p^2\right)^{-(\alpha+\frac{m-1}{2})}\right) : \right. \\ & \left. \quad : f\left(\vec{\theta}, p\right) = f\left(-\vec{\theta}, -p\right) \right\}\,. \end{aligned}$$

An orthogonal basis of $H_{-\alpha}$ is given by the functions

$$\begin{aligned} P_{n,k}^{\alpha i} &= (1-r^2)^\alpha C_{n,m,k}^\alpha(\vec{x}) P_k^i(\vec{x}) \\ &= (D)^n (1-r^2)^{\alpha+n} P_k^i(\vec{x}) \end{aligned}$$

The closure of D as operator from $H_{-\alpha}$ to $H_{-\alpha+1}$ based on the formula $DP^{\alpha i}_{n,k} = P^{\alpha-1i}_{n+1,k}$ will also be denoted by D. The operator $D_{-\alpha} = (1+\vec{x}^2)^{\alpha} D (1+\vec{x}^2)^{-\alpha+1}$ is the adjoint of D and $D_{-\alpha}D$ has eigenfunctions $P^{\alpha i}_{n,k}$. We can also introduce the operators $D_p : h_{-\alpha-1} \to h_{-\alpha}$ and $D_{p,-\alpha-1} : h_{-\alpha} \to h_{-\alpha-1}$ as the closure of the operators $\vec{\theta}\partial_p$ with domain $C_{-\alpha-1}$ and $(1-p^2)^{\alpha+\frac{m+1}{2}}\vec{\theta}\partial_p(1-p^2)^{-(\alpha+\frac{m-1}{2})}$ with domain $C_{-\alpha}$, where

$$C_{-\alpha} = \{f \in h_{-\alpha} : (1-p^2)^{\alpha+\frac{m-1}{2}} f \text{ has continuous derivative w.r.t. } p\}.$$

Here use is made of the orthogonal basis of $h_{-\alpha}$

$$\begin{aligned}\{Q^{\alpha i}_{nk} : Q^{\alpha i}_{nk} &= (\vec{\theta}\partial_p)^n (1-p^2)^{\alpha+\frac{m-1}{2}+n} \vec{\theta}^k P^{(i)}_k(\vec{\theta}), \\ &\qquad n,\, k \in \mathbf{N},\, i \le K(m,k)\}.\end{aligned}$$

The functions $Q^{\alpha i}_{nk}$ are connected to the Gegenbauer polynomials by Rodrigues' formula (for the basic properties of the Gegenbauer polynomials we refer to [7]) giving

$$\begin{aligned}Q^{\alpha i}_{nk}(\vec{\theta},p) &= \frac{(-1)^n 2^n \Gamma(\lambda+\frac{1}{2}+n)\Gamma(2\lambda) n!}{\Gamma(\lambda+\frac{1}{2})\Gamma(2\lambda+n)} (1-p^2)^{\lambda-\frac{1}{2}} \times \\ &\qquad \times C^{\lambda}_n(p)\vec{\theta}^{n+k} P^{(i)}_k(\vec{\theta})\end{aligned} \tag{3}$$

with $\lambda = \alpha + \frac{m}{2}$. The norm of $Q^{\alpha i}_{nk}(\vec{\theta},p)$ follows immediately from this relation and the norm $\langle P^{(i)}_k, P^{(i)}_k \rangle = 1$:

$$\begin{aligned}\|Q^{\alpha i}_{nk}\|^2 &= \left(\frac{(-1)^n\, 2^n \Gamma\left(\lambda+\frac{1}{2}+n\right)\Gamma(2\lambda)\, n!}{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(2\lambda+n)} \right)^2 \times \\ &\qquad \times \|C^{\lambda}_n\|^2_{L_2\left([-1,1],(1-p^2)^{\alpha+\frac{m-1}{2}}\right)} \cdot \frac{\omega_m}{2} \\ &= \frac{2^{2n}\Gamma\left(\lambda+\frac{1}{2}+n\right)^2 \Gamma(2\lambda)^2 (n!)^2}{\Gamma\left(\lambda+\frac{1}{2}\right)^2 \Gamma(2\lambda+n)^2} \times \\ &\qquad \times \frac{\sqrt{\pi}\Gamma(2\lambda+n)\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(2\lambda)\, n!\, (n+\lambda)\Gamma(\lambda)} \frac{\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}\end{aligned}$$

$$= \frac{2^{2n} n! \Gamma\left(\lambda + \frac{1}{2} + n\right)^2 \Gamma(2\lambda) \pi^{\frac{m+1}{2}}}{\Gamma\left(\lambda + \frac{1}{2}\right) \Gamma(2\lambda + n)(n+\lambda) \Gamma(\lambda) \Gamma\left(\frac{m}{2}\right)} \tag{4}$$

$$= \frac{2^{2\lambda+2n-1} n! \Gamma\left(\lambda + \frac{1}{2} + n\right)^2 \pi^{\frac{m}{2}}}{(n+\lambda)\Gamma(2\lambda+n)\Gamma\left(\frac{m}{2}\right)}. \tag{5}$$

