BOUNDING THE ORDER OF AUTOMORPHISMS OF CERTAIN CURVES (*)

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SOMMARIO. - *Studiamo il limite superiore sull'ordine degli automorfismi delle curve X soddisfacenti almeno una delle seguenti ipotesi: 1) X é un rivestimento di grado m di esattamente una curva di genere γ, dove m é primo; 2) il centro del gruppo degli automorfismi di X é non banale.*

SUMMARY. - *We study upper bounds on the order of automorphisms of curves X satisfying at least one of the following hypothesis: 1) X is an m-sheeted covering of exactly one curve of genus γ, where m is prime; 2) the center of the group of automorphisms of X is non-trivial.*

NOTATION. Throughout this paper, by a curve we mean a non-singular, irreducible and projective algebraic curve defined over an algebraically closed field k of characteristic p. Let X be a curve and $P \in X$;

- $k(X)$ and $\mathrm{Aut}(X)$ will denote, respectively, the field of rational functions and the group of automorphisms of X. The symbol $\mathrm{div}_\infty(f)$ will stand for the polar divisor of $f \in k(X)$.
- For $\tau \in \mathrm{Aut}(X)$, $\mathrm{ord}(\tau)$, $\mathrm{Fix}(\tau)$ and $v(\tau)$ will denote, respectively, the order, the set of fixed points and the number of fixed points of τ. k_τ and g_τ will denote, respectively, the field of rational functions and the genus of the quotient curve $X/\langle\tau\rangle$. π_τ will denote the natural morphism $X \to X/\langle\tau\rangle$.

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- $H(P)$ and $G(P)$ will denote, respectively, the Weierstrass semigroup and the set of gaps at P.

Introduction.

In this paper we study upper bounds on the order of automorphisms of curves satisfying at least one of the following hypothesis.

$H_1(m,\gamma)$: X is an m-sheeted covering of exactly one curve $\tilde{X}$ of genus γ, where m is prime.

H_2 : The center of $\mathrm{Aut}(X)$ is non-trivial.

Let X be a curve of genus $g \geq 1$ and let τ be an automorphism of X. We assume $v(\tau) \geq 1$ if $g = 1$. It is known that the order of τ satisfies

$$\mathrm{ord}(\tau) \leq \begin{cases} 2g+1 & \text{if } \mathrm{ord}(\tau) \text{ is prime} \\ 2(2g+1) & \text{otherwise,} \end{cases} \tag{1}$$

except for some exceptional cases occurring for wildly ramified extensions $k(X) \,|\, k_\tau$ (Wiman [Wi], Harvey [Har], Singh [S, Thms. 3.3, 3.3'], Stichtenoth [St, §4]).

Suppose that X satisfies $H_1(m,\gamma)$. The following discussion follows from Accola's [A, §4] (see also [A2, Chapters 4, 5]). Let $G(X|\tilde{X})$ be the group of covering transformations of $X \to \tilde{X}$, and let $\tau \in \mathrm{Aut}(X) \backslash G(X|\tilde{X})$. Then τ induces an automorphism $\tilde{\tau} \in \mathrm{Aut}(\tilde{X})$ whose order is the smallest $\tilde{n} \in \mathbb{N}$ such that $\tau^{\tilde{n}} \in G(X \,|\, \tilde{X})$. If $\gamma \geq 2$ or $v(\tilde{\tau}) \geq 1$ if $\gamma = 1$, from (1) we have upper bounds for $\tilde{n}$ and hence for $\mathrm{ord}(\tau)$. For instance if $\mathrm{ord}(\tau)$ is a prime different from $\#G(X \,|\, \tilde{X})$, then

$$\mathrm{ord}(\tau) \leq 2\gamma + 1. \tag{2}$$

(see §2). We remark that one can also obtain information about $\#\mathrm{Aut}(X)$ because $\mathrm{Aut}(X)/G(X \mid \tilde{X})$ is isomorphic to a subgroup H of $\mathrm{Aut}(\tilde{X})$. For example Accola (loc. cit.) used this to give an explicit construction of curves admitting of only the identity as an automorphism. On the other hand if $k(X) \mid k(\tilde{X})$ is a Galois extension, then

$$\#\mathrm{Aut}(X) = m\#H.$$

Thus if X has many automorphisms, then either X does not satisfy $H_1(m,\gamma)$ with $\gamma \geq 1$ (e.g. Hermitian curves, see [St]), or if X does, then $\gamma = 0$ (e.g. the Klein curve, see [Hur]; the curve $y^2 = x^p + x$, see [Ro]), or H has many automorphisms (see [Mac]).

We also remark that $H_1(m,\gamma)$ is satisfied if X is an m-sheeted covering of a curve of genus γ and $g > 2m\gamma+(m-1)^2$. The hypothesis on g implies the uniqueness property of $H_1(m,\gamma)$ by means of one of Castelnuovo's genus bound (see 1.1). The existence of an m-sheeted covering from X to a curve of genus γ can be characterized by means of the existence of certain Weierstrass semigroups as well as the existence of certain linear series on X (see [T]).

Now suppose that X satisfies H_2. Fix σ_0 in the center of $\mathrm{Aut}(X)$ with $m := \mathrm{ord}(\sigma_0)$ being a prime. Let $\tau \in \mathrm{Aut}(X) \setminus \langle\sigma_0\rangle$. We bound $\mathrm{ord}(\tau)$ by using the data (m, g_{σ_0}). As the main consequence of H_2 we can "pushdown" the data $(\mathrm{ord}(\tau), v(\tau))$ on X to $(\mathrm{ord}(\tilde{\tau}), v(\tilde{\tau}))$ on $\tilde{X} := X/\langle\sigma_0\rangle$, where $\tilde{\tau}$ is the pushdown of τ to $\tilde{X}$. Moreover, $X/\langle\sigma_0,\tau\rangle$ is isomorphic to $\tilde{X}/\langle\tilde{\tau}\rangle$, and $v(\tau)$ satisfies an equation of type

$$v(\tau) = mu + f,$$

where $u \in \mathbb{N}$ and $f = \#\mathrm{Fix}(\sigma_0) \cap \mathrm{Fix}(\tau)$. In particular, if $m \nmid \mathrm{ord}(\tau)$ and $\mathrm{Fix}(\tau^d) = \mathrm{Fix}(\tau)$, for $d \mid \mathrm{ord}(\tau)$, $d < \mathrm{ord}(\tau)$ we find

$$2g_{\sigma_0} - 2 + u + f = \mathrm{ord}(\tau)(2g_{\tau_1} - 2 + u + f),$$

where $\tau_1 := \sigma_0 \circ \tau$. If X also fulfils $H(m, g_{\sigma_0})$ the above relation improves (2) (see §3).

Typical examples of curves satisfying both the hypothesis above are the 2-sheeted coverings having genus large enough. Assume that X is a 2-sheeted covering of a curve of genus γ, and let J_γ be an involution on X whose orbits are the fibers of the 2-sheeted covering. Then J_γ is unique provided $g > 4\gamma+1$ (Farkas [F, Corollary 2], Accola [A, Lemma 5]). Also in this case J_γ belongs to the center of $\mathrm{Aut}(X)$ (Farkas, [F, Thm. 2]; Accola [A1, Application 4]). Furthermore Farkas (loc.cit.) showed that

$$v(\tau) \leq 4\gamma + 4$$

for $\tau \in \mathrm{Aut}(X) \setminus \langle J_\gamma\rangle$. For the case of hyperelliptic curves ($\gamma = 0$) of genus $g > 1$ it is well known that all the possibilities for $v(\tau)$

in $\{0,1,2,3,4\}$ occur and the unique restriction on $\mathrm{ord}(\tau)$ is the Riemann-Hurwitz formula for $k(X) \mid k_\tau$ (cf. Hurwitz [Hur]). However, if $\gamma \geq 1$, $g > 4\gamma + 1$, and if we assume $v(\tau) \geq 1$ for $\gamma = 1$ the situation for both $v(\tau)$ and $\mathrm{ord}(\tau)$ is different as we can see from (2) and the above relation involving u and f. For instance it was announced by Yoshida [Yo] that if $\gamma = 1$ and $g > 5$, then the possibilities for $(\mathrm{ord}(\tau), v(\tau))$ are

$$\begin{array}{llllllll} (3,3), & (3,5), & (3,4), & (3,2), & (5,2), & (5,3), & & \\ (7,3), & (9,2), & (12,1), & (8,2), & (6,1), & (6,2), & (4,4), & \end{array}$$

provided $\mathrm{Fix}(J_1) \cap \mathrm{Fix}(\tau) \neq \emptyset$.

The prototypes of our results are the following rather simple examples. They also illustrate the methods used here.

EXAMPLE 1. Let X be a hyperelliptic curve of genus $g > 1$ defined over k with $p \neq 2$. Let $\tau \in \mathrm{Aut}(X)$ such that $\mathrm{ord}(\tau)$ is an odd prime different from p. Set $f := \#\mathrm{Fix}(J_0) \cap \mathrm{Fix}(\tau)$. Then $(v(\tau), f) \in \{(4,0),(3,1),(2,2)\}$.

