MORE ON MILD CONTINUITY (*)

by Julian Dontchev (in Helsinki)
and Maximilian Ganster (in Graz)(**)

SOMMARIO. - *Lo scopo di questo articolo è quello di presentare alcuni nuovi risultati sulla continuità "mild" come pure quello di dare una chiara decomposizione di continuità via la continuità "mild" cioè senza alcuna condizione sul dominio nè sul rango. Viene inoltre introdotta la nozione di precontinuità estrema. Tra i molti risultati, si prova che una funzione è α-continua se e solo se essa è continua in modo "mild" ed estremamente precontinua.*

SUMMARY. - *The aim of this paper is to present some new results on mild continuity as well as to give a pure decomposition of continuity via mild continuity that is without any assumptions on the domain and the range. The notion of extremal precontinuity is introduced. Among several results we prove that a function is α-continuous if and only if it is mildly continuous and extremally precontinuous.*

1. Introduction.

The decomposition of continuity is a classical problem of Real Analysis.

In 1922, Blumberg [4] introduced the concept of near continuity on Euclidean spaces by using the term *densely approaching* and proved that every function $f: \mathbf{R} \to \mathbf{R}$ is nearly continuous on a dense set of $\mathbf{R}$. Nowadays, near continuity is better known in the topological community as precontinuity. It is well-known that every linear function from one Banach space to another is nearly continuous. Nearly continuous function are of importance in Functional Analysis in connection with the well-known closed graph and open mapping theorems.

Quasi-continuity is even an older concept than the one of near continuity. In 1899, Baire stated that first Volterra observed the fact that every separately continuous function $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is quasi-continuous. It was Kempisty [18], who later on called this property of separately continuous functions quasi-continuity. It is well-known that near continuity and quasi-continuity imply continuity if the range is regular.

In 1954, Klee and Utz [19] proved that a function $f: \mathbf{R} \to \mathbf{R}$ is continuous if and only if f preserves compact sets and f preserves connected sets. Originally they proved the result in the settings of metric spaces with the domain locally connected.

In 1961, Levine [20] proved that a function is continuous if and only if it is weakly continuous and weak-* continuous. His result was improved in 1978 by Rose [28], who replaced weak-* continuity in Levine's theorem with a weaker form of continuity named local weak-* continuity. In 1990, weak continuity was reduced to weak α-continuity again by Rose [29].

In 1985, Reilly and Vamanamurthy [27] proved that a function is α-continuous if and only if it is semi-continuous and precontinuous. Note that quasi-continuity is same as semi-continuity.

In 1986, Tong [30] produced an independent of Levine's decomposition. He showed that a function is continuous if and only if it is α-continuous and $\mathcal{A}$-continuous. Three years later Tong [32] decomposed continuity into $\mathcal{B}$-continuity and precontinuity but his result was improved in 1991 by Ganster, Gressl and Reilly in [13] as $\mathcal{B}$-continuity was replaced with weak $\mathcal{B}$-continuity.

Some years later, Ganster and Reilly [11] improved Tong's first decomposition with reducing $\mathcal{A}$-continuity to LC-continuity and furthermore LC-continuity was reduced to sub-LC-continuity in [10]. In the 1990 paper mentioned above, Ganster and Reilly decomposed $\mathcal{A}$-continuity.

In 1993, Przemski [25] obtained some new decompositions of continuity as well as of α-continuity via some newly defined classes of sets. Several new result can be found in [2, 9, 25].

The theory of the decomposition of continuity was investigated recently by Yalvaç in [33]. In 1987, Tong [31] obtained a decomposition of fuzzy continuity. A decomposition of pairwise continuity was constructed by Jelić in [16, 17]. The decomposition of quasi-continuity was studied in [6] by Borsìk and J. Doboš. For more historical background on the problem of decomposition of continuity the reader may refer to [23].

In this note we give some new results related to mild continuity as well as we search for the dual of mild continuity to α-continuity and furthermore to continuity.

2. Mild continuity.

DEFINITION 1. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called *almost continuous* [14] ($=$ precontinuous [21] $=$ nearly continuous [26]) at a point $x \in X$ if for each neighborhood V of $f(x)$, the set $\mathrm{Cl}f^{-1}(V)$ is a neighborhood of x. If the function f is almost continuous at every $x \in X$, then it is called *almost continuous*. Since the term precontinuous is most often used in literature, throughout the sequel we will call almost continuous functions *precontinuous*.

DEFINITION 2. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called *quasi-continuous* [18] at a point $x \in X$ if for each neighborhood U of x and each neighborhood V of $f(x)$ there exists a non-empty open set $G \subseteq U$ such that $f(G) \subseteq V$. If the function f is quasi-continuous at every $x \in X$, then it is called *quasi-continuous*. Quasi-continuous functions are well-known under the name of *semi-continuous*.