According to the definition of the functions $Q_{nk}^{\alpha i}$ we have that

$$D_p Q_{nk}^{\alpha,i} = Q_{n+1,k}^{\alpha-1,i}$$

while from the derivation formula

$$\frac{d}{dp} C_n^{\lambda}(p) = 2\lambda C_{n-1}^{\lambda+1}(p)$$

it follows that

$$D_{p,-\alpha} Q_{n+1,k}^{\alpha-1,i} = n(n+2\alpha+m-2) Q_{nk}^{\alpha i}.$$

Hence $D_{p,-\alpha}$ is the adjoint of D_p. For the singular value decomposition we shall make use of some properties of the image of the Dirac operator under the Radon transformation.

LEMMA 2.2. *If $f \in H_{-\alpha-1}$ with $\alpha > -1$ is such that Df exists (D looked upon as a closed operator between Hilbert modules) and if moreover $\mathcal{R}f \in \operatorname{dom}(D_p)$, then $\mathcal{R}Df = D_p\mathcal{R}f$.*

Proof. For $\varphi \in \mathcal{D}(\mathbb{R}^m)$ we have (cfr. [5])

$$\mathcal{R}\partial_{x_i}\varphi = \theta_i \partial_p \mathcal{R}\varphi$$

where θ_i is the i-th coordinate of $\vec{\theta}$. Multiplication at left with e_i and summing over i completes the proof for these φ. The general case follows by a density argument using the adjoint operators. $\diamondsuit$

It is now possible to give a singular value decomposition for the Radon transformation.

THEOREM 2.3. *The basis $\{P_{nk}^{\alpha i}\}$ is mapped by $\mathcal{R}$ to an orthogonal system where*

$$\mathcal{R}\left(P_{nk}^{\alpha i}\right) = \frac{\pi^{\frac{m-1}{2}} \Gamma(\alpha+n+1)}{2^k \Gamma\left(\alpha+n+1+\frac{m-1}{2}+k\right)} Q_{n+k,k}^{\alpha i}. \tag{6}$$

Proof. Using the explicit calculation of $\mathcal{R}\left((1+\vec{x}^2)^\beta\right)$ and theorem 2.2 for the Gegenbauer transformation we obtain

$$\begin{aligned}\mathcal{R}\left(\left(1+\vec{x}^2\right)^{\alpha+n} P_k^i(\vec{x})\right) &= \frac{2\pi^{\frac{m-1}{2}+k}\Gamma(\alpha+n+1)}{\Gamma\left(\alpha+n+1+\frac{m-1}{2}+k\right)} \times \\ &\quad \times \left(\frac{-1}{2\pi}\partial_p\right)^k \left(1-p^2\right)^{\alpha+\frac{m-1}{2}+n} \\ &\quad P_k^i\left(\vec{\theta}\right).\end{aligned}$$

For $j<n$, $\mathcal{R}\left(D^j\left(1+\vec{x}^2\right)^{\alpha+n} P_k^i(\vec{x})\right) \in \mathrm{dom}(D_p)$ and we can apply lemma 3.2. We get

$$\begin{aligned}\mathcal{R}\left(P_{nk}^{\alpha i}(\vec{x})\right) &= \mathcal{R}\left(D^n\left(1+\vec{x}^2\right)^{\alpha+n} P_k^i(\vec{x})\right) \\ &= D_p^n \mathcal{R}\left(\left(1+\vec{x}^2\right)^{\alpha+n} P_k^i(\vec{x})\right) \\ &= \frac{2\pi^{\frac{m-1}{2}+k}\Gamma(\alpha+n+1)}{\Gamma\left(\alpha+n+1+\frac{m-1}{2}+k\right)} \times \\ &\quad \times \left(\frac{-1}{2\pi}\right)^k \vec{\theta}^n (\partial_p)^{k+n} \left(1-p^2\right)^{\alpha+\frac{m-1}{2}+n} P_k^i\left(\vec{\theta}\right) \\ &= \frac{\pi^{\frac{m-1}{2}}\Gamma(\alpha+n+1)}{2^k\Gamma\left(\alpha+n+1+\frac{m-1}{2}+k\right)} Q_{k+n,k}^{\alpha i}\end{aligned}$$

