

SPANNED VECTOR BUNDLES ON ALGEBRAIC CURVES AND LINEAR SERIES (*)

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SOMMARIO. - *Sia X una curva complessa compatta di genere g . In questo lavoro usiamo risultati molto fini sulle serie lineari di X per studiare i fibrati vettoriali generati da sezioni globali su X (ad esempio per quali d, n esiste un fibrato vettoriale stabile di rango n e grado d su X generato da sezioni globali). Risultati più completi sono dimostrati nel caso di fibrati di rango 2.*

SUMMARY. - *Let X be a complex genus g smooth complete curve. Here we use a detailed knowledge and very refined results on the linear series on X to study the spanned vector bundles on X (e. g. for which integers d, n there are degree d spanned and stable rank n vector bundles on X). More detailed results are proved for rank 2 vector bundles. Their existence (for suitable degrees) depends strongly on the gonality of X .*

In this paper we continue (with other tools) the program started in [BR] whose aim is the study of spanned vector bundle on algebraic curves. Let X be a smooth complex projective curve of genus g . In this paper we will link the numerical and geometric properties of the sections of rank $n > 1$ vector bundles on X (the so called Brill-Noether theory for rank n) to the properties of special divisors on

The author was partially supported by MURST and GNSAGA of CNR (Italy). He is a member of the VBAC Research group of Europroj and this work form part of a project "Moduli of bundles in algebraic geometry" funded by the EU International Scientific Cooperation Initiative.

(*) Pervenuto in Redazione il 16 Settembre 1995.

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X (the classical Brill-Noether theory). There are many subtleties in the case $n > 1$. First of all, we are mainly interested in stable and semi-stable bundles. Set $\rho(g, r, d) := (g - (r + 1))(g + r + d)$ (the Brill-Noether number) and $\rho(n, d, r, g) := n^2(g - 1) + 1 - (r + 1)(r + 1 - d + n(g - 1))$ (the Brill-Noether number for rank n degree d bundles with at least $r + 1$ sections). In the case $n > 1$ the condition $\rho(n, d, r, g) \geq 0$ is not sufficient for the existence of rank n stable bundles on X (even for general X) with degree d and at least $r + 1$ sections; see [Te1], [KN], [BGN] and the updated problem list on the subject distributed through Europroj by P. Newstead. But fixing $h^0(E)$ is not enough to understand the geometry of E because $H^0(E)$ may span a subsheaf J of E with $\text{rank}(J) < \text{rank}(E)$ (see Definition 8.2), while if $H^0(E)$ spans a proper subsheaf T of E with $\text{rank}(T) < \text{rank}(E)$ T may be unstable. Thus, contrary to the usual rule for special divisors, for rank $n > 1$ bundles it is not easy to reduce to the case of spanned bundles. In this paper (for the first time in this area, as far as we know) we will make an essential use of very refined results and definitions on the rank 1 case (see in particular [CKM1] and [CKM2]). The aim is to find the “building blocks” for the spanned bundles (see Definition 1.2). In section 1 we introduce the notion of “primitive vector bundles” and a few related notions and study their elementary properties. A spanned vector bundle E on X is primitive if and only if there is no bundle F with $E \subset F$, $\text{length}(F/E) = 1$ and $h^0(F) = h^0(E) + 1$ (see Remark 1.3). In section 2 we study the primitive bundles on the hyperelliptic curves. Then in section 3 we generalize the constructions made in section 2 just assuming the existence and a cohomological property of a g_t^1 , $t \geq 2$, on X . These generalizations will be used in section 4 to study trigonal curves and in section 5 to study “generic” curves. In section 6 we introduce and study the Clifford index for vector bundles. In section 7 we give other results on primitive bundles. In the last section we study (in terms of the numerical properties of special divisors on the curve X) the dimensional properties of the rank 2 bundles with a given number of sections. In particular we point out the deep difference from this point of view between the case in which $H^0(E)$ spans a rank 1 subsheaf of E and the case in which $H^0(E)$ generically spans E . This is the section in which in a very extremistic way we use the following general strategy and rule of this paper.

We assume as a datum the distribution of the “rank 1” linear series on X and we use this knowledge for the construction (or proof of nonexistence) of suitable vector bundles on X . For instance, when we study in section 5 curves with general moduli we state explicitly what assumptions on the linear series on X (assumptions satisfied by curves with general moduli) are used in each statement. More precisely, we prove the following results.

PROPOSITION 0.1. *Assume that X has a g_x^1 if and only if $\rho(g, 1, x) \geq 0$ and a g_y^2 if and only if $\rho(g, 2, y) \geq 0$. Fix an integer $d \leq g$, such that there is no g_{d+1}^3 on X (hence $4d < 3g + 12$) and there is a base point free g_d^2 on X . Then there is a stable spanned primitive degree d rank 2 bundle E on X . Furthermore, $h^0(E) = 3$ for every such bundle.*

THEOREM 0.2. *Assume that X has a spanned $g_d^r L$ and that for all integers c with $1 \leq c \leq r - 1$, X has no $g_{[cd/r]}^c$. Then there is a rank r stable spanned vector bundle F on X with $\det(F) \cong L$. If L is primitive, then F is primitive.*

Concerning the relation between stability and spannedness for vector bundles on curves with general moduli, we will prove in section 6 the following result.

THEOREM 0.3. *Let $W_n^{r,d}(X)$ be the scheme of a stable vector bundles, E , on X with $\text{rank}(E) = n$, $\text{deg}(E) = d$ and $h^0(E) \geq r + 1$. Let X be a genus $g \geq 2$ curve with general moduli. Assume that the statement of [Te1], Th. 1, gives the existence of a non empty component $W(c, b)$ of dimension $\rho(n, c, b, g)$ of $W_n^{c,b}(X)$ for all pairs of integers $(c, b) = (d, r)$, $(d - 1, r)$ and $(d + 1, r + 1)$. Then the general $E \in W(d, r)$ is a primitive rank n stable bundle with $\text{deg}(E) = d$ and $h^0(E) = r + 1$.*

1. In this paper we work over an algebraically closed base field with characteristic 0. Here we fix a few notations and introduce the key notion of primitive vector bundle. Let X be a smooth genus g curve. For any sheaf T on X , $H^i(T)$ and $h^i(T) := \dim(H^i(T))$ will denote

the cohomology groups on X . Let E a rank 2 vector bundle on X . Set $L := \det(E)$ and $d := \deg(E) = \deg(L)$. We fix an exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow M \longrightarrow 0 \quad (1)$$

with A, M line bundles and set $a := \deg(A)$, $m := \deg(M) = d - a$. We are interested in the case in which E (and hence M) is spanned.