For a subset A of (X, τ), the preinterior of A and the semi-interior of A are defined as follows: $\mathrm{Pint}A = A \cap \mathrm{Int}\overline{A}$ and $\mathrm{Sint}A = A \cap \overline{\mathrm{Int}A}$,

respectively.

For a function $f\colon (X, \tau) \to (Y, \sigma)$ let us define the following subsets of X:

P_f - the set of all points of precontinuity (= almost continuity),

Q_f - the set of all points of quasi-continuity (= semi-continuity),

C_f - the set of all points of continuity.

Clearly $C_f \subseteq P_f \cap Q_f$. In addition one easily checks that $x \in P_f$ if and only if $x \in \mathrm{Pint} f^{-1}(V)$ for each neighborhood V of $f(x)$, and $x \in Q_f$ if and only if $x \in \mathrm{Sint} f^{-1}(V)$ for each neighborhood V of $f(x)$.

DEFINITION 3. [5, Definition 5.] Let $f\colon (X, \tau) \to (Y, \sigma)$. Set $S_f = \{x \in X\colon$ there is a base $\mathcal{A}$ of neighborhoods of $f(x)$ such that for every $V \in \mathcal{A}$ and for every neighborhood U of x the set $f^{-1}(V) \setminus \mathrm{Int} f^{-1}(V)$ is not dense in $U\}$. If the set S_f is dense in X, then f is called *mildly continuous*.

REMARK 2.1. One easily verifies that $x \in S_f$ if and only if there exists a base $\mathcal{A}$ of neighborhoods of $f(x)$ such that

$$x \notin \mathrm{Int}\,\overline{f^{-1}(V) \setminus \mathrm{Int} f^{-1}(V)} \quad \text{for every} \quad V \in \mathcal{A}.$$

In order to obtain an alternative description of the set S_f, we first prove the following:

LEMMA 2.2. *Let A be a subset of (X, τ) and let $x \in A$. Then the following conditions are equivalent:*

(1) $x \in \mathrm{Int}(\overline{A \setminus \mathrm{Int} A})$.

(2) $x \in \mathrm{Pint} A \setminus \mathrm{Sint} A$.

Proof. (1) $\Rightarrow$ (2) Let $x \in \mathrm{Int}(\overline{A \setminus \mathrm{Int} A})$. Then $x \in \mathrm{Int}\overline{A}$ and thus $x \in \mathrm{Pint} A$. Since $x \in \mathrm{Int}(X \setminus \mathrm{Int} A) = X \setminus \overline{\mathrm{Int} A}$, then we have $x \notin \mathrm{Sint} A$.

(2) $\Rightarrow$ (1) Let $x \in \mathrm{Pint} A \setminus \mathrm{Sint} A$. Then there exists an open neighborhood U of x such that $U \subseteq \overline{A}$ and $U \cap \mathrm{Int} A = \emptyset$. Next we show that $U \subseteq \overline{A \setminus \mathrm{Int} A}$. Let $x \in U$. Assume that for some open V containing x we have $V \cap (A \setminus \mathrm{Int} A) = \emptyset$. Then $W = U \cap V$ is an open neighborhood of x disjoint from $A \setminus \mathrm{Int} A$ such that $W \subseteq$

$\overline{A}$. Then clearly W must be a subset of $\mathrm{Int}\,A$, which is impossible since even U is disjoint from $\mathrm{Int}\,A$. Hence $U \subseteq \overline{A \setminus \mathrm{Int}\,A}$ and thus $x \in \mathrm{Int}(\overline{A \setminus \mathrm{Int}\,A})$. $\Diamond$

THEOREM 2.3. *Let* $f\colon (X,\tau) \to (Y,\sigma)$ *be a function. Then* $S_f = (X \setminus P_f) \cup Q_f$.

Proof. Let $x \in S_f$ and, according to Remark 2.1, let $\mathcal{A}$ be a base of neighborhoods of $f(x)$ such that $x \notin \mathrm{Int}\overline{f^{-1}(V) \setminus \mathrm{Int}f^{-1}(V)}$ for each $V \in \mathcal{A}$. Suppose that $x \in P_f$ and let W be a neighborhood of $f(x)$. Choose $V \in \mathcal{A}$ with $V \subseteq W$. Since $x \in \mathrm{Pint}f^{-1}(V)$, by Lemma 2.2 we have $x \in \mathrm{Sint}f^{-1}(V) \subseteq \mathrm{Sint}f^{-1}(W)$, i.e. $x \in Q_f$. We have thus shown that $S_f \subseteq (X \setminus P_f) \cup Q_f$.