The hypothesis on g implies that J_0 and τ commute with each other. Hence if $P \in \mathrm{Fix}(\tau) \setminus \mathrm{Fix}(J_0)$, then $J_0(P) \in \mathrm{Fix}(\tau)$. Thus there exists $u \in \mathbb{N}$ such that $v(\tau) = 2u + f$. Let $\tilde{\tau}$ be the pushdown of τ to $X/\langle J \rangle$. We have that $\mathrm{ord}(\tau) = \mathrm{ord}(\tilde{\tau})$ because $\mathrm{ord}(\tau)$ is odd. Hence the Riemann-Hurwitz formula applied to $\pi_{\tilde{\tau}}$ gives

$$-2 + u + f = \mathrm{ord}(\tau)(-2 + u + f),$$

and so $u + f = 2$, which establishes the example.

EXAMPLE 2. Let X be a 2-sheeted covering of an elliptic curve $\tilde{X}$ defined over k with $p \neq 2$. Suppose that the genus of X satisfies $g > 5$, and let $\tau \in \mathrm{Aut}(X)$ with $\mathrm{Fix}(\tau) \neq$ and $\mathrm{ord}(\tau)$ an odd prime different from p. Then $(\mathrm{ord}(\tau), v(\tau)) \notin \{(5,2),(3,2),(5,3),(7,3),(9,2)\}$.

Suppose that such a τ exists. Let $\tilde{\tau}$ be the pushdown of τ to $\tilde{X}$. Then $\mathrm{ord}(\tau) = \mathrm{ord}(\tilde{\tau})$ because $\mathrm{ord}(\tau)$ is odd. Now since 3 is the only possible odd order for a non-trivial automorphism of $\tilde{X}$ fixing a point (this follows from the well known group structure of automorphism

fixing a point on elliptic curves; see Silverman [Sil, Thm. 10.1]), we reduce the example to analyze the case $(\mathrm{ord}(\tau), v(\tau)) = (3, 2)$. With the notation from the above example we have $v(\tau) = 2u + f = 2$ and so $u + f \in \{1, 2\}$. Consequently applying Riemann-Hurwitz to $\pi_{\tilde{\tau}}$ we find

$$u + f = 3(2\tilde{g} - 2 + u + f),$$

where $\tilde{g}$ stands for the genus of $k(\tilde{X})/\langle\tilde{\tau}\rangle$. This is a contradiction.

In particular, we see that not all the cases listed by Yoshida can occur. We will also see that most of the known results on automorphisms of hyperelliptic curves (e.g. Farkas-Kra [F-K; III.7.11, V.2.13]) will emerge as simple corollaries of ours (see 5.1). Yoshida and Farkas - Kra use Lewittes' results concerning representations of the group of automorphisms as linear maps of differential spaces (see [L]). To compute diagonal matrices, here one uses the sequence of Weierstrass gaps at fixed points. Then, by means of the character of the representation, one produces an equation $(*)$ involving the genus of the curve, the order of the automorphism and the number of its fixed points. This equation and the Riemann-Hurwitz formula imply restrictions for the order and the number of fixed points. When the curve satisfies H_2 we obtain an analogous of $(*)$ by pushing down the automorphism to an appropriated curve. The advantage of this equation is that it does not involve gaps sequence at fixed points.

The contents of the paper are as follows. In §1 we summarize the results needed for the results stated here. We mainly based our computations on one of Castelnuovo's genus bound (1.1), the Riemann-Hurwitz formula (1.3) and on some results involving Weierstrass semigroups (1.2).

In §2 and §3 we bound the order on automorphisms of curves satisfying hypothesis $H_1(m, \gamma)$ and H_2 respectively. In §4 we consider necessary and sufficient conditions for automorphisms having large number of fixed points. In 4.3 we improve Farkas' [F, Thm. 1].

In §5 we specialize §2 and §3 to the case of double coverings of curves. In 5.4 we consider automorphisms of elliptic-hyperelliptic curves. In 5.5 we deal with automorphisms of certain double coverings of hyperelliptic curves, and we finish with 5.6 where we indicate how to obtain results similar to those of 5.4 and 5.5 for certain double covering of trigonal curves.

1. Preliminary results.

1.1. Castelnuovo's lemma ([C], [St1]).

Let X be a curve of genus g. Let k_1 and k_2 be two subfields of $k(X)$ with compositum equal to $k(X)$. Let $n_i = [k(X) : k_i]$ and g_i be the genus of k_i. Then

$$g \leqq n_1 g_1 + n_2 g_2 + (n_1 - 1)(n_2 - 1).$$

1.2. Remarks on Weierstrass semigroups.

Let X be a curve and $\tau \in \mathrm{Aut}(X)$ with $p \nmid \mathrm{ord}(\tau)$.

(i) If $P \in \mathrm{Fix}(\tau)$ and $h \in \mathbb{N}$, then

$$\mathrm{ord}(\tau)h \in H(P) \Leftrightarrow h \in H(\pi_\tau(P)).$$

This is included in an implicit way in Kato's [K, p. 393] (see also [T, Lemma 3.4]). Consequently ([Sch])

$$g_\tau = \#\{\ell \in G(P) : \ell \equiv 0 \pmod{\mathrm{ord}(\tau)}\}.$$

(ii) The above remark implies the following. Let $\sigma \in \mathrm{Aut}(X)$ such that $\mathrm{ord}(\sigma) = \mathrm{ord}(\tau)$ and $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\tau) \neq \emptyset$. Then

$$g_\sigma = g_\tau.$$

(iii) Let $P \in \mathrm{Fix}(\tau)$ and $\ell \in G(P)$ such that $\ell \equiv 0 \pmod{\mathrm{ord}(\tau)}$. Since $H(\pi_\tau(P)) \supseteq \{2g_\tau, 2g_\tau + 1, \ldots\}$, then (i) also implies

$$\ell \leq (2g_\tau - 1)\mathrm{ord}(\tau).$$

In particular, if $\mathrm{ord}(\tau) = 2$ then $H(P)$ has g_τ odd non-gaps $\leq 2g - 1$. Moreover, let $U_1 < \ldots < U_{g_\tau}$ be such a non-gaps. Then $U_1 \geq 2g - 4g_\tau + 1$ and

$$H(P) = \langle 2m_1, \ldots, 2m_{g_\tau}, 4g_\tau + 2, U_1, \ldots, U_{g_\tau} \rangle,$$

where the m_i are the first g_τ positive non-gaps at $\pi_\tau(P)$ ([T, Lemmas 2.1, 2.3]).

1.3. The Riemann-Hurwitz formula.

Let X be a curve of genus g, and $\tau \in \mathrm{Aut}(X)$. Assume $p \nmid n := \mathrm{ord}(\tau)$. We will use the following version of the Riemann-Hurwitz formula for π_τ ([F-K, p. 274])

$$2g - 2 = n(2g_\tau - 2) + \sum_{d|n, d<n} \varphi(n/d) v(\tau^d),$$

where φ is the Euler function. In the formulae of §3 we will also use the number

$$\Lambda_\tau := \sum_{d|n, d<n} \varphi(n/d)(v(\tau^d) - v(\tau)).$$

The following definitions allow us to have a way of computing Λ_τ ([F-K, p.261]). The ramification set of π_τ can be partitioned into a disjoint union of subsets B_d with $d \mid n$ and $d < n$ where

$$\begin{aligned} B_1 &= \mathrm{Fix}(\tau), \\ B_d &= \{P \in X : \tau^d(P) = P,\ \tau^i(P) \neq P \text{ for } 0 < i < d\} \text{ for } d > 1. \end{aligned}$$

Let $x_d = x_d(\tau) := \#\pi_\tau(B_d)$. Then

$$\Lambda_\tau = \sum_{d|n, 1<d<n} (n - d) x_d.$$

2. Consequences of $H_1(m, \gamma)$.

Let X be a curve of genus g satisfying $H_1(m, \gamma)$. Let $\pi : X \to \tilde{X}$ be the m-sheeted covering of X over a curve of genus γ, and let $G = G(X \mid \tilde{X})$ be the group of cover transformations of π. We have $\#G = 1$ or $\#G = m$ and the last case occurs if and only if $k(X) \mid k(\tilde{X})$ is a Galois extension. Let $\tau \in \mathrm{Aut}(X)$. By the uniqueness of π the pushdown $\tilde{\tau}$ of τ to $\tilde{X}$ is an automorphism of $\tilde{X}$. By means of the data (m, γ) and by using (1), we will set up upper bounds on the order of $\tau \in \mathrm{Aut}(X) \setminus G$. If $\gamma = 1$ we assume $v(\tau) \geq 1$. (Since $\tilde{\tau} \circ \pi = \pi \circ \tau$, this implies $v(\tilde{\tau}) \geq 1$.)