We will use often the following well-known lemma.

LEMMA 1.1. *If a spanned vector bundle F on an integral complete variety Y has \mathcal{O}_Y as a quotient, then F has \mathcal{O}_Y as a direct factor.*

Proof. Since F is spanned we have a surjection $t\mathcal{O}_Y \longrightarrow F$ which induces a surjection $u : t\mathcal{O}_Y \longrightarrow \mathcal{O}_Y$. Since Y has only the constants as global sections, the surjection u splits and induces a splitting of the surjection $F \longrightarrow \mathcal{O}_Y$. \diamond

The following definition in the rank 1 case is equivalent to the classical definition of primitive line bundle (see e. g. [CKM1] and Remark 1.3 below). Recall (see the introductions of [CKM1] or [CKM2]) that in the rank 1 case the primitive linear series are the “building blocks” for all special linear systems.

DEFINITION 1.2. Let E be a rank r vector bundle on X with $h^1(E) \neq 0$. E is said to be primitive if it is spanned and if for every rank r vector bundle F on X with $\deg(F) = \deg(E) + 1$ and such that E is a subsheaf of F , we have $h^0(F) = h^0(E)$ (or, equivalently, F is not spanned).

REMARK 1.3. Let E be a rank r vector bundle on X . We will check here that E is primitive if and only if both E and $K_X \otimes E^*$ are spanned. Indeed $K_X \otimes E^*$ is not spanned if and only if there is a rank r bundle T contained in $K_X \otimes E^*$, with $\text{length}(K_X \otimes E^*/T) = 1$ and with $h^0(T) = h^0(K_X \otimes E^*)$. Set $F := K_X \otimes T^*$. E is a subsheaf of F with $\text{length}(F/E) = 1$. By Riemann-Roch we have $h^0(F) = h^0(E) + 1$.

REMARK 1.4. By Remark 1.3 if E is primitive, then $\deg(E) \leq (2g - 2)\text{rank}(E)$.

LEMMA 1.5. *Assume $\text{rank}(E) = 2$ and that E is given by (1) with $h^0(A) \neq 0$. If E is primitive, then $h^1(A) \neq 0$.*

Proof. Assume $h^1(A) = 0$. Take a general $P \in X$ and consider the push out by the inclusion $A \subset A(P)$ of the exact sequence (1). Equivalently, consider the map $H^1(\text{Hom}(M, A)) \rightarrow H^1(\text{Hom}(M, A(P)))$ induced by the inclusion $\text{Hom}(M, A) \rightarrow \text{Hom}(M, A(P))$. We obtain an exact sequence

$$0 \rightarrow A(P) \rightarrow E' \rightarrow M \rightarrow 0 \quad (2)$$

and an inclusion $E \subset E'$ with $\text{length}(E'/E) = 1$ and $h^0(E') = h^0(E) + 1$. \diamond

REMARK 1.6. Assume $\text{rank}(E) = 2$ and E primitive. By Remark 1.3 and Lemma 1.5 there is an exact sequence (1) such that $h^0(K \otimes M^*) \neq 0$ i. e. $h^1(M) \neq 0$.

DEFINITION 1.7. Let E be a rank r spanned vector bundle on X with $h^1(E) \neq 0$. E is called maximally stable primitive (resp. maximally semi-stable primitive) if it is stable (resp. semi-stable) and if for every rank r vector bundle F with E subsheaf of F with $\text{length}(F/E) = 1$ and $h^0(F) = h^0(E) + 1$, F is not stable (resp. F is not semi-stable).

PROPOSITION 1.8. *Fix an integer $x \geq g + 2$ (resp. $x \geq g + 1$) such that there is a primitive $M \in \text{Pic}^x(X)$. Then there is a rank 2 maximally stable primitive (resp. maximally semi-stable primitive) bundle E on X with $\text{deg}(E) = 2x - 1$ (resp. $2x$) and which is not primitive.*

Proof. Fix a spanned line bundle A with $h^1(A) = 0$ and $\text{deg}(A) = x - 1$ (resp. $\text{deg}(A) = x$). Take as E a bundle fitting in a non splitted exact sequence (1). Since $h^1(M) \neq 0$, we have $h^1(E) \neq 0$. Since $h^1(A) = 0$, E is spanned. By Lemma 1.5 E is not primitive. First assume $\text{deg}(A) = \text{deg}(M)$. Hence E is semi-stable but not stable. Fix a rank 2 bundle F with E subsheaf of F , $\text{length}(F/E) = 1$ and $h^0(F) = h^0(E) + 1$. If A is saturated in F , F/A is a line bundle V containing M , with $\text{deg}(V) = x + 1$ and $h^0(V) = h^0(M) + 1$ because $h^1(A) = 0$. This is impossible because M is primitive.

Hence the saturation, B , of A in F is a line bundle with $\deg(B) > \deg(A) = \deg(M)$ and $F/B \cong M$. Thus F is not semi-stable and E is maximally semi-stable primitive. Now assume $\deg(A) = \deg(M) - 1$. Since (1) does not split, we see that E has no subbundle of degree $\geq x$. Thus E is stable. Fix a rank 2 bundle F with E subsheaf of F , $\text{length}(F/E) = 1$ and $h^0(F) = h^0(E) + 1$. If A is saturated in F , F/A is a line bundle V containing M , with $\deg(V) = x + 1$ and $h^0(V) = h^0(M) + 1$ because $h^1(A) = 0$. This is impossible because M is primitive. Hence the saturation, U , of A in F is a line bundle with $\deg(U) \geq \deg(A) + 1 = \deg(M)$ and $F/U \cong M$. Thus F is not stable and E is maximally stable primitive. \diamond

LEMMA 1.9. *Assume E spanned and with $h^1(E) \neq 0$. If $\det(E)$ is primitive, then E is primitive.*

Proof. Take a rank r bundle F containing E and with $\text{length}(F/E) = 1$. Since $\det(F) = \det(E)(P)$ for some $P \in X$, and $\det(E)$ is primitive, $\det(F)$ is not spanned. Hence F is not spanned. \diamond

2. Here we consider the case X hyperelliptic. Let R be the g_2^1 on X . Let E be a spanned degree d rank 2 vector bundle with $h^1(E) \neq 0$. We fix an exact sequence (1) with $h^0(A) \neq 0$. Since $h^0(E) \neq 0$, there is at least one such exact sequence.