$\diamondsuit$

Starting from this we can calculate the singular values. The squared norm of the image is given by

$$\begin{aligned}A_{nk}^\alpha &= \frac{\pi^{m-1}\Gamma^2(\alpha+n+1)}{2^{2k}\Gamma^2\left(\alpha+n+\frac{m-1}{2}+k+1\right)} \|Q_{n+k,k}^{\alpha i}\|^2 \\ &= \frac{\pi^{m-1}\Gamma^2(\alpha+n+1)}{2^{2k}\Gamma^2\left(\alpha+n+\frac{m-1}{2}+k+1\right)} \times\end{aligned}$$

$$\times \frac{2^{2\alpha+m+2n+2k-1}(n+k)!\Gamma\left(\alpha+\frac{m}{2}+\frac{1}{2}+n+k\right)^2 \pi^{\frac{m}{2}}}{\left(n+k+\alpha+\frac{m}{2}\right)\Gamma(2\alpha+m+n+k)\Gamma\left(\frac{m}{2}\right)}$$

$$= \frac{\pi^{\frac{3m-2}{2}} 2^{2n+2\alpha+m-1}(n+k)!\Gamma^2(\alpha+n+1)}{\Gamma(2\alpha+m+n+k)\left(\alpha+\frac{m}{2}+n+k\right)\Gamma\left(\frac{m}{2}\right)}$$

where $\|Q^{\alpha i}_{n+k,k}\|^2$ is given by (4). The last transition can be obtained applying the doubling formula for the Γ-function for $z=\alpha+\frac{m}{2}$.

Since the norm of $P^{\alpha i}_{nk}$ squared is given by by γ^{α}_{nk} (see [3]), the singular value decomposition is given by

$$\mathcal{R}\left(\frac{P^{\alpha i}_{nk}}{\|P^{\alpha i}_{nk}\|}\right) = K^{\alpha}_{nk}\frac{Q^{\alpha i}_{n+k,k}}{\|Q^{\alpha i}_{n+k,k}\|}$$

with, for n even,

$$\begin{aligned}
(K^{\alpha}_{nk})^2 &= \frac{A^{\alpha}_{nk}}{\gamma^{\alpha}_{nk}} \\
&= \frac{\pi^{\frac{3m-2}{2}} 2^{2n+2\alpha+m-1}(n+k)!\Gamma^2(\alpha+n+1)}{\Gamma(2\alpha+m+n+k)\left(\alpha+\frac{m}{2}+n+k\right)\Gamma\left(\frac{m}{2}\right)} \times \\
&\qquad \times \frac{\Gamma\left(\alpha+\frac{n}{2}+\frac{m}{2}+k\right)\Gamma\left(\alpha+\frac{n}{2}+1\right)}{2^{2n}\pi^{\frac{m}{2}}\left(\frac{n}{2}\right)!\Gamma^2(\alpha+n+1)} \times \\
&\qquad \times \frac{\Gamma\left(\frac{m}{2}\right)\left(\alpha+n+\frac{m}{2}+k\right)}{\Gamma\left(\frac{m}{2}+k+\frac{n}{2}\right)} \\
&= \frac{2^{2\alpha+m-1}\pi^{m-1}(n+k)!\Gamma\left(\alpha+\frac{n}{2}+\frac{m}{2}+k\right)}{\left(\frac{n}{2}\right)!\Gamma(2\alpha+m+n+k)} \times \\
&\qquad \times \frac{\Gamma\left(\alpha+\frac{n}{2}+1\right)}{\Gamma\left(\frac{m}{2}+k+\frac{n}{2}\right)}.
\end{aligned}$$

For n odd we have

$$A^{\alpha}_{n,k} = 4(\alpha+n+1)^2 A^{\alpha}_{n-1,k+1}$$

$$\gamma^{\alpha}_{n,k} = 4(\alpha+n+1)^2 \gamma^{\alpha}_{n-1,k+1}$$

and hence

$$(K^{\alpha}_{n,k})^2 = (K^{\alpha}_{n-1,k-1})^2.$$

From (5) it follows that the $K^\alpha_{n,k}$ are positive and hence are the singular values.