We have that $\mathrm{ord}(\tilde{\tau}) \mid \mathrm{ord}(\tau)$, and $\mathrm{ord}(\tilde{\tau})$ is the smallest positive integer $\tilde{n}$ such that $\tau^{\tilde{n}} \in G$. Thus,

$$\frac{\mathrm{ord}(\tau)}{\mathrm{ord}(\tilde{\tau})} \mid \#G.$$

We consider two cases.

2.1. $\mathrm{ord}(\tau) = \mathrm{ord}(\tilde{\tau})$.

(This is the case if $\#G = 1$ or $m \nmid \mathrm{ord}(\tau)$.) Here by (1) we have

$$\mathrm{ord}(\tau) \leq \begin{cases} 2\gamma + 1 & \text{if } \mathrm{ord}(\tau) \text{ is prime} \\ 2(2\gamma + 1) & \text{otherwise.} \end{cases}$$

2.2. $\mathrm{ord}(\tau) \neq \mathrm{ord}(\tilde{\tau})$.

Here we have $\mathrm{ord}(\tau) = \mathrm{ord}(\tilde{\tau})m$, and hence $G = \langle \tau^{\mathrm{ord}(\tilde{\tau})} \rangle$. Thus (1) implies

$$\mathrm{ord}(\tau) \leq \begin{cases} (2\gamma + 1)m & \text{if } \frac{\mathrm{ord}(\tau)}{m} \text{ is prime} \\ 2(2\gamma + 1)m & \text{otherwise.} \end{cases}$$

Once we know that X admits an m-sheeted covering over a curve of genus γ, we have the following criterion for the uniqueness of this covering:

2.3. Claim.

Let X be a curve of genus g, m a prime and γ a natural. If

$$g > 2m\gamma + (m-1)^2,$$

then X admits at most one m-sheeted covering over a curve of genus γ.

Proof. Direct application of Castelnuovo's genus bound (1.1). $\diamondsuit$

3. Consequences of H_2.

Let X be a curve of genus g. Throughout this section we fix σ_0 in the center of $\mathrm{Aut}(X)$ with a prime order $m := \mathrm{ord}(\sigma_0)$. Let $\tilde{X} := X/\langle\sigma_0\rangle$ and $\tau \in \mathrm{Aut}(X)$. Then the pushdown $\tilde{\tau}$ of τ to $\tilde{X}$ defines an automorphism of $\tilde{X}$ and thus we can apply §2 to π_{σ_0}. (We assume $v(\tau) \geq 1$ if $g_{\sigma_0} = 1$.) However, if

$$p \nmid \mathrm{ord}(\sigma_0)\mathrm{ord}(\tau),$$

we can obtain more precise information on $\mathrm{ord}(\tau)$.

The hypothesis on the center implies that $X/\langle\sigma_0, \tau\rangle$ is isomorphic to $\tilde{X}/\langle\tilde{\tau}\rangle$. Then by means of the following particular equations for the number of fixed points, we can pushdown the data $(\mathrm{ord}(\tau), v(\tau))$ on X to the data $(\mathrm{ord}(\tilde{\tau}), v(\tilde{\tau}))$ on $\tilde{X}$.

3.1. Let $\tau \in \mathrm{Aut}(X)$ and set $n := \mathrm{ord}(\tau)$. For $d \mid n$ let

$$f_d := \#\mathrm{Fix}(\sigma_0) \cap \mathrm{Fix}(\tau^d).$$

For $P \in \mathrm{Fix}(\tau^d)$, H_2 implies $\{\sigma_0(P), \ldots, {\sigma_0}^{m-1}(P)\} \subseteq \mathrm{Fix}(\tau^d)$. Since m is prime we have m points in the above set, unless $P \in \mathrm{Fix}(\sigma_0)$. Consequently, there exists a non-negative integer $u_d = u_d(\sigma_0, \tau)$ such that

$$v(\tau^d) = mu_d + f_d. \tag{3}$$

3.2. Bounding the order I.

Suppose

$$\langle \sigma_0 \rangle \not\subseteq \langle \tau \rangle.$$

Here $\mathrm{ord}(\tilde{\tau}) = \mathrm{ord}(\tau)$. Thus if $g_{\sigma_0} \geq 2$ or $v(\tau) \geq 1$ for $g_{\sigma_0} = 1$, 2.1 implies

$$n = \mathrm{ord}(\tau) \leq \begin{cases} 2g_{\sigma_0} + 1 & \text{if } \mathrm{ord}(\tau) \text{ is prime} \\ 2(2g_{\sigma_0} + 1) & \text{otherwise.} \end{cases}$$

Moreover, for any g_{σ_0} we have the following

3.2.1. Lemma.

$$v(\tau) \leq \frac{2mg_{\sigma_0}}{n-1} + 2m.$$

Proof. From 1.3 we have $n(2g_\tau - 2) + (n-1)v(\tau) \leq 2g - 2$ $(*)$ (here we used $v(\tau^d) \geq v(\tau)$ and $\sum_{d|n, d>1} \varphi(n/d) = n-1$). Now since m is prime we have $k(X) = k_{\sigma_0} k_\tau$ and thus the lemma follows from $(*)$ and Castelnuovo's genus bound (1.1). ♢

Remark. The proof above only uses an inequality from the Riemann-Hurwitz formula for π_τ. Hence the lemma is also valid when

$$p \mid \mathrm{ord}(\sigma_0)\mathrm{ord}(\tau).$$

Next we will improve the upper bound on n. We consider two cases according as $m \nmid n$ or $m \mid n$. We always assume $p \nmid n$.

3.2.2. $m \nmid n$.

Here we have $v(\tilde{\tau}^d) = u_d + f_d$ for $d \mid n$, $d < n$. Let $\tau_1 := \sigma_0 \circ \tau$. Then $\langle \sigma_0, \tau \rangle = \langle \tau_1 \rangle$, and by applying the Riemann-Hurwitz formula to $k_{\sigma_0} \mid k_{\sigma_0}/\langle \tilde{\tau} \rangle$ we find

$$\begin{aligned} 2g_{\sigma_0} - 2 + u_1 + f_1 &= n(2g_{\tau_1} - 2 + u_1 + f_1) + \\ &\quad + \sum_{d \mid n, d < n} \varphi(n/d)(u_d + f_d - u_1 - f_1). \end{aligned} \tag{4}$$

In particular this implies

$$\begin{aligned} v(\tau) &= \tfrac{2mg_{\sigma_0}}{n-1} + 2m + \\ &\quad - \frac{2mng_{\tau_1} + \Lambda_\tau + (m-1) \sum_{d \mid n, d < n} \varphi(n/d) f_d}{n-1}. \end{aligned} \tag{5}$$

(Λ_τ was defined in 1.3.) Equation (5) yields to the following considerations.

3.2.2.1. $\Lambda_{\tilde{\tau}} = 0$.

Here (5) becomes

$$2g_{\sigma_0} - 2 + u_1 + f_1 = n(2g_{\tau_1} - 2 + u_1 + f_1). \tag{6}$$

(i) Suppose

$$2g_{\sigma_0} - 2 + u_1 + f_1 = 0.$$

Thus either $g_{\sigma_0} = 1$, $u_1 + f_1 = 0$, or $g_{\sigma_0} = 0$ and $u + f_1 = 2$. In the first case we have $g - 1 = n(g_\tau - 1)$, and in the second case

(1) $g - 1 + m = n(g_\tau + m - 1)$, $v(\tau) = 2m$, $f_1 = 0$,
(2) $2g + m - 1 = n(2g_\tau + m - 1)$, $v(\tau) = m + 1$, $f_1 = 1$, or
(3) $g = ng_\tau$, $v(\tau) = f_1 = 2$.

From (5) we notice that if $g_{\sigma_0} = 0$, then $u_1 + f_1 = 2 \Leftrightarrow \Lambda_\tau = 0$. Consequently the above three statements generalizes a result on automorphisms of prime order on hyperelliptic curves stated in [F-K, V.2.13].

(ii) Now suppose

$$2g_{\sigma_0} - 2 + u_1 + f_1 \neq 0.$$

Here we have $g_{\sigma_0} \geq 1$ and $g_{\sigma_0} = 1 \Rightarrow v(\tau) \geq 1$. Hence $n \leq 2g_{\sigma_0} + 1$ for n prime. This upper bound fulfils for any n:

CLAIM.

(1) $n \leq 2g_{\sigma_0} + 1$.