2.1. Here we assume that in the exact sequence (1) we have $h^1(A) = 0$. Thus $\deg(A) = h^0(A) + g - 1$. Since $h^1(E) \neq 0$, we have $h^1(M) \neq 0$. Since M is spanned we have the following possibilities.

2.1.2. $M \cong \mathbf{O}$. By Lemma 1.1 we have $E \cong A \oplus \mathbf{O}$.

2.1.2. $\deg(M) > 0$. Hence there is an integer $u > 0$ with $M = R^{\otimes u}$. Thus $d = h^0(A) + g - 1 + 2u$, $h^0(E) = h^0(A) + u + 1$. In particular there are such bundles for all integers $d \geq g + 3$ (taking A spanned). If $\deg(A) \geq 2u$ (resp. $> u$), no such bundle is semi-stable (resp. stable). If $\deg(A) < 2u$, there are non trivial exact sequences (1). If $\deg(A) \geq g + 1$ every such bundle is spanned. By Lemma 1.5 no such bundle is primitive.

2.2. Here we assume $h^1(A) \neq 0$ and $h^1(M) = 0$. By Remark 1.6 no such bundle is primitive. Since M is spanned, we have $\deg(M) \geq g + 1$.

2.3. Here we assume $h^1(A) \neq 0$, $h^1(M) \neq 0$. Since M is spanned, there is an integer $b \geq 0$ with $M = R^{\otimes b}$.

2.3.1. First assume $h^1(L) \neq 0$ i. e. $a + b \leq g - 1$. Note that $h^0(L) = h^0(A) + h^0(B) + 1$. Set $u := [(a + b + 1)/2]$, $Q := R^{\otimes u} \oplus R^{\otimes(a+b-u)}$. Note that both in the case $a + b$ even and in the case $a + b$ odd we have $h^0(\text{End}(Q)) = 4$ and $h^0(Q) = a + b + 2 = h^0(L) + 1$. Consider the possible spanned E with fixed $L := \det(E)$. Since any rank r spanned bundle on a smooth curve is spanned by $r + 1$ sections, all such bundles are given by an exact sequence

$$0 \longrightarrow L^* \longrightarrow 3\mathbf{O} \longrightarrow E \longrightarrow 0 \quad (3)$$

Since

$$\begin{aligned} \dim(G(3, h^0(L))) &= \dim(G(3, h^0(Q))) + 3 \\ &= \dim(G(3, h^0(Q))) + h^0(\text{End}(Q)) + \\ &\quad - h^0(\text{End}(L)), \end{aligned}$$

we see that an open subset of the set all bundles E fitting in (3) is given by bundles isomorphic to Q . Since stability is an open condition, we see that no such bundle is stable. In summary we have proved the following result.

PROPOSITION 2.3.1.1. *Assume X hyperelliptic. There is no stable rank 2 spanned bundle on X with $\det(E)$ special. In particular there is no rank 2 stable spanned bundle on X with degree $\leq g + 1$.*

PROPOSITION 2.4. *Assume X hyperelliptic of genus g . Fix an integer x with $g \leq x \leq 2g - 2$ and set $L := R^{\otimes x}$ and $u := [x/2]$. Then there exists a family of stable spanned rank 2 vector bundles on X with determinant L and which have the splitted bundle $R^{\otimes u} \oplus R^{\otimes(x-u)}$ in its closure.*

Proof. It is sufficient to prove that the general bundle E fitting in the exact sequence (3) is stable. The proof will be a dimensional

count similar to the one used in the proof of 2.3.1.1 and will be divided into 4 steps.

- Step 1. Note that for all positive integers v, w the product map $H^0(R^{\otimes v}) \otimes H^0(R^{\otimes w}) \rightarrow H^0(R^{\otimes(v+w)})$ is surjective.
- Step 2. Fix any exact sequence (1) with $A \cong R^{\otimes i}$ and $M \cong R^{\otimes j}$. The extension class f of (1) is given by an element of $H^0(R^{\otimes(j-i+g-1)})^*$ by Serre duality and the coboundary map ∂ of (1) induces a map $H^0(R^{\otimes j}) \rightarrow H^0(R^{\otimes(g-1-i)})^*$ (again by Serre duality). $\text{Ker}(\partial)$ is the subspace of $H^0(R^{\otimes j})$ which lifts to $H^0(E)$.
- Step 3. Among the bundles fitting in (3) there is the splitted bundle $R^{\otimes u} \oplus R^{\otimes(x-u)}$. Assume that the general E given by (3) is not stable and let (1) be a destabilizing sequence of E . We have $M \cong R^{\otimes u}$ because M is spanned. Hence $A \cong R^{\otimes(x-u)}$. Set $t := \dim(H^0(R^{\otimes u})/\text{Ker}(\partial))$. Note that $\dim(G(3, h^0(E))) \leq -3t + \dim(G(3, h^0(A))) + h^0(M) \leq -3t + \dim(G(3, h^0(L))) + 3$ because $h^0(L) > x + 1$ by the assumption on x and Riemann-Roch.
- Step 4. Set $i = x - u$ and $j = u$. Consider the multiplication map $H^0(M) \otimes H^0(K_X \otimes A^*) \rightarrow H^0(K_X \otimes M \otimes A^*)$. By Step 1 we see that if $i = j$ only the trivial extension lifts non zero sections of M to E , while if $j = i + 1$ for every $t > 0$ the set of all extensions with $\dim(H^0(R^{\otimes u})/\text{Ker}(\partial)) = t$ has dimension ≤ 1 . By Steps 2 and 3 we get a contradiction. \diamond

3. Here we generalize the construction made in the case X hyperelliptic. We fix a genus g smooth curve X , an integer $t \geq 2$ and a base point free complete $g_t^1 R$. There is an integer $e(1, R) \geq 1$ such that $h^0(R^{\otimes j}) = v + 1$ if and only if $0 \leq j \leq e(1, R)$. Since $e(1, R) + 1 \geq t(e(1, R)) + 1 - g$ by Riemann-Roch, we have $e(1, R) \leq [g/(t - 1)]$. By [B] we have $e(1, R) = [g/(t - 1)]$ if X is a general t -gonal curve. Assume $L := \det(E) \cong R^{\otimes x}$ with $1 \leq x \leq e(1, R)$. Set $u := [x/2]$, $A' := R^{\otimes u}$, $M' := R^{\otimes(x-u)}$ and $Q := A' \oplus M'$. As in 2.3.1 we see that an open subset of the bundles fitting in an exact

sequence (3) are isomorphic to Q . In particular no such bundle is stable. Thus we have proved the following result.