COROLLARY 2.4. *The Radon transformation* $\mathcal{R} : H_{-\alpha} \to h_{-\alpha}$ *has a range which is not dense in* $h_{-\alpha}$. *Moreover* $\mathcal{R}$ *is compact, and so the generalized inverse* $\mathcal{R}^\dagger$ *is not continuous.*

Proof. The image of $H_{-\alpha}$ under $\mathcal{R}$ is given by

$$\begin{aligned}\mathrm{Range}(\mathcal{R}) \quad = \quad & \Bigg\{ f \in h_{-\alpha} : f = \sum \frac{Q^{\alpha i}_{n+k,k}}{\|Q^{\alpha i}_{n+k,k}\|} a^{\alpha i}_{n+k,k}, \\ & \qquad\qquad \sum \frac{|a^{\alpha i}_{n+k,k}|^2}{(K^\alpha_{n,k})^2} < +\infty \Bigg\}.\end{aligned}$$

The orthogonal complement of this clearly is given by

$$\mathrm{span}_{\mathcal{A}}\{Q^{\alpha i}_{n,k} : n < k\}.$$

Hence $\mathcal{R}(H_{-\alpha})$ is not dense.

To prove compactness we have to prove that the singular values tend to zero or that for each $\varepsilon > 0$ only a finite number of $K^\alpha_{n,k}$ are greater than ε. First we prove that

$$\frac{K^\alpha_{n,k+1}}{K^\alpha_{n,k}} < 1\,.$$

We have, for n even that

$$\begin{aligned}\left(\frac{K^\alpha_{n,k+1}}{K^\alpha_{n,k}}\right)^2 &= \frac{(n+k+1)\left(\alpha + \frac{m}{2} + \frac{n}{2} + k\right)}{(2\alpha + m + n + k)\left(\frac{m}{2} + \frac{n}{2} + k\right)} \\ &= \frac{k^2 + \left(\alpha + \frac{3n}{2} + \frac{m}{2} + 1\right) k + (n+1)\left(\alpha + \frac{n}{2} + \frac{m}{2}\right)}{k^2 + \left(2\alpha + \frac{3n}{2} + \frac{3m}{2}\right) k + \left(\frac{n}{2} + \frac{m}{2}\right)(n + 2\alpha + m)}.\end{aligned}$$

Since $\alpha > -1$ and $m \geq 2$ both the coefficients for k and k^0 of the numerator are smaller than the corresponding ones in the denominator, proving this first part.

Moreover for n fixed we have that $\lim_{k\to+\infty} K^\alpha_{n,k} = 0$. Indeed from Stirling's asymptotic formula for the Γ-function,

$$\lim_{x\to+\infty} \frac{\Gamma(x)}{e^{-x}x^{x-\frac{1}{2}}} = \sqrt{2\pi},$$

we can deduce that there exists a constant C, independent of k, such that

$$\begin{aligned}
&\frac{n+k!\,\Gamma(\alpha+\frac{n}{2}+\frac{m}{2}+k)}{\Gamma(2\alpha+m+n+k)\Gamma(\frac{m}{2}+k+\frac{n}{2})} \leq \\
&\quad \leq \frac{Ce^{-(n+k+1)}(n+k+1)^{n+k+1}}{e^{-(2\alpha+m+n+k)}(2\alpha+m+n+k)^{2\alpha+m+n+k}} \times \\
&\quad\quad \times \frac{e^{-(\alpha+\frac{n}{2}+\frac{m}{2}+k)}(\alpha+\frac{n}{2}+\frac{m}{2}+k)^{\alpha+\frac{n}{2}+\frac{m}{2}+k}}{e^{-(\frac{m}{2}+k+\frac{n}{2})}(\frac{m}{2}+k+\frac{n}{2})^{\frac{m}{2}+k+\frac{n}{2}}} \\
&\quad = Ce^{(\alpha+m-1)}\frac{(n+k+1)^{n+k+1}(\alpha+\frac{n}{2}+\frac{m}{2}+k)^{\alpha+\frac{n}{2}+\frac{m}{2}+k}}{(2\alpha+m+n+k)^{2\alpha+m+n+k}(\frac{m}{2}+k+\frac{n}{2})^{\frac{m}{2}+k+\frac{n}{2}}}.
\end{aligned}$$