(2) $n = 2g_{\sigma_0} + 1 \Leftrightarrow g_{\tau_1} = 0$ and $u_1 + f_1 = 3$. Consequently

$$(v(\tau), f_1) \in \{(3m, 0), (2m + 1, 1), (m + 2, 2), (3, 3)\}.$$

Proof. From (6) we have $n \leq 2g_{\sigma_0} - 2 + u_1 + f_1$, and we can also write

$$u_1 + f_1 = \frac{2g_{\sigma_0} - 2ng_{\tau_1}}{n - 1} + 2. \tag{7}$$

Then $n(n-1) \leq 2ng_{\sigma_0} - 2ng_{\tau_1}$, which implies statement (1). To prove (2) we notice that $n = 2g_{\sigma_0} + 1$ implies $g_{\tau_1} = 0$ except for $g_{\sigma_0} = 1$, $u_1 + f_1 = 0$ (eliminated by our assumption). In fact if $g_{\tau_1} \geq 1$, by (7) we have $u_1 + f_1 \leq (n - 3)/(n - 1)$, and hence $u_1 + f_1 = 0$. Thus once again by (8) we have $n = 3$ and so $g_{\sigma_0} = 1$. ♢

REMARKS.

(i) Equation (7) implies

$$n < 2g_{\sigma_0} + 1 \quad \Rightarrow \quad n \leq g_{\sigma_0} + 1,$$

unless the case $g_{\tau_1} = 1$, $u_1 + f_1 = 1$ where n could be $2g_{\sigma_0} - 1$.

(ii)

$$n = g_{\sigma_0} + 1 \quad \Leftrightarrow \quad u_1 + f_1 = 4 \text{ and } g_{\tau_1} = 0.$$

Consequently

(ii.1) $(v(\tau), f_1) \in \{(4m, 0), (3m+1, 1), (2m+2, 2), (m+3, 3), (4, 4)\}$.

(ii.2) $(g_{\sigma_0}+1)m \in H(P)$ for $P \in \mathrm{Fix}(\sigma_0) \cap \mathrm{Fix}(\tau)$.

Let $n = g_{\sigma_0}+1$. Then (7) gives

$$u_1 + f_1 = 4 - \frac{2ng_{\tau_1}}{(n-1)} \qquad (*)$$

If $g_{\tau_1} \geq 1$, then $u_1 + f_1 \leq 1$. If $u_1 + f_1 = 1$ (resp. 0), Equation $(*)$ above gives $g_{\sigma_0} = 2$ (resp. 1). The first case leads to a contradiction because an automorphism of prime order cannot have just one fixed point ([Gue], [F-K, Thm. V.2.11]), and the second one is eliminated by hypothesis.

3.2.2.2. $\Lambda_{\tilde{\tau}} > 0$.

By 1.3, $\Lambda_{\tilde{\tau}} = \sum_{d|n, 1<d<n}(n-d)x_d \geq n/2$. Hence by (4) and 2.1, we obtain

Claim.

(1) Let $g_{\tau_1} \geq 1$. Then

$$n \leq \begin{cases} 2(2g_{\sigma_0}+1)/7 & \text{if } u_1+f_1 \geq 3 \\ 4g_{\sigma_0}/5 & \text{if } u_1+f_1 = 2 \\ 2(2g_{\sigma_0}-1)/3 & \text{if } u_1+f_1 = 1 \\ 4(g_{\sigma_0}-1) & \text{if } u_1+f_1 = 0. \end{cases}$$

(2) Let $g_{\tau_1} = 0$ (recall that $g_{\sigma_0} = 1 \Rightarrow u_d + f_d \geq 1$). Then

$$n \leq \begin{cases} 4(g_{\sigma_0}+1)/5 & \text{if } u_1+f_1 \geq 4 \\ 2(2g_{\sigma_0}+1)/3 & \text{if } u_1+f_1 = 3 \\ 4g_{\sigma_0} & \text{if } u_1+f_1 = 2 \\ 2(2g_{\sigma_0}+1) & \text{if } g_{\sigma_0} \geq 1 \text{ and } u_1+f_1 \leq 1. \end{cases}$$

$\Diamond$

3.2.3. $m \mid n$.

Unlike the above case here $\langle \sigma_0, \tau \rangle$ is not cyclic. However, we can use 3.2.1 and 2.1 to bound n. We find

$$n \leq \begin{cases} g_{\sigma_0} + 1 & \text{if } v(\tau) \geq 4 \\ 2g_{\sigma_0} + 1 & \text{if } v(\tau) = 3 \\ 2(2g_{\sigma_0} + 1) & \text{if } g_{\sigma_0} \geq 1,\ v(\tau) \leq 2. \end{cases}$$

3.2.3.1. Remarks.

(i) Suppose $g > 2mg_{\sigma_0} + (m-1)^2$, let $d \mid n$, $d < n$. If $f_d = \#\mathrm{Fix}(\sigma_0) \cap \mathrm{Fix}(\tau^d) \geq 1$, then $m \nmid \frac{n}{d}$.

Indeed, if $d \mid \frac{n}{m}$, then $\tau^{n/m} = \tau^{dr}$ for certain $r \in \mathbb{N}$. If $f_d \geq 1$, by 1.2 (ii) the genus of $\tau^{n/m} = g_{\sigma_0}$ and hence by $H_1(m, g_{\sigma_0})$ (the hypothesis on g implies this; see 2.3) we have $\langle \sigma_0 \rangle \subseteq \langle \tau \rangle$, a contradiction.

(ii) By the Riemann-Hurwitz formula applied to $k_{\tau^{n/m}} \mid k_\tau$ we find

$$n(2g_\tau - 2) + (n-m)v(\tau) + \Lambda_\tau^{(1)} = m(2g_{\tau^{n/m}} - 2),$$

where $\Lambda_\tau^{(1)} := \sum_{d \mid n, d < n, d \neq n/m} \varphi(n/d)(v(\tau^d) - v(\tau))$. In particular for $n > m$ we obtain

$$v(\tau) = \frac{2mg_{\tau^{n/m}} - 2ng_\tau - \Lambda_\tau^{(1)}}{n-m} + 2.$$

(iii) Let $n = m^x q$ with $m \nmid q$. From the Riemann-Hurwitz formula for $k_{\tau^m} \mid k_\tau$ we get

$$mq(2g_\tau - 2) + (m-1)\sum_{d \mid q} \varphi(q/d)v(\tau^d) = q(2g_{\tau^m} - 2). \qquad (*)$$

Thus

$$v(\tau) = \frac{2g_{\tau^m} - 2mg_\tau - \Lambda_\tau^{(2)}}{m-1} + 2,$$

where $\Lambda_\tau^{(2)} := (m-1)\sum_{d|q} \varphi(q/d)(v(\tau^d) - v(\tau))$. In particular if $n = m^x$ we have

$$v(\tau) = \frac{2g_{\tau^m} - 2mg_\tau}{m-1} + 2.$$

Now if we compute $v(\tau^q)$ by using the above formula and replacing it in $(*)$, we get

$$\begin{aligned}(m-1)\sum_{d|q,d<q} \varphi(q/d)v(\tau^d) \quad &= \quad 2(m-1)(q-1) + 2qg_{\tau^m} + \\ &\quad - 2g_{\tau^{mq}} + 2mg_{\tau^q} - 2mqg_\tau.\end{aligned}$$

(iv) Finally we consider the extension $k_{\tau^q} \mid k_\tau$. Let $n = m^x q$ with $m \nmid q$. We find

$$\sum_{d|n,q\nmid d} \varphi(n/d)v(\tau^d) = 2m^x(q-1) + 2m^x g_{\tau^q} - 2ng_\tau.$$

Thus if $q > 1$ we have

$$v(\tau) = \frac{2g_{\tau^q} - 2qg_\tau - \Lambda_\tau^{(3)}}{q-1} + 2,$$

where $\Lambda_\tau^{(3)} := \sum_{d|n,q\nmid d} \varphi(n/d)(v(\tau^d) - v(\tau))$.

3.3. Bounding the order II.

Now suppose

$$\langle \sigma_0 \rangle \subseteq \langle \tau \rangle.$$

Here we have $\mathrm{ord}(\tilde{\tau}) = n/m$, $g_{\tau^{n/m}} = g_{\sigma_0}$, $v(\tau) = f_1$ and hence by the Riemann-Hurwitz formula for $k(\tilde{X}) \,|\, k(\tilde{X})/\langle\tilde{\tau}\rangle$ (or by 3.2.3.1 (ii)) we find

$$n(2g_\tau - 2) + (n-m)v(\tau) + \sum_{d|\frac{n}{m}, d<\frac{n}{m}} \varphi(n/d)(v(\tau^d) - v(\tau)) = m(2g_{\sigma_0} - 2). \tag{8}$$

In particular we have:

$$v(\tau) \leq \frac{2mg_{\sigma_0}}{n-m} + 2.$$

Then by (8) and 2.2 (recall our assumption: $g_{\sigma_0} = 1 \Rightarrow v(\tau) \geq 1$) we obtain

3.3.1. Claim.

(1) If $g_\tau \geq 2$, then

$$n \leq \frac{m(2g_{\sigma_0} - 2 + v(\tau))}{v(\tau) + 2}.$$

(2) If $g_\tau = 1$ and $v(\tau) \geq 1$, then

$$n \leq \frac{m(2g_{\sigma_0} - 2 + v(\tau))}{v(\tau)}.$$

(3) If $g_\tau = 0$ and $v(\tau) \geq 3$, then

$$n \leq \frac{m(2g_{\sigma_0} - 2 + v(\tau))}{v(\tau) - 2}.$$

(4) Suppose that $g_{\sigma_0} \geq 1$. If $g_\tau = 1$ and $v(\tau) = 0$, or $g_\tau = 0$ and $v(\tau) \leq 2$, then

$$n \leq 2(2g_{\sigma_0} + 1)m.$$

3.3.2. Remarks.

Let $\tau \in \mathrm{Aut}(X)$ and set $n := \mathrm{ord}(\tau)$.