PROPOSITION 3.1. *Let R be a base point free complete g_t^1 on X . Set $L := R^{\otimes x}$ with $1 \leq x \leq e(1, R)$. Then there is no stable rank 2 spanned bundle E on X with $\det(E) = L$.*

In the case $x = 2$ more is true.

PROPOSITION 3.2. *Let R be a base point free complete g_t^1 on X . Assume $e(1, R) \geq 2$. Set $L := R^{\otimes 2}$. Then every spanned rank 2 bundle E with $\det(E) \cong L$ is either isomorphic to $R \oplus R$ or to $\mathbf{O} \oplus L$.*

Proof. If $h^0(E^*) \neq 0$, then $E \cong \mathbf{O} \oplus L$ by Lemma 1.1. Assume $h^0(E^*) = 0$. Since $h^0(L) = 3$ the dual of the map $3\mathbf{O} \rightarrow H^0(L)$ induced by the exact sequence (3) is an isomorphism. Hence E is the unique bundle (up to isomorphism) which fits in the exact sequence (3) and with $h^0(E^*) = 0$. Since $R \oplus R$ fits in that exact sequence, we conclude. \diamond

In the opposite range we have the following result.

PROPOSITION 3.3. *Let R be base point free complete g_t^1 on X . Set $L := R^{\otimes x}$ with $x > e(1, R)$ and x even. Set $u := x/2$ and assume $u \leq e(1, R)$. Then there is a spanned rank 2 semistable vector bundle E with $\det(E) = L$ and either $E \cong U \oplus V$ with $U \neq V$, $h^0(V) = h^0(R^{\otimes u}) = u + 1$ or $h^0(E) \geq 2u + 1$ and E is simple. In particular, if $R^{\otimes u}$ is the unique g_{tu}^u on X (or at least if $R^{\otimes u}$ is an isolated point of the reduction of the scheme of all g_{tu}^u 's on X), E is simple.*

Proof. The same dimensional count made to prove Proposition 3.1 now shows that a general E fitting in the exact sequence (3) is not isomorphic to $Q := R^{\otimes u} \oplus R^{\otimes u}$. Since Q is semi-stable and semi-stability is an open condition, we see that E is semi-stable. Furthermore, by semicontinuity we have $h^0(E) \leq h^0(Q)$. The same dimensional count gives that either $h^0(E) = h^0(Q)$ and $h^0(\text{End}(E)) < h^0(\text{End}(Q))$, or $h^0(E) = h^0(Q) - 1$ and $h^0(\text{End}(E)) \leq h^0(\text{End}(Q)) - 3 = 1$ \diamond

4. Here we consider the case of a trigonal curve X of genus $g \geq 5$. Let R be the unique g^1_3 on X . We have $(g-1)/3 \leq e(1, R) \leq g/2$. If $0 \leq y \leq e(1, R)$ we have $h^0(R^{\otimes y}) = y+1$; if $e(1, R) < y < g-e(1, R)$ we have $h^0(R^{\otimes y}) = e(1, R) + 1 + 2(y - e(1, R)) = 2y - e(1, R) + 1$; if $y \geq g - e(1, R)$ we have $h^1(R^{\otimes y}) = 0$, i. e. $h^0(R^{\otimes y}) = 3y - g + 1$. Set $Q := R^{\otimes a} \oplus R^{\otimes a}$ and $L := R^{\otimes 2a}$. Set $\$(a, e) = h^0(L) - h^0(Q)$. Note that if $e(1, R) < a < g - e(1, R)$ we have $2a \geq g - e(1, R)$.

4.1.1. If $0 < a \leq e(1, R)$ and $e(1, R) < 2a < g - e(1, R)$, we have $\$(a, e) = 2a - e(1, R) - 1$.

4.1.2. If $e(1, R) < a < g - e(1, R)$ we have $\$(a, e) = 2a - g + 2e(1, R) - 1 \geq (g-1)/3$.

REMARK 4.2. Let A be a spanned line bundle on X with $h^1(A) \neq 0$. Then either $A \cong R^{\otimes y}$ for some $y \geq 0$ or $A \cong K_X \otimes R^{\otimes z}$ for some $z < 0$. In particular if $\deg(A) = 3a$ for some integer a we have $-z = 2(g-1)/3 - a$. Thus if $g-1$ is not divisible by 3, then $A \cong R^{\otimes a}$.

THEOREM 4.3. Assume $0 < a \leq g - e(1, R) - 1$, $2a \geq e(1, R)$ and $\$(a, e) \geq (g-1)/3$ i. e. by 4.1.1 and 4.1.2 either $a \leq e(1, R)$, $e(1, R) < 2a < g - e(1, R)$ and $2a - e(1, R) - 1 \geq (g-1)/3$ or $2a \geq g - e(1, R)$. If $g \equiv 1 \pmod{3}$ assume that $R^{\otimes 4(g-1)/3}$ is not isomorphic to $K_X^{\otimes 2}$. Then there is a flat family of stable spanned rank 2 vector bundles on X with determinant $L := R^{\otimes 2a}$ and with $Q := R^{\otimes a} \oplus R^{\otimes a}$ as flat limit.