The difference between the exponents in numerator and denominator is $1-\alpha-m$ and since $\alpha>-1$ and $m\geq 2$ this is negative. Taking the limit at both sides for $k\to+\infty$ gives

$$\lim_{k\to+\infty} K^{\alpha}_{n,k} = 0.$$

We finally have to prove that $\lim_{n\to+\infty} K^{\alpha}_{n,0} = 0$. By a direct calculation

$$\begin{aligned}
\left(\frac{K^{\alpha}_{n+2,0}}{K^{\alpha}_{n,0}}\right)^2 &= \frac{(n+1)(n+2)\left(\alpha+\frac{n}{2}+\frac{m}{2}\right)\left(\alpha+\frac{n}{2}+1\right)}{\left(\frac{n}{2}+1\right)(2\alpha+m+n)(2\alpha+m+n+1)\left(\frac{n}{2}+\frac{m}{2}\right)} \\
&= \frac{(n+1)(2\alpha+n+2)}{(n+2)(2\alpha+m+n+1)}.
\end{aligned}$$

Since $m\geq 2$ we have that $(2\alpha+n+2)<(2\alpha+m+n+1)$ and so, for n even

$$\left(\frac{K^{\alpha}_{n,0}}{K^{\alpha}_{0,0}}\right)^2 < \prod_{j=0}^{\frac{n-2}{2}} \frac{2j+1}{2j+m}.$$

The limit for n going to $+\infty$ for the right hand side is zero. $\diamondsuit$

REMARK. Similar singular value decompositions for the Radon transformation have been constructed by Marr ([12]) for the special case $m=2$ and $\alpha=0$, for the general case by Davison ([4]) using the spectral decomposition of $\mathcal{R}\mathcal{R}^*$, later followed by Louis([10]), who used the connection between the Radon transformation and the Fourier transformation.

2.3. The Radon transformation on $L_2(\mathbb{R}^m, e^{\frac{r^2}{2}})$.

In a complete analogical way a singular value decomposition for $\mathcal{R} : H \to h$, with

$$H = L_2(\mathbb{R}^m, e^{\frac{r^2}{2}})$$

and

$$h = \{f \in L_2(S^{m-1} \times \mathbb{R}, e^{\frac{p^2}{2}}) : f(\vec{\theta}, p) = f(-\vec{\theta}, -p)\}$$

is constructed.

Using the generalized Hermite polynomials $H_{n,m,k}$ one obtains that an orthogonal basis for H is described by $\{P^i_{nk} : n, k \in \mathbf{N}\ ,\ i \le K(m,k)\}$ where

$$\begin{aligned} P^i_{nk}(\vec{x}) &= e^{\frac{-r^2}{2}} H_{n,m,k}(\vec{x}) P_k^{(i)}(\vec{x}) \\ &= (-D)^n e^{\frac{-r^2}{2}} P_k^{(i)}(\vec{x})\,, \end{aligned}$$

while an orthogonal basis of h is given by $\{Q^i_{nk} : n, k \in \mathbf{N}\ ,\ i \le K(m,k)\}$ where

$$\begin{aligned} Q^i_{nk}(\vec{\theta}, p) &= (-\vec{\theta}\partial_p)^n e^{-\frac{p^2}{2}} \vec{\theta}^k P_k^{(i)}(\vec{\theta}) \\ &= e^{-\frac{p^2}{2}} H_n(p) \vec{\theta}^{n+k} P_k^{(i)}(\vec{\theta})\,. \end{aligned}$$

Here H_n is the n-th Hermite polynomial. The norm of these basic functions is given by

$$\|Q^i_{nk}\|^2 = \frac{2\pi^{\frac{m}{2}} n!}{\Gamma(\frac{m}{2})}\,.$$

The closure of $\vec{\theta}\partial_p$ again will be denoted by D_p, while its adjoint D_p^+ is defined as the closure of the operator $e^{-\frac{p^2}{2}}\vec{\theta}\partial_p e^{\frac{p^2}{2}}$ in a similar way as before. From the definition of the basic functions we obtain immediately that

$$D_p Q^i_{nk} = -Q^i_{n-1,k}$$

while from the derivation formula for the Hermite polynomials

$$\frac{d}{dp} H_n(p) = n H_{n-1}(p)$$

it follows that
$$D_p^+ D_p Q^i_{nk} = n Q^i_{nk}\,.$$
The adjoint $\mathcal{R}^*$ of $\mathcal{R}$ is given by
$$\mathcal{R}^* g(\vec{x}) = e^{r^2/2} \int_{S^{m-1}} g(\vec{\omega}, \langle \vec{\omega}, \vec{x} \rangle) e^{-p^2/2} dS$$
for $g \in \mathcal{D}(\mathbb{R}^m)$ and hence we have in this case the counterpart of Lemma 2.2.:

LEMMA 2.5. *If $f \in H$, Df exists in H and $\mathcal{R}f \in \mathrm{dom}(D_p)$ then*
$$\mathcal{R}Df = D_p \mathcal{R} f.$$

Hence the singular value decomposition follows from the following theorem, the proof of which is similar to the preceding one.