(i) Let $g > 2mg_{\sigma_0} + (m-1)^2$. From 3.2.3.1 (i) we have the following criterion for the hypothesis of this section. If $m \mid n$ and $f_1 \geq 1$, then $\langle \sigma_0 \rangle \subseteq \langle \tau \rangle$.

Now suppose $n = m^x q$, with $m \nmid q$. For any g_{σ_0} and $v(\tau) \in \{1, 2\}$ we can use 1.2 (i), (iii) and 3.2.3.1 to obtain more information on n:

(ii) Let $v(\tau) = 2$. Then either $g_{\tau^q} \neq 0$, or all the powers of τ, different from $\tau^{n/m}$, whose order is prime also have two fixed points.

Suppose that $g_{\tau^q} = 0$. Then by 3.2.3.1 (iv) we have

$$\sum_{d \mid n, q \nmid d} \varphi(n/d) v(\tau^d) = 2m^x(q-1),$$

which proves the remark.

(iii) Let $\mathrm{Fix}(\tau) = \{P\}$ (hence $q > 1$ by [Gue] or [F-K, V.2.11]). Suppose that $x \geq 2$ and let $\bar{q}$ be the smallest proper divisor of q, with $\bar{q} = 1$ if q is prime. If $m^{x-1}\bar{q} > 2g_{\tau^{n/m}} + 1$, then $m^{x-1} \in G(P)$.

By the formula for $v(\tau)$ in 3.2.3.1 (ii) and the hypothesis on $\bar{q}$ we have $v(\tau^d) \leq 2$. Then by using the last equation in 3.2.3.1 (iii) we get

$$(m-1)\varphi(q) \leq -2qg_{\tau^m} + 2g_{\tau^{mq}} - 2mg_{\tau^q} + 2mqg_{\tau}.$$

Consequently $g_{\tau^{mq}} \neq 0$ and hence by 1.2 (i), $m^{x-1} = \mathrm{ord}(\tau^{mq}) \in G(P)$.

4. Additional remarks.

Throughout this section X is a curve of genus g and $\sigma_0 \in \mathrm{Aut}(X)$ with $m := \mathrm{ord}(\sigma_0)$ being a prime.

4.1. Suppose that X satisfies $H(m, g_{\sigma_0})$, and let $\tau \in \mathrm{Aut}(X)$ with $\mathrm{ord}(\tau) = m$. Assume that $p \nmid m$. Then either $\tau \in \langle \sigma_0 \rangle$, $f_1 = \#\mathrm{Fix}(\sigma_0) \cap \mathrm{Fix}(\tau) = 0$, or $v(\tau) = mu_1 \leq \frac{2mg_{\sigma_0}}{m-1} + 2m$.

This is an immediate consequence of 1.2 (ii) and 3.2.1.

4.2. Suppose that σ_0 belongs to the center of $\mathrm{Aut}(X)$. Let $\tau \in \mathrm{Aut}(X) \setminus \langle \sigma_0 \rangle$ and set $n := \mathrm{ord}(\tau)$. Assume that $p \nmid mn$. This remark is concerned with the maximum value for $v(\tau)$ in the following cases

I. $m \nmid n$, and II. $\langle \sigma_0 \rangle \subseteq \langle \tau \rangle$.

I. From (5) we have $v(\tau) \leq \frac{2mg_{\sigma_0}}{n-1} + 2m - (m-1)f_1$. From this equation we also have

(1)
$$v(\tau) = \frac{2mg_{\sigma_0}}{n-1} + 2m$$
$$\Leftrightarrow$$
$$\Lambda_\tau = 0, \quad \mathrm{Fix}(\sigma_0) \cap \mathrm{Fix}(\tau) = \emptyset, \quad g_{\tau_1} = 0.$$

(2) If $f_1 \geq 1$, then $v(\tau) \leq \frac{2mg_{\sigma_0}}{n-1} + m + 1$ and
$$v(\tau) = \frac{2mg_{\sigma_0}}{n-1} + m + 1$$
$$\Leftrightarrow$$
$$\Lambda_\tau = 0, \quad f_d = 1 \text{ for } d \mid n, d < n, \quad g_{\tau_1} = 0.$$

II. By 3.3 we have $v(\tau) \leq \frac{2mg_{\sigma_0}}{n-m} + 2$, and
$$v(\tau) = \frac{2mg_{\sigma_0}}{n-m} + 2$$
$$\Leftrightarrow$$
$$g_\tau = 0, \ v(\tau) = v(\tau^d) \text{ for } \ d \mid n, \ d < n, \ d \neq \frac{n}{m}.$$

The equivalence follows from 3.2.3.1 (ii) (recall that $g_{\sigma_0} = g_{\tau^{n/m}}$).

4.3. A theorem of Farkas.

This remark is concerned with Farkas' [F, Thm. 1].

(1) Let $\tau \in \mathrm{Aut}(X)$ such that $v(\tau) > m(2g_{\sigma_0} + 1)$. Then $\tau \in \langle \sigma_0 \rangle$.

In particular, $\mathrm{Fix}(\sigma_0) \subseteq \mathrm{Fix}(\tau)$ and if $\tau \neq 1$, then $\mathrm{ord}(\tau) = \mathrm{ord}(\sigma_0)$.

(2) If $g > 2mg_{\sigma_0} + (m-1)^2$, then $\langle \sigma_0 \rangle$ is normal in $\mathrm{Aut}(X)$. In particular if $m = 2$, σ_0 belongs to the center of $\mathrm{Aut}(X)$.

Proof. If $\tau \notin \langle \sigma_0 \rangle$, then either by 3.2.1 we have $v(\tau) \leq 2m(g_{\sigma_0}+2)$ or by 3.3, $v(\tau) \leq 2g_{\sigma_0} + 2$. This proves (1).

Now we prove (2). By 2.3, k_{σ_0} is the only subfield of $k(X)$ having index m and genus g_{σ_0}. Let $\tau \in \mathrm{Aut}(X)$. Then since $\tau^{-1} \circ \sigma_0 \circ \tau$ also has order m and genus g_{σ_0} we must have $\tau^{-1} \circ \sigma_0 \circ \tau \in \langle \sigma_0 \rangle$ and we are done. ♢

NOTE.

Let $\tau \in \mathrm{Aut}(X) \setminus \langle \sigma_0 \rangle$. The proof of (1) above shows that if $v(\tau) = m(2g_{\sigma_0} + 2)$, then $\langle \sigma_0 \rangle \not\subseteq \langle \tau \rangle$. If in addition $g_{\sigma_0} \geq 1$, then $\mathrm{ord}(\tau) = 2$.

5. Double coverings.

In this section we specialize our results to the case $m = 2$. In what follows $\pi : X \to \tilde{X}$ is a double covering of curves of genus g and γ respectively. We assume

$$g > 4\gamma + 1.$$

Hence there exists a unique involution J_γ belonging to the center of $\mathrm{Aut}(X)$ and such that $\tilde{X} = X/\langle J_\gamma \rangle$ (see 2.3 and 4.3 (2)). This involution will take the place of σ_0 in §3. For $\tau \in \mathrm{Aut}(X)$ we write $n = \mathrm{ord}(\tau)$. We recall that $v(\tau) = 2u_1 + f_1$ where $u_1 \in \mathbb{N}$ and $f_1 = \#\mathrm{Fix}(J_\gamma) \cap \mathrm{Fix}(\tau)$.

Moreover

$$v(\tau) \leq \begin{cases} \dfrac{4\gamma}{\mathrm{ord}(\tau)-1} + 4 & \text{if } \langle J_\gamma \rangle \not\subseteq \langle \tau \rangle \\ \dfrac{4\gamma}{\mathrm{ord}(\tau)-2} + 2 & \text{otherwise} \end{cases}$$

(see 4.2).

To begin with we can reprove well known results on automorphisms of hyperelliptic curves ([Hur], [F-K, Thm. V.2.13]).

5.1. Proposition.

Let X be a hyperelliptic curve of genus $g > 1$. Let $\tau \in \mathrm{Aut}(X) \setminus \langle J_0 \rangle$, set $n := \mathrm{ord}(\tau)$ and suppose $p \nmid 2n$.

(i) If n is odd then $v(\tau^d) = v(\tau)$ for $d \mid n$, and there are three possibilities:

(1) $g + 1 = n(g_\tau + 1)$, $v(T) = 4$, $f_1 = 0$,
(2) $2g + 1 = n(2g_\tau + 1)$, $v(T) = 3$, $f_1 = 1$, or
(3) $g = ng_\tau$, $v(T) = 2$, $f_1 = 2$.