Proof. We will use the set-up for the dimensional count introduced in the proof of 2.3.1. Let E be a general bundle fitting in the exact sequence (3). Here by the assumption on $\$(a, e)$ we see that E is not isomorphic to Q . Since Q is semi-stable E is semistable. Assume by contradiction that E is not stable and let M be a degree $3a$ quotient line bundle of E . Since M is spanned, by Remark 4.2 either $M \cong R^{\otimes a}$ or $g \equiv 1 \pmod{3}$ and $M \cong K_X \otimes R^{\otimes z}$ with $z = -a + (g-1)/3$. First assume $M \cong R^{\otimes a}$. Hence E is given by an exact sequence (1) with $A \cong M$. We claim that $h^0(E) < h^1(Q)$, i. e. not all sections of M lift to sections of E . The claim is well-known and easy: indeed it is proved in [BR, Prop. 1.3], the much stronger

assertion that $h^0(E) = h^0(A) + \max(0, h^1(M) - h^1(A))$. Hence by assumption on $\$(a, e)$ we have $\dim(G(3, h^0(L))) > \dim(G(3, h^0(E)))$ and we conclude. Now assume $M \cong K_X \otimes R^{\otimes z}$ with $-z = -a + 2(g-1)/3$. Thus E fits in an exact sequence (1) with either $A \cong M$ or $A \cong R^{\otimes a}$. Since $A \otimes M \cong L$, if $A \cong R^{\otimes a}$, then $M \cong R^{\otimes a}$ while if $A \cong M$ we have $R^{\otimes 4(g-1)/3} \cong K_X^{\otimes 2}$, contradiction. \diamond

Now we consider the case in which $L = R^{\otimes(2a+1)}$ for some integer a .

THEOREM 4.4. *Fix integers $g \geq 5$, $e(1, R)$, a with $0 < a < g - e(1, R)$ and set $\$(g, e(1, R)) := 3(h^0(R^{\otimes(2a+1)}) - h^0(R^{\otimes a}) - h^0(R^{\otimes(a+1)})) - 3$. Assume $\$(g, e(1, R)) \geq g - 1$ and let X be a trigonal curve whose $g_3^1 R$ has invariant $e(1, R)$. If $g \equiv 1 \pmod{3}$ assume that $R^{\otimes(4g-4)/3}$ is not isomorphic to $K_X^{\otimes 2}$. Then there is a flat family of stable spanned rank 2 vector bundles with $Q' = R^{\otimes a} \oplus R^{\otimes(a+1)}$ as flat limit.*

Proof. The proof is exactly the same as the one of Theorem 4.3. Indeed, note again that if E is not stable by the semicontinuity of the Harder-Narasimhan filtration and the assumption when $g \equiv 1 \pmod{3}$ any destabilizing quotient M of E must be isomorphic to $R^{\otimes a}$. \diamond

5. Here we discuss the case of spanned vector bundles on a smooth curve X which is “general” in a very weak sense.

Fix an exact sequence (1) with $h^0(A) \neq 0$. Since E is spanned, if $M \neq A \oplus \mathcal{O}$, then $h^0(M) \geq 2$. Note that if d is such that there is a base point free g_d^2 , say L , we obtain a spanned rank 2 bundle E with $\det(E) = L$ and fitting in the exact sequence (3). For instance this is the case if d is the first integer such that $\rho(g, 2, d) \geq 0$. Since for such d we have $\rho(g, 1, d/2) < 0$, for this particular example every A fitting in (1) has $h^0(A) = 1$. Furthermore, since the existence of a g_z^3 implies the existence of a g_{z-1}^2 , in this particular example we have $A \cong \mathcal{O}$. Since E is not trivial, $h^0(E) > 2$. Thus in any exact sequence

$$0 \longrightarrow A' \longrightarrow E \longrightarrow M' \longrightarrow 0 \quad (4)$$

we have $\deg(A') \leq 0$. Hence for this particular d , E is stable and with Harder-Narasimhan-Lange-Segre invariant d (see e. g. [LN]). If d is larger we have at least $\deg(A') \leq d - d'$, where d' is the first integer with $\rho(g, 2, d') \geq 0$. Note that $d < 2d'$. Note that if there is a spanned g_d^r, L , on X but no g_{d+1}^{r+1} , then L is primitive. Hence Lemma 1.9 and the discussion just given prove Proposition 0.1.

Proof of Theorem 0.2. Take F given by the following exact sequence:

$$0 \longrightarrow L^* \longrightarrow (r+1)\mathcal{O} \longrightarrow F \longrightarrow 0 \quad (5)$$

Assume by contradiction the existence of a rank c ($1 \leq c \leq r-1$) quotient bundle U of F with $\deg(U) \leq c \deg(F)/r$. Since F is spanned, U is spanned. Hence $\det(U)$ is spanned and not trivial. Hence there is a $g_{\deg(U)}^c$ on X , contradiction. If L is primitive, then F is primitive by 1.9. \diamond

REMARK 5.1. Note that we have a spanned g_d^r on X if $\dim W_{d-1}^r(X) \leq \dim W_d^r(X) - 2$ and there is a primitive g_d^r if we have also $\dim W_d^r(X) > \dim W_{d+1}^{r+1}(X)$.

6. In this section we consider 5 generalizations (5 among the many possible ones) of the Clifford index (and the geometric theory behind it) to the case of vector bundles of rank $n > 1$. No unique definition seems to be the best possible one for all aims and targets (stable or semistable vector bundles, inductive proofs, classifications of extremal cases and study of nice examples). We fix a smooth genus g curve X . Let $W_n^{r,d}(X)$ be the scheme of stable vector bundles, E , on X with $\text{rank}(E) = n$, $\deg(E) = d$ and $h^0(E) \geq r+1$.

DEFINITION 6.1. Let E be a rank n vector bundle on X with $h^0(E) \neq 0$. Set $\text{Cliff}(E) := \deg(E) - 2h^0(E) + 2n$. $\text{Cliff}(E)$ is called the Clifford index of E .

REMARK 6.2. We have $\text{Cliff}(\mathcal{O}) = 0$, $\text{Cliff}(A \oplus B) = \text{Cliff}(A) + \text{Cliff}(B)$; if $h^1(E) \neq 0$ we have $\text{Cliff}(E) = \text{Cliff}(E^* \otimes K_X)$.

DEFINITION 6.3. Set $c(n, X) := \inf\{\text{Cliff}(E) \text{ with } E \text{ rank } n \text{ spanned indecomposable vector bundle on } X \text{ with } h^0(E) \neq 0, h^1(E) \neq 0\}$.

Set $s(n, X) := \inf\{\text{Cliff}(E) \text{ with } E \text{ rank } n \text{ spanned stable vector bundle with } h^0(E) \neq 0 \text{ and } h^1(E) \neq 0\}$.