THEOREM 2.6. *The basis $\{P^i_{nk}\}$ of H is mapped by $\mathcal{R}$ to an orthogonal system where*
$$\mathcal{R}(P^i_{nk}) = (-1)^k (\sqrt{2\pi})^{m-1} Q^i_{n+k,k}\,.$$

The squared norm of the image of a basic vector $P^{(i)}_{nk}$ under $\mathcal{R}$ is given by
$$\| (-1)^k \left(\sqrt{2\pi}\right)^{m-1} Q^i_{n+k,k} \|^2 = (2\pi)^{m-1} \sqrt{2\pi} (n+k)! \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}\,.$$
Hence the singular value decomposition is given by
$$\mathcal{R}\left(\frac{P^i_{nk}}{\|P^i_{nk}\|}\right) = K_{nk} \frac{(-1)^k Q^i_{n+k,k}}{\|Q^i_{n+k,k}\|}$$
where for n even
$$\begin{aligned} K^2_{nk} &= \frac{2^{m-\frac{1}{2}} \pi^{\frac{3m-1}{2}} (n+k)! \Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}) (\frac{n}{2})! \pi^{\frac{m}{2}} 2^{n+\frac{m}{2}+k} \Gamma(\frac{m}{2}+k+\frac{n}{2})} \\ &= \frac{2^{\frac{m}{2}-n-k-\frac{1}{2}} \pi^{m-\frac{1}{2}} (n+k)!}{(\frac{n}{2})! \Gamma(\frac{m}{2}+k+\frac{n}{2})} \end{aligned}$$

while for n odd we have that $K^2_{nk} = K^2_{n-1,k-1}$. Again these results are similar to those obtained by Davison.

REMARK. For image reconstruction in the case of real valued functions it is possible to take linear combinations of the Clifford valued functions corresponding to the same singular values in such a way that a new orthogonal set, but this time of real valued functions is obtained. For the first case the functions $P^{\alpha i}_{nk}$ and $P^{\alpha i}_{n-1,k-1}$, for n even and fixed and k arbitrary and fixed can be combined. If we take $H^{(i)}_k, i = 1, \ldots, N(m,k)$ is an orthogonal basis of the module $\mathcal{H}_k$, consisting of real valued spherical harmonics we get the real set

$$T^{\alpha i}_{nk}(\vec{x}) = (1-r^2)^\alpha C^\alpha_{nk}(\vec{x}) H^{(i)}_k(\vec{x})$$

(n even, k arbitrary) and in the module $h_{-\alpha}$ the orthogonal basis of real valued functions

$$\begin{aligned} S^{\alpha i}_{n,k}(\vec{\theta}, p) &= \frac{(-1)^n 2^n \Gamma(\lambda + \frac{1}{2} + n)\Gamma(2\lambda) n!}{\Gamma(\lambda+\frac{1}{2})\Gamma(2\lambda+n)} \times \\ &\quad \times (1-p^2)^{\lambda-\frac{1}{2}} C^\lambda_n(p) \vec{\theta}^{n+k} H^{(i)}_k(\vec{\theta}) \end{aligned}$$

($n+k$ even) and where the mapping relations are given by

$$\mathcal{R}\left(T^{\alpha i}_{nk}\right) = \frac{\pi^{\frac{m-1}{2}} \Gamma(\alpha+n+1)}{2^k \Gamma\left(\alpha + n + 1 + \frac{m-1}{2} + k\right)} S^{\alpha i}_{n+k,k}.$$

For the second case we have analogically the orthogonal, real valued bases

$$T^i_{nk}(\vec{x}) = e^{\frac{-r^2}{2}} H_{n,m,k}(\vec{x}) H^{(i)}_k(\vec{x})$$

where again n is even and k arbitrary and

$$S^i_{nk}(\vec{\theta}, p) = e^{-\frac{p^2}{2}} H_n(p) H^{(i)}_k(\vec{\theta})$$

where $n+k$ is even. The transformation here is given by

$$\mathcal{R}(T^i_{nk}) = (-1)^k (\sqrt{2\pi})^{m-1} S^i_{n+k,k}\,.$$

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