In cases (2) and (3), $X/\langle \tau \rangle$ is hyperelliptic.

(ii) If n be even, then $f_1 \leq 2$ and we have

(1) $f_1 = v(\tau) = 2 \;\Rightarrow\; v(\tau^d) = 2$ for $d \mid n$, $d < n$, $d \neq n/2$.
(2) $f_1 = v(\tau) = 1 \;\Rightarrow\; n = 2q$ with q being an odd.
(3) $f_1 = 0 \;\Rightarrow\; v(\tau) \in \{0, 2, 4\}$,

In cases (1) and (2), $X/\langle \tau \rangle$ has genus 0.

Proof. (i) n odd. Equation (4) becomes

$$2(n-1) = (u_1 + f_1)(n-1) + \Lambda_{\tilde{\tau}} \qquad (*),$$

where $\tilde{\tau}$ is the pushdown of τ to $\tilde{X}$. Thus $u_1 + f_1 \leq 2$.

Claim. $u_1 + f_1 = 2$.

Now (i) is a particular case of 3.2.2.1 (i). The statement on hyperellipticity follows from 1.2 (i).

Proof of the claim. Suppose $u + f_1 = 1$. Then $(*)$ and 1.3 implies $n - 1 = \sum_{d|n, 1<d<n}(n-d)x_d$ for certain $x_d \in \mathbb{N}$. Since $n - d \geq 2n/3$ then $d = 1$, a contradiction.

A similar argument also shows that $u_1 + f_1 = 0$ is impossible.

(ii) n even. If $\langle J_0 \rangle \not\subseteq \langle \tau \rangle$, then $f_1 = 0$ and $v(\tau) = 2u_1 \leq 4$ (see 4.3). Let $\langle J_0 \rangle \subset \langle \tau \rangle$. Then by 3.3 $v(\tau) \leq 2$. Now equation (8) becomes

$$v(\tau) = 2 - \frac{\Lambda_\tau^{(1)}}{n-2}.$$

This implies (1). Now let $v(\tau) = f_1 = 1$ and set $n = 2^x q$ with q odd. If $x \geq 2$ by 3.3.2 (iii) we would have $2^{x-1} \in G(P)$, a contradiction. ♢

REMARK. The examples in [Hur], [Ho] and [F-K] show that all the cases of the proposition occur.

From now on we assume $\gamma > 0$.

5.2. Proposition.

Let $\pi : X \to \tilde{X}$ be a 2-sheeted covering of curves of genus g and γ respectively. Suppose $\gamma > 0$ and $g > 4\gamma + 1$. Let $\tau \in \mathrm{Aut}(X) \setminus \langle J_\gamma \rangle$ such that $v(\tau) \geq 1$ if $\gamma = 1$. Set $n := \mathrm{ord}(\tau)$ and assume $p \nmid 2n$.

(i) Let n be odd.

(1) If $v(\tau^d) = v(\tau)$ for all $d \mid n$, then $n \leq 2\gamma + 1$.

(2) Set $\tau_1 := J_\gamma \circ \tau$. If $v(\tau^d) \neq v(\tau)$ for some $d \mid n$, then

$$n \leq \begin{cases} 3(2\gamma+1)/11 & \text{if } u_1 + f_1 \geq 3 \\ 3\gamma/4 & \text{if } u_1 + f_1 = 2 \\ 3(2\gamma-1)/5 & \text{if } u_1 + f_1 = 1 \\ 3(\gamma-1) & \text{if } u_1 + f_1 = 0. \end{cases}$$

provided $g_{\tau_1} \geq 1$, and

$$n \leq \begin{cases} 3(\gamma+1)/4 & \text{if } u_1 + f_1 \geq 4 \\ 3(2\gamma+1)/5 & \text{if } u_1 + f_1 = 3 \\ 3\gamma & \text{if } u_1 + f_1 = 2 \\ 2(2\gamma+1)-1 & \text{if } u_1 + f_1 \leq 1. \end{cases}$$

otherwise.

(ii) Let n be even.

(1) If $\langle J_\gamma \rangle \not\subseteq \langle \tau \rangle$, then $f_1 = 0$ and

$$n \leq \begin{cases} \gamma+1 & \text{if } v(\tau) \geq 8 \\ 2\gamma & \text{if } v(\tau) = 6 \\ 2(2\gamma+1) & \text{if } v(\tau) \leq 4. \end{cases}$$

(2) If $\langle J_\gamma \rangle \subseteq \langle \tau \rangle$, then

$$n \leq \begin{cases} 2(2\gamma-2+v(\tau))/(v(\tau)+2) & \text{if } g_\tau \geq 2 \\ 2(2\gamma-2+v(\tau))/v(\tau) & \text{if } g_\tau = 1 \text{ and } v(\tau) \geq 1 \\ 2(2\gamma-2+v(\tau))/(v(\tau)-2) & \text{if } g_\tau = 0 \text{ and } v(\tau) \geq 3 \\ 4(2\gamma+1) & \text{if } g_\tau = 1 \text{ and } v(\tau) = 0, \text{or} \\ & \quad g_\tau = 0 \text{ and } v(\tau) \leq 2. \end{cases}$$

Proof. If n is odd (1) follows from 3.2.2.1 while (2) follows from (4) and 2.1 (notice that here $\Lambda_{\tilde{\tau}} \geq 2n/3$, where $\tilde{\tau}$ is the pushdown of τ to $\tilde{X}$). If n is even the bounds follow from 3.2.3 and 3.3. ♢

REMARK. Let τ be as in 5.2 and suppose $f_1 \geq 1$. If $n \geq 2\gamma$, then $X/\langle \tau \rangle$ is hyperelliptic. If n is even and $n \geq 4\gamma$, then $X/\langle \tau \rangle$ has genus zero.

This remark follows from 1.2 (i), (iii).

5.3. Let $\tau \in \mathrm{Aut}(X)$ whose order n is odd and assume $v(\tau) \geq 1$ if $\gamma = 1$. Assume further that $\Lambda_\tau = 0$ and $p \nmid 2n$. We state some remarks on $v(\tau)$ in the case where n is large enough. By §3.2.2.1 this means $n = 2\gamma+1$ or $n = \gamma+1$. Thus $g_{\tau_1} = 0$ ($\tau_1 = J_\gamma \circ \tau$).

(i) $n = 2\gamma + 1$. By the claim of 3.2.2.1 we have the following table.

Case	I	II	III	IV
$v(\tau)$	6	5	4	3
f_1	0	1	2	3

We notice that in Case II from the Riemann-Hurwitz formula for π_τ we have $2g - 4\gamma + 1 = (2g_\tau + 1)(2\gamma + 1)$ and hence for $P \in \mathrm{Fix}(J_\tau) \cap \mathrm{Fix}(\tau)$ we find

$$H(P) = \langle 2m_1, \ldots 2m_\gamma, 4\gamma + 2, 2g - 4\gamma + 1 \rangle,$$

where $m_1, \ldots, m_\gamma = 2\gamma$ are the first γ positive non-gaps at $\pi(P)$ (see 1.2 (i) (iii)).

In Case III we have $2g - 2\gamma + 1 = (2\gamma + 1)(2g_\tau + 1) \in H(P)$ for $P \in \mathrm{Fix}(J_\gamma) \cap \mathrm{Fix}(\tau)$.

(ii) $n = \gamma + 1$. By Remark (ii) of 3.2.2.1 we have $2(\gamma + 1) \in H(P)$ for $P \in \mathrm{Fix}(J_\gamma) \cap \mathrm{Fix}(\tau)$, and the following table.

Case	I	II	III	IV	V
$v(\tau)$	8	7	6	5	4
f_1	0	1	2	3	4

Moreover in Case II, $H(P)$ is as in Case II above; in Case III, $2g - 3\gamma + 1 \in H(P)$; in Case IV, $2g - 2\gamma + 1 \in H(P)$; and in Case V, $2g - \gamma + 1 \in H(P)$.

5.4. Elliptic-hyperelliptic curves.

Let X be a 2-sheeted covering of an elliptic curve $\tilde{X}$. Let $\tau \in \mathrm{Aut}(X) \setminus \langle J_1 \rangle$ with $v(\tau) \geq 1$, and set $n := \mathrm{ord}(\tau)$.

I. n odd. By 5.2 (i) we have $n = 3$ and hence the possibilities for $(v(\tau), f_1)$ are those of the table in 5.3 (i). However, as we will see

in the remark below, the cases $(6,0)$ and $(4,2)$ are not possible. Hence the possibilities are listed below.

Case	1	2
$v(\tau)$	5	3
f_1	1	3

II. n even. By 5.2 (ii) $n \leq 12$. Moreover $n \neq 10$ because X does not admit automorphisms of order 5 fixing a point.