Set $p(n, X) := \inf\{\text{Cliff}(E) \text{ with } E \text{ rank } n \text{ primitive stable vector bundle with } h^0(E) \neq 0 \text{ and } h^1(E) \neq 0\}$.

Set $ms(n, X) := \inf\{\text{Cliff}(E) \text{ with } E \text{ rank } n \text{ maximally stable primitive vector bundle with } h^0(E) \neq 0 \text{ and } h^1(E) \neq 0\}$.

Set $gs(n, X) := \inf\{\text{Cliff}(E) \text{ with } E \text{ rank } n \text{ vector bundle with } h^1(E) \neq 0 \text{ and such that } H^0(E) \text{ spans a rank } n \text{ subsheaf of } E\}$.

EXAMPLE 6.4. Fix an integer $n \geq 2$. Assume the existence on X of a degree t pencil R such that $n \leq e(1, R)$ (i. e. $h^0(R^{\otimes n}) = n + 1$) and that there is no g_{nt-1}^n on X . Assume that $s(n, X)$ is computed by bundles, E , with $h^0(E) = n + 1$. Then we have $s(n, X) \geq nt - 2n - 2$. Indeed, take a spanned rank n vector bundle F computing $s(n, X)$ and set $L := \det(F)$. Since F is spanned by $n + 1$ sections and $h^0(F^*) = 0$ by the stability of F , F fits in an exact sequence (5) (taking $r = n$) and we have $h^0(L) \geq n + 1$. Thus $\deg(L) \geq nt$. Hence $\text{Cliff}(F) \geq nt - 2n - 2$. Furthermore, if we assume that $R^{\otimes n}$ is the unique g_{nt}^n on X , then we have $s(n, X) > nt - 2n - 2$ for the following reason. With the notations of the first part, assume by contradiction $s(n, X) = nt - 2n - 2$. Then we have $L \cong R^{\otimes n}$. If a bundle F which fits in (5) (with $r = n$) has $h^0(F^*) = 0$ (and in particular if it is semistable with degree > 0 or stable of degree ≥ 0), then the induced map $V^* \rightarrow H^0(L)$ is an isomorphism. Hence the unique such bundle which fits in (5) (with $r = n$) is $R^{\oplus n}$ which is not stable.

Now we will consider again the case of curves with general moduli with very different methods with respect to the ones of §5. The main tool is the existence theorem (and the proof that it has the expected dimension $\min\{\rho(n, d, r, g), n^2(g - 1) + 1\}$ for $W_n^{r,d}(X)$ when X has general moduli and ρ is not too small (see [Te1], Th. 1, for the precise statement). Here we consider the problem of the spannedness of the corresponding bundle and prove Theorem 0.3.

Proof of Theorem 0.3. The main point is to show that a general $E \in W(d, r)$ is spanned. First, note that the proof of [Te1, Th. 1],

implies that for a general $F \in W(c, b)$ with $c \geq n + 1$, the bundle F is generically spanned; this is proved in §4 and §5 of [Te1] (see in particular the explicit data for the existence part at page 397, Proof of Theorem 1). Note that the schemes $W(c, b)$'s for all pairs of integers (c, b) come from the same construction. Hence if a general $E \in W(d, r)$ is not spanned we would have $\dim(W(d - 1, r)) > \rho(n, d - 1, r, g)$, contradiction. Similarly, if a general $E \in W(d, r)$ is not primitive, we would have $\dim(W(d + 1, r + 1)) > \rho(n, d + 1, r + 1, r, g)$, contradiction. \diamond

REMARK 6.5. Note that the vector bundle E whose existence was proved in Theorem 0.3 has $\text{Cliff}(E) = d - 2r - 2 + 2n$.

7. For the existence of primitive bundles with a given determinant the following result is often useful.

PROPOSITION 7.1. *Assume the existence of a primitive $g_k^1 A$ on X with $k \leq g - 1$. Then there is a unique rank $g - k$ primitive bundle F with $\det(F) = K_X \otimes A^*$ and $h^0(F) = g - k + 1$.*

Proof. By Serre duality $K_X \otimes A^*$ is a g_{2g-2-k}^{g-k} . By definition of primitive line bundle, $K_X \otimes A^*$ is spanned. Hence we may define F using the exact sequence (5) taking $K_X \otimes A^*$ as L and $g - k$ instead of r . F is primitive by Lemma 1.9. Viceversa, any such F must be given by (5) with $g - k$ instead of r and $L \cong \det(F)$. \diamond

COROLLARY 7.2. *X has a primitive g_k^1 (hence a primitive rank $g - k$ bundle) for all integers k with $g/2 + 1 \leq k \leq g - 1$ if either X has general moduli or X is a general t -gonal curve for some integer t with $3 \leq t \leq g/2$.*

Proof. If X has general moduli, the result follows from the classical Brill-Noether theory ([ACGH]) and the fact that for these integers k we have $\rho(g, 1, k - 1) < \rho(g, 1, k) + 1$. If X is a general t -gonal curve, the result is [CKM2, Th. 3.1]. \diamond

Now we study the stability and semi-stability of the primitive

bundles we obtain.

PROPOSITION 7.3. *Fix an integer t with $3 \leq t \leq g/2+1$. Assume that X has 2 different base point free $g_t^1 R$ and R' , no g_x^1 with $x \leq 2t/3$ and no g_y^2 with $y \leq 4t/3$. Then there is a stable rank 3 vector bundle F with $\text{rank}(F) = 3$ and with $\det(F) \cong R \otimes R'$. If $R \otimes R'$ is primitive, then F is primitive.*

Proof. Since R and R' are not compounded with the same rational involution, by the strong form of a lemma of Segre-Hopf we have $h^0(R \otimes R') \geq 4$. Take 4 general sections of $R \otimes R'$ and take as F the bundle induced by the exact sequence (5) (with $r = 3$). Assume that F is not stable. Since $\text{rank}(F) = 3$, either F has a quotient line bundle N with $\deg(N) \leq \deg(F)/3 = 2t/3$ or a quotient stable rank 2 bundle Z with $\deg(Z) \leq 4t/3$. Since F is spanned, N and Z should be spanned. Since X has no pencil of degree $\leq 2t/3$, N cannot exist. We claim that $h^0(\det(Z)) \geq 3$. Since Z is stable and spanned, it has a section $s \neq 0$ vanishing at a general point P . Let D be the zero locus of s ; we have $P \in D$. Since Z is stable, $\deg(D) < 2t/3$. Since Z is spanned, both $\det(Z)$ and $\det(Z)(-D)$ are spanned. Hence we get the claim. Hence Z cannot exist. The last assertion is Lemma 1.9. \diamond