II.1 $\langle J_1 \rangle \not\subseteq \langle \tau \rangle$. Here $f_1 = 0$, $n \leq 6$ and $2 \leq v(\tau) = 2u_1 \leq 4 + 4/(n-1)$. If $n = 2$ then $v(\tau) \in \{4, 8\}$. The case $n = 6$ is not possible. All these statements will be proved in the remark below.

II.2 $\langle J_1 \rangle \subseteq \langle \tau \rangle$. Here $f_1 = v(\tau)$, $n \leq 12$ and we have $g_\tau = 0$ and $1 \leq v(\tau) \leq 4$. Equation (8) becomes

$$(n-2)v(\tau) + \Lambda_\tau^{(1)} = 2n.$$

Then

(1) $n = 4 \Leftrightarrow v(\tau) = 4$;

(2) $n = 6 \Rightarrow v(\tau) \in \{1, 3\}$. In fact, we have $v(\tau) + v(\tau^2) = 6$. By the odd case $v(\tau^2) \in \{5, 3\}$ and hence the result. Conversely $v(\tau) = 3$ gives $(n-6) + \Lambda_\tau^{(1)} = 0$ and hence $n = 6$;

(3) $n = 8 \Leftrightarrow v(\tau) = 2$. If $n = 8$, $2v(\tau) + v(\tau^2) = 8$. By 3.2.3.1 (iii), $v(\tau^2) = 4 - 4g_{\tau^2}$. Since this number is positive then we must have $v(\tau^2) = 4$ and thus the result. The implication "$\Leftarrow$" follows from the other cases.

(4) $n = 12 \Rightarrow v(\tau) = 1$. Here we find $2v(\tau) + v(\tau^2) + v(\tau^3) + v(\tau^4) = 12$ $(*)$ and $v(\tau^3) = 4 - 4g_{\tau^3}$. If this number is 0 then $v(\tau^2) + v(\tau^4) = 12$. But since $v(\tau^2) \leq 4$, $v(\tau^4) \leq 5$ this is a contradiction. Hence $(*)$ becomes $2v(\tau) + v(\tau^2) + v(\tau^4) = 8$. By the case $n = 6$ we have $v(\tau^2) + v(\tau^4) = 6$ and thus the result.

We now summarize this discussion in a table.

Case	3	4	5	6	7	8
n	2	4	4	6	8	12
$v(\tau)$	$\{4, 8\}$	$\{2, 4\}$	4	{1,3}	2	1
f_1	0	0	4	{1,3}	2	1

REMARKS.

(i) Let $\tau \in \mathrm{Aut}(X) \setminus \langle J_1 \rangle$ with $n = \mathrm{ord}(\tau) = 3$. We show that $(v(\tau), f_1) \notin \{(6,0),(4,2)\}$. Let $P_0 \in \mathrm{Fix}(J_1)$. Since $2 \in H(\pi(P_0))$ we have $4 \in H(P)$. Let $x \in k(X)$ such that $\mathrm{div}_\infty(x) = 4P_0$. Then by Castelnuovo's genus bound (Lemma 1.1) we have

$$k(x) \subseteq k(\tilde{X}).$$

Let r (resp. s) be the number of fixed points of J_1 which are (resp. not) totally ramified for $k(X) \,|\, k(x)$. Let $2t$ be the number of points $P \in X \setminus \mathrm{Fix}(J_1)$ such that $\pi(P)$ is totally ramified for $k(\tilde{X}) \,|\, k(x)$.

CLAIM. s is even, $r + s = 2g - 2$ and $r + t = 4$.

Proof. By [M-P, 7.5], $k(X) \,|\, k(x)$ is a Galois extension and then s is even and $r + s$ is the number of fixed points of J_1 which is $2g - 2$. Now by the Riemann-Hurwitz formula we have

$$2g - 2 = 4(-2) + 3r + s + 2t,$$

from where it follows that $r + t = 4$. ♢

Let $B_\pi := \pi(\mathrm{Fix}(J_1))$. Since τ commute with J_1 we have $\tilde{\tau}(B_\pi) = B_\pi$. In particular, the claim above shows that at least $f_1 \geq 1$. This eliminates the case $(v(\tau), f_1) = (6,0)$. Now suppose $(v(\tau), f_1) = (4,2)$. Thus $g \equiv 2 \pmod 3$ $(*)$. Moreover, we can assume that P_0 is fixed by τ or the image under π of its two fixed points not fixed by J_1 is a totally ramified for $k(\tilde{X}) \,|\, k(x)$. In the first case there exists $Q \in \mathrm{Fix}(\tau)$, such that $\pi(Q)$ is a point of ramification 2 for $k(\tilde{X}) \,|\, k(x)$. Let $x^{-1}(x(Q)) = \{Q, Q_1\}$. Then $\tilde{\tau}(\pi(Q_1)) = \pi(Q_1)$ and, since $\gcd(2,3) = 1$, we have $\tau(Q_1) = Q_1$. This implies $f_1 \geq 3$. In the second case $2g - 6 \equiv 0 \pmod 3$. This is impossible under $(*)$ and we also eliminated $(v(\tau), f_1) = (4,2)$.

(ii) Here we illustrate Case 1 and Case 6: $n = 6$, $v(\tau) = f_1 = 1$.

Let g be a multiple of 3, $q(x)$ a separable polynomial over k such that $q(0) \neq 0$, $\deg(q(x)) = (g-3)/3$, and let $p(x) := 4x^3 - A$,

$A \in k^*$. Suppose that $\gcd(p(x), q(x))) = 1$. Now consider the curve X whose plane model is given by

$$z^4 = p(x)(q(x^3))^2.$$

Clearly X is a double covering of the curve $y^2 = P(x)$ which is an elliptic curve. By Riemann-Hurwitz X has genus g and the unique point P_∞ over $x = \infty$ is a fixed point of J_1. Moreover, the four points over $x = 0$ are not fixed by J_1. Let ϵ be a 3-root of unity. The automorphism τ given by

$$(x, z) \longmapsto (\epsilon x, z),$$

has five fixed points, namely P_∞ and the points over $x = 0$. Considering $J_1 \circ \tau$ we illustrate the case with $n = 6$.

(iii) Here we illustrate Case 2 and Case 6: $n = 6$, $v(\tau) = f_1 = 3$.

Let $g \equiv 1 \pmod 3$, $q(x)$ a separable polynomial over k such that $q(0) \neq 0$, $\deg(q(x) = (g-4)/3$ and $p(x) = 4x^3 - A$, $A \in k^*$. Suppose that $\gcd(p(x), q(x)) = 1$. Let X be the curve given by

$$z^4 = p(x)(q(x^3))^2 x^2.$$

Then X is a double covering of $y^2 = p(x)$; the unique point P_∞ over $x = \infty$ and the two points over $x = 0$ are fixed by J_1. Let ϵ be a 3-root of unity and τ the automorphisms defined by

$$(x, z) \longmapsto (\epsilon^2 x, \epsilon z).$$

Then the fixed points of τ are P_∞ and the two ones over $x = 0$. By considering $J_1 \circ \tau$ we illustrate the case where $n = 6$.

(iv) Let n even and suppose that $\langle J_1 \rangle \not\subseteq \langle \tau \rangle$. Thus $n \in \{2, 4, 6\}$. We prove that $n = 2 \Rightarrow v(\tau) \in \{4, 8\}$ and $n = 6$ is not possible. Let $n = 2$. Hence $g = 2g_\tau - 1 + v(\tau)/2$. Since $\mathrm{ord}(\tilde{\tau}) = 2$ from the proof of the claim in (i), $g - 3$ must be an even number. This eliminates $v(\tau) \in \{2, 6\}$. Now let $n = 6$. Riemann-Hurwitz gives $g - 3 = 6g_\tau - 8 + v(\tau) + v(\tau^2) + v(\tau^3)/2$. By the cases $n = 2$ and $n = 3$, $g - 3$ is odd. This eliminates $n = 6$.

Next we show that our results are sharp. Let $g \in \mathbb{Z}^+$ such that $g \equiv 3 \pmod 4$. Let $p(x) = x^4 - A$ irreducible in $k[x]$, $q(x)$ a separable polynomial of degree $(g-3)/4$ such that $\gcd(p(x), q(x)) = 1$. Let X be the curve given by

$$z^4 = p(x)(q(x^4))^2.$$

Then X is a double covering of the curve $y^2 = p(x)$ and the four points over $x = \infty$ are not fixed by J_1. X admits the automorphisms $\tau : (x, z) \longmapsto (\epsilon x, z)$ and τ^2, where ϵ is a 4-root of unity.

If $q(0) \neq 0$ then τ^2 has eight fixed points and τ has two fixed points. If $q(0) = 0$, $v(\tau^2) = v(\tau) = 4$.

(v) Finally we consider $\langle J_1 \rangle \subseteq \langle \tau \rangle$. We can assume $n \in \{4, 8, 12\}$. Case 5 is illustrated by the automorphism $(x, z) \to (x, \epsilon z)$, ϵ a 4-root of unity, defined on any of the above curves. Now we illustrate Case 7. Let $g \in \mathbb{Z}^+$ such that $g \equiv 4 \pmod 4$. Let $p(x)$, $q(x)$ be as in (iv) except that $\deg(q(x)) = (g-4)/4$ and $q(0) \neq 0$. The curve

$$z^4 = p(x)(xq(x^4))^2,$$

is a double covering of $y^2 = p(x)$ and admits the automorphism $(x, z) \mapsto (\epsilon^2 x, \epsilon z)$, where ϵ is a 8-root of unity. It fixs the two points over $x = \infty$.

Now we consider Case 8. The curve in (ii) admits of the automorphism $(x, z) \mapsto (\epsilon^4 x, \epsilon^3 z)$, where ϵ is a 12-root of unity. Its unique fixed point is the only one over $x = \infty$.

(vi) Suppose that X admits of an automorphism as listed in Cases 1-8 and let x be as in (i). Then, since $k(X) \,|\, k(x)$ is Galois [M-P, 7.5], X is defined by a model plane as in the examples above. To see this one uses the well known group structure of automorphisms fixing a point on elliptic curves (see e.g. Silverman [Sil, Thm. 10.1]) and Kato's [K1, §6] or Garcia's [G, Lemma 7].

5.5. Certain double coverings of hyperelliptic curves.

Here we generalize the previous example. We consider curves X such that there exists $P_0 \in X$ with $4 \in H(P_0)$ and such that X is a double covering of a curve $\tilde{X}$ of genus $\gamma \geq 2$. Let g be the genus of X, J_γ an involution such that $X/\langle J_\gamma \rangle = \tilde{X}$, and let $x \in k(X)$ such that $\mathrm{div}_\infty(x) = 4P_0$. Then by Castelnuovo's inequality (1.1) and 1.2 (i) we obtain

5.5.1. Claim.

If $g > 2\gamma + 3$, then
(1) $P_0 \in \mathrm{Fix}(J_\gamma)$.
(2) $\gamma = \{\ell \in G(P_0) : \ell \text{ even}\}$.

In particular $\tilde{X}$ is hyperelliptic. In what follows we assume $g > 4\gamma + 1$. Let $\pi : X \to \tilde{X}$, r (resp. s) the number of fixed points of J_γ which are (resp. not) totally ramified in $k(X) \mid k(x)$ and let $2t$ be the number of points $P \in X \setminus \mathrm{Fix}(J_\gamma)$ such that $\pi(P)$ is totally ramified for $k(\tilde{X}) \mid k(x)$. By [M-P, 7.5], $k(X) \mid k(x)$ is Galois (here it is enough to assume $g \geq 3\gamma$). Hence as in the claim of 5.4 Remark (i), we obtain

Claim. s is even, $r + s = 2g - 4\gamma + 2$ and $r + t = 2\gamma + 2$.

Next we only consider automorphisms τ such that $n = \mathrm{ord}(\tau)$ is prime and equals to either $2\gamma + 1$ or $\gamma + 1$.

Case $n = 2\gamma + 1$.

As in Remark (i) of 5.1 here we also eliminate the cases I and III of 5.3 (i). We illustrate the remaining cases.

Let $g \in \mathbb{Z}^+$ such that $g \equiv -2 \pmod{(2\gamma + 1)}$. Let $q(x)$ be a separable polynomial over k such that $q(0) \neq 0$ and $\deg(q(x)) = (g - 3\gamma)/(2\gamma + 1)$. Let $p(x) := x^{2\gamma+1} - A$ be an irreducible polynomial over $k[x]$ such that $\gcd(p(x), q(x)) = 1$. Consider the curve

$$z^4 = p(x)(g(x^{2\gamma+1}))^2.$$

Then the unique point P_0 over $x = \infty$ satisfies $4 \in H(P)$; X is a

double covering over the hyperelliptic curve $y^2 = p(x)$; the automorphism

$$(x, z) \longmapsto (\epsilon x, z),$$

where ϵ is a $(2\gamma + 1)$-root of unity, has five fixed points namely P_0 and the four ones over $x = 0$. This illustrate case II of 5.3 (i).

Now take $g \in \mathbb{Z}^+$ such that $g \equiv \gamma \pmod{(2\gamma + 1)}$, $q(x)$, $p(x)$ as above but with $\deg(q(x)) = (g - 3\gamma - 1)/(2\gamma + 1)$. Then the curve given by

$$z^4 = p(x)(q(x^{2\gamma+1}))^2 x^2,$$

is a double covering of $y^2 = p(x)$, $4 \in H(P_0)$ where P_0 is the unique point over $x = \infty$. The automorphism

$$(x, z) \longmapsto (\epsilon^2 x, \epsilon z),$$

where ϵ is a $(2\gamma+1)$-root of unity, has three fixed points: P_0 and the two ones over $x = 0$. This illustrate Case IV of 5.3 (i).

CASE $n = \gamma + 1$.

As in the previous examples here one can show that Cases I, II and IV of 5.3 (ii) are not possible. The remaining cases are illustrated below.

Let x be a transcendental element over k, $g \in \mathbb{Z}^+$ such that $g \equiv -2 \pmod{(\gamma + 1)}$. Let $q(x)$, $p(x)$ be separable polynomials over $k[x]$ such that $\deg(q(x)) = (g - 3\gamma - 1)/(\gamma + 1)$, $\deg(p(x)) = 2$, $q(0) \neq 0$, $p(0) \neq 0$ and $\gcd(p(x), q(x)) = 1$. Consider the curve X defined by

$$z^4 = p(x^{\gamma+1})(q(x^{\gamma+1})x)^2.$$

Then X is a double covering of $y^2 = p(x^{\gamma+1})$; the unique point P_0 over $x = a$, for $a \in k$ a root of $p(x)$, satisfies $4 \in H(P_0)$, and it has two or four points over $x = \infty$ according as $g - 2\gamma + 1$ is odd or even. Let ϵ be a $(\gamma + 1)$-root of unity and consider the automorphism of X given by

$$(x, z) \longmapsto (\epsilon^2 x, \epsilon z).$$

Its fixed points are those over $x = 0$ or $x = \infty$. This illustrate cases III and V of 5.3 (ii).

REMARK. Let X be a hyperelliptic curve admitting of an automorphism τ of order n. Then Hurwitz [Hur] showed that X and τ can be defined by $y^2 = f(x^n)$ and $\tau : (x, y) \mapsto (\epsilon x, \pm y)$, or by $y^2 = xf(x^n)$ and $\tau : (x, y) \mapsto (\epsilon x, \epsilon^{1/2} y)$, where ϵ is a n-root of unity.

Let x be as in the above examples. Using the fact that $K(X) \mid k(x)$ is Galois, the mentioned Hurwitz's results and Komeda's [Ko, §4], one can show that the curves of this section admitting of an automorphism satisfying Cases III and V of 5.3 (ii), can be defined by a model plane as those of the examples stated above.

5.6. Certain double coverings of trigonal curves.

To finish this paper, let consider a curve X admitting of a point P_0 such that $6 \in H(P_0)$ and such that X is a double covering of a curve $\tilde{X}$ of genus γ. Let J_γ be an involution such that $X/\langle J_\gamma \rangle = \tilde{X}$ and let $x \in k(X)$ such that $\mathrm{div}_\infty(x) = 6P_0$. As in 5.5.1 we have that $P_0 \in \mathrm{Fix}(J_\gamma)$ and γ is the genus of $\tilde{X}$, provided $g > 2\gamma + 5$. In particular $\tilde{X}$ is trigonal. We further assume

$$g \geq \max(4\gamma + 2, 2\gamma + \rho),$$

where $\rho := \{\ell \in G(P_0) : \ell \equiv 0 \pmod 3\}$. Under this condition it follows from [M-P, Thm 7.1] that $k(X) \mid k(x)$ is a Galois extension. Then, it is not difficult to see that $H(P_0) = \langle 6, 2g - 4\gamma + 1 + 2i_2 + 4i_4 + 4i_5, 4\gamma + 4 - 2i_1 - 2i_4, 2g - 4\gamma + 1 + 2i_1 + 2i_5, 4\gamma + 4 - 2i_2 - 2i_5, 2g - 4\gamma + 1 + 4i_1 + 4i_2 + 2i_4 \rangle$, where the $i'_j s$ are non-negative integers satisfying $i_1 + i_2 + i_4 + i_5 = \gamma + 1$, $i_1 + i_3 + i_5 = 2r + 1$ and $i_1 + 2i_2 + i_4 + 2i_5 \not\equiv 0 \pmod 3$. Moreover, X admits a model plane of type

$$z^6 = \prod_{j=1}^{5} \prod_{i=1}^{i_j} (x - a_{ij})^j,$$

where the $a'_{ij} s$ are pairwise different elements of k.

From this fact one can prove results similar to 5.4 and 5.5. We leave these to the reader.

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