PROPOSITION 7.4. *Assume that X has two base point free $g_t^1 R$ and R' ; we allow the case $R = R'$. Assume $R \otimes R'$ primitive. Assume that X has no pencil of degree $< t$. Then there is a semistable primitive rank 2 bundle E on X with $\det(E) \cong R \otimes R'$.*

Proof. Set $U := R \oplus R'$ and take a general 4-dimensional subspace of $H^0(U)$ spanning U . Hence we may define the bundle E by the exact sequence

$$0 \longrightarrow U^* \longrightarrow V \otimes \mathcal{O} \longrightarrow E \longrightarrow 0 \quad (6)$$

By construction E has no trivial factor. Hence any quotient line bundle of E has at least 2 sections. Hence E has no destabilizing quotient line bundles. \diamond

REMARK 7.5. Note that by [CKM2, Prop. 1.1], if X is a general t -gonal curve, $3 \leq t \leq g/2$, and R is its g_t^1 , then $R^{\otimes x}$ is primitive for

all $x < (g - 2)/(t - 1)$.

8. In this section we will give estimates for the dimension of families of rank 2 vector bundles with sections and with particular properties on a curve X . For instance we will study the case in which the sections generate a rank 1 subsheaf of the bundle E . The estimates will be in terms of the numerical invariants of the schemes of the special divisors on the fixed curve X . As a byproduct of these bounds the interested reader may easily extend [Te2, Prop. 1.3 and Th. 2] (and other similar results!) from the case of curves with general moduli to the case of curves with too many g_x^1 and g_y^2 . We will introduce the following notations. Let X be a smooth genus g curve. Let $W_n^{r,d}(X, \text{gen})$ be the subset of $W_n^{r,d}(X)$ parametrizing spanned vector bundles. Unless otherwise stated, the subsets of $W_n^{r,d}(X)_{\text{red}}$ will be taken with the reduce structure and the induced topology. Set $w(n, r, d) := \dim(W_n^{r,d}(X))$; if $u \geq g + r$, we set $w(1, r, u) := g$.

The following elementary lemma is due to B. Feinberg for $r = 2$ (see also [Te2, Lemma 1.1 and Cor. 1.2]).

LEMMA 8.1. *Let E be a rank r vector bundle on X and $W \subseteq H^0(E)$ a linear subspace with $\dim(W) \geq 2$ and such that for every $P \in X$ there is $s \in W$ with $s(P) \neq 0$. Then either W spans a rank 1 subsheaf of E or a general $s \in W$ has no zero.*

Proof. Assume that W spans a subsheaf of rank $t \geq 2$. For every $P \in X$, set $W(-P) := \{s \in W : s(P) = 0\}$. By assumption $W(-P) \neq W$ for every $P \in X$. Furthermore, for a general $P \in X$ the linear space $W(-P)$ has codimension $t \geq 2$. Hence, since the algebraically closed base field is infinite, a general $s \in W$ has no zero. \diamond

Motivated by Lemma 8.1 we introduced the following definition.

DEFINITION 8.2. A rank r vector bundle E on X with $h^0(E) \geq 2$ is said to be of *line bundle type* if $H^0(E)$ spans a rank 1 subsheaf of E . More generally a pair (E, W) with $W \subseteq H^0(E)$ and $\dim(W) \geq 2$ is said to be of *line bundle type* if W spans a rank 1 subsheaf of E . Call $W_n^{r,d}(X; \text{lin})$ the subset of $W_n^{r,d}(X)$ formed by the vector bundles of line bundle type.

We want to show that the structure of the subscheme $W_n^{r,d}(X)_{\text{red}}$ is essentially determined from the structure of $W_1^{a,b}(X)$ with $a \leq r$. However, from the point of view of the (rational) maps from X to the Grassmannian $G(n, v)$ we are essentially interested to the components of the subscheme $W_n^{r,d}(X, \text{gen})$ (hence to the subvarieties of $W_n^{r,d}(X)_{\text{red}}$ whose general element is spanned by its global sections).

Now we will introduce other notations. Set $x(1) = 0$. If $z \geq 1$ let $x(z+1)$ be the minimal integer x such that X has a g_x^z ; hence $x(2)$ is the gonality of X . Fix a line subbundle A of E ; often A will be a maximal degree subbundle of E or the saturation of a rank 1 subsheaf spanned by $V \subseteq H^0(E)$. Set $M := E/A$, i. e. take E as an extension given by (1) with $A = A''(D)$, $M = R(B)$, D base locus of $H^0(A)$, $a := \deg(A)$, $a'' := \deg(A'')$, $b'' := \deg(D)$; B base locus of $H^0(M)$, $b := \deg(B)$.

REMARK 8.3. Note that $0 \leq b < g$; if $a > g$, then $D = \emptyset$, i. e. $a = a''$; if $d - a > g$, then $b = 0$; if $u = r + z + g$, with $z \geq 0$ then we set $w(1, r, u) := g$.

We need the following well known observation.

REMARK 8.4. If we are looking for families of simple bundles, E , fitting in the exact sequence (1) we may assume $H^0(\text{Hom}(M, A)) = 0$. Hence by Riemann-Roch for fixed M and A the set of all possible extensions has dimension $d - 2a + g - 1$ and the corresponding set of isomorphism classes of bundles has dimension $\leq d - 2a + g - 2$.

PROPOSITION 8.5. *Let T be an integral variety parametrizing finite to one rank 2 vector bundles on X not of line bundle type and simple with invariants $r \geq 2$, d , b , a and fitting in an exact sequence (1). We have:*

- (a) *Assume $h^0(A) = 0$. If $a = g - 1$, then $\dim(T) \leq d - 2g + 2 + w(1, r, d - b - g + 1) + b + g$. If $a \neq g - 1$, then $\dim(T) \leq d - a + w(1, r, d - a - b) + b - 1$.*
- (b) *Assume $h^0(A) = 1$. If $a = g$, then $\dim(T) \leq d - g + b + w(1, r, d - g - b) + b$. If $a \neq g$ then $\dim(T) \leq d - a + w(1, r - 1, d - a - b) + b - 1$.*
- (c) *Assume $h^0(A) = x \geq 2$. If $a = g + x - 1$, then $\dim(T) \leq g + d - 2a + b + w(1, r - x, d - 2a) + b$. If $a \neq g + x - 1$, then*

$$\dim(T) \leq d - 2a + w(1, x, a) + w(1, r - x, d - 2a) + b - 1.$$

Proof. We will check part (b), since the same proof works almost verbatim for part (a) and part (c). Assume $h^0(A) = 1$. Here we have $h^1(A) = 0$ if and only if $a = g$. Since $\dim(\text{Im}(V)) = \dim(V) - 1 = r$, in this case $\dim(T)$ is bounded by the dimension of all possible A (which is $a := g$) plus the dimension of all possible M (which is $w(1, r, d - 2a - b) = w(1, r, d - 2g - b) + b$, because the base locus varies in a b -dimensional family) plus the dimension Ext^1 of the extensions in (1) with A and M fixed (which is $d - 2a + g$ by Riemann-Roch since we assumed $h^0(\text{Hom}(M, A)) = 0$). If $a \neq g$, to the previous computation we add Remark (see [BR, Prop. 1.3], for much more) that for every fixed A and M and a general extension in (1) we have $\dim(\text{Im}(V)) < h^0(E) + h^0(A)$. Since $w(1, r - 1, d - 2a - b) - 1 \geq w(1, r, d - 2a - b)$ we conclude. \diamond

By Proposition 8.5 we have the following corollary.

COROLLARY 8.6. *Let T be an integral variety parametrizing finite to one rank 2 vector bundles on X not of line bundle type and simple with invariants $r \geq 2$, d , b , a and fitting in an exact sequence (1). If $a \leq g - 2$, then $\dim(T) \leq d - a + w(1, r, d - a - b) + b - 1$. Assume $a = g + y - 1$ with $y \geq 0$; Then $\dim(T) \leq d - 2a + g + w(1, r - y, d - a - b) + b$.*

REMARK 8.7. Let T be an integral variety parametrizing finite to one rank 2 vector bundles on C not of line bundle type, not decomposable and not simple, fitting in an exact sequence (1) with invariants $r \geq 2$, d , b , a . Fix an exact sequence (1) and assume $w := h^0(\text{Hom}(M, A)) > 0$. By Riemann-Roch we have $h^1(\text{Hom}(M, A)) = w + g - 1 + d - 2a$. Assume $a \leq g - 1$; if $h^0(A) = 0$, the set of all possible line bundles A has dimension g , the set of all possible M has dimension $\leq w(1, r, d - a - b) + b$ and the set of extensions with fixed A and M has dimension $d - 2a + g - 1 + w$; hence $\dim(T) \leq -1 + d - 2a + g - 1 + w + g + w(1, r, d - a - b) + b$; if $h^1(A) > 0$ we obtain better bounds. However, there is no such extension unless $d - 2a \geq x(w)$. Now assume $a \geq g$ and set $\alpha := a + 1 - g$. To have $T \neq \emptyset$ we have again the necessary condition $d - 2a \geq x(w)$; if $h^1(A) = 0$ we have $\dim(T) \leq -1 + g + w(1, r - \alpha, d - a - b) + b + w + g - 1 + d - 2a$;

if $h^1(A) > 0$ we have a better bound.

PROPOSITION 8.8. *Let T be an integral variety parametrizing finite to one rank 2 splitted vector bundles on X not of line bundle type with invariants $r \geq 2, d, b, a$. Then $\dim(T) \leq \dim\{A\} + \dim\{M\} = \max\{w(1, a, s) + w(1, d - a, s')\}$ for all possible s and s' with $s + s' \geq r + 1, s' > 0, s \geq 0$.*

PROPOSITION 8.9. *Let T be an integral variety parametrizing finite to one rank 2 vector bundles on X of line bundle type with invariants $r \geq 2, d, b, a, a = a'' + b'', b'' = \deg(D)$. Then $\dim(T) \leq b'' + w(1, a'', r) + g$ with $d - (a'' + b'') \leq g - 1$.*

LEMMA 8.10. *Fix invariants $a = a'' + b'', d, r, b$. Let T be an integral variety parametrizing finite to one simple rank 2 vector bundles of line bundle type on X with $h^0 = r + 1$. Then $\dim(T) \leq w(1, r, a'') + b'' + g + d - 2a + g - 2$ and $T = \emptyset$ unless $d \leq 2g + r - 1$.*

Proof. In the extension (1) the condition “no section of M is lifted to a section of E'' implies $h^1(A) \geq h^0(M)$. Since $h^1(A) = r - a + g$ and $h^0(M) \geq (d - a) + 1 - g$, we have the emptyness statement. The dimension of the set of all possible $M \in \text{Pic}^{(d-a)}(C)$ is bounded by g , while for fixed A and M we may apply Remark 8.4. \diamond

LEMMA 8.11. *Fix invariants $a = a'' + b'', d, r, b$. Let T be an integral variety parametrizing finite to one non simple rank 2 vector bundles of line bundle type on X with $h^0 = r + 1$. $T = \emptyset$ unless $d \leq 2g + r - 1$. Let w be the maximal integer > 0 such that $d - 2a \geq x(w)$; if there is no such w , then $T = \emptyset$. Assume $T \neq \emptyset$. Then $\dim(T) \leq w(1, r, a'') + b'' + g + d - 2a + g - 2 + w$.*

Here we will give an elementary observation on the existence part in Brill-Noether theory.

LEMMA 8.12. *Fix integers g, d, r, n such that for a general genus g curve X the scheme $W_n^{r,d}(X)$ is not empty and has a component of dimension $w \geq 0$. Then for every smooth genus g curve Y there is a non empty family $T(Y)$ of semistable bundles on Y of degree d*

and rank n such that every $E \in T(Y)$ has $h^0(Y, E) \geq r + 1$.

In particular if d and n are coprime we have $W_n^{r,d}(Y) \neq \emptyset$ and $\dim(W_n^{r,d}(Y)) \geq w$.

Proof. The result follows from the properness of the semistability condition in families of curves and from the semicontinuity of cohomology. \diamond

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