Now let $x \notin S_f$, i.e. for each base $\mathcal{A}$ of neighborhoods of $f(x)$ there exists $V \in \mathcal{A}$ with $x \in \mathrm{Int}\overline{f^{-1}(V) \setminus \mathrm{Int}f^{-1}(V)}$. If W is a neighborhood of $f(x)$, let $\mathcal{A} = \{V \subseteq Y\colon V \text{ is a neighborhood of } f(x) \text{ and } V \subseteq W\}$. Hence there exists $V \in \mathcal{A}$ with $x \in \mathrm{Pint}f^{-1}(V) \subseteq \mathrm{Pint}f^{-1}(W)$, by Lemma 2.2. This shows that $x \in P_f$. Since $x \notin \mathrm{Sint}f^{-1}(V)$ we have $x \notin Q_f$ by Lemma 2.2. Thus $S_f = (X \setminus P_f) \cup Q_f$. $\Diamond$

Thus Lemma 1 $(Q_f \subseteq S_f)$ and Lemma 2 $(S_f \subseteq Q_f$, if f is precontinuous) in [5] are obvious.

COROLLARY 2.4. *Every quasi-continuous function is mildly continuous.*

REMARK 2.5. See Example 2.9 below. The function there is mildly continuous but not quasi-continuous.

COROLLARY 2.6. *If* $P_f \subseteq Q_f$, *then* $S_f = X$.

Recall [10] that a function $f\colon (X,\tau) \to (Y,\sigma)$ is called *sub-LC-continuous* if there is a subbase (or equivalently a base) $\mathcal{B}$ for (Y,σ) such that $f^{-1}(V)$ is locally closed in (X,τ) for each $V \in \mathcal{B}$. A set A is called *locally closed* [7] if A can be represented as the intersection of an open and a closed set.

Sub-LC-continuity plays an important role in the theory of de-

composition of continuity: it is the dual of precontinuity, that is:

THEOREM 2.7. [10, Theorem 3] *A function $f: (X, \tau) \to (Y, \sigma)$ is continuous if and only if it is precontinuous and sub-LC-continuous.*

For the different properties of LC- and sub-LC-continuous functions the reader may check [10].

Next we show how sub-LC-continuity is related to mild continuity.

COROLLARY 2.8. *Every sub-LC-continuous function is mildly continuous.*

Proof. Assume that $f: (X, \tau) \to (Y, \sigma)$ is sub-LC-continuous. Then $P_f \subseteq C_f \subseteq Q_f$ and so it follows from Theorem 2.3 that $S_f = X$. This shows that f is mildly continuous. $\Diamond$

EXAMPLE 2.9. Not every mildly continuous function is sub-LC-continuous. Let X be the real line and let $\mathcal{I}$ denote the set of all irrational numbers. Set $\tau = \{\emptyset, X\}$ and $\sigma = \{\emptyset, \mathcal{I}, X\}$. Let $f: (X, \tau) \to (X, \sigma)$ be the identity function. Note that (for example) the origin of (X, τ) is a point of quasi-continuity. By Theorem 2.3, $0 \in S_f$. Thus S_f is dense in (X, τ) or equivalently f is mildly continuous. To see that f is not sub-LC-continuous, in the notion of Theorem 2.7, it is enough to check that f is precontinuous but not continuous.

COROLLARY 2.10. *If $f: (X, \tau) \to (Y, \sigma)$ has closed graph and Y is locally compact, then f is mildly continuous.*

Proof. Since the inverse image of each compact subset of Y is closed in X, f is sub-LC-continuous and thus mildly continuous. $\Diamond$

3. Extremal precontinuity.

In 1988, Borsìk and Doboš decomposed continuity by assuming regularity of the range of the function. They proved the following:

THEOREM 3.1. [5, Theorem 2] *Let (Y, σ) be regular space. Then $f: (X, \tau) \to (Y, \sigma)$ is continuous if and only if it is precontinuous and mildly continuous.*

Example 2.9 above shows that regularity cannot be removed as an assumption. The function from Example 2.9 is both precontinuous and mildly continuous. However it fails to be continuous and Y is not regular.

In what follows, we will try to produce a more general result, i.e. we will try to decompose continuity via mild continuity without any assumptions on the domain and the range. For that, we need the following definition:

DEFINITION 4. A function $f: (X, \tau) \to (Y, \sigma)$ is called *extremally precontinuous* (= e-precontinuous) if f is precontinuous and S_f is a closed subset of (X, τ).

E-precontinuous functions need not be mildly continuous as the following example shows:

EXAMPLE 3.2. Consider the classical Dirichlet function $f: \mathbf{R} \to \mathbf{R}$, where $\mathbf{R}$ is real line with the usual topology:

$$f(x) = \begin{cases} 1, & x \in \mathbf{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily observed that f is precontinuous. Moreover S_f coincides with the void set and hence f is e-precontinuous but not mildly continuous.

On the other hand Example 2.9 provides a mildly continuous function, which is not e-precontinuous; note that there S_f coincides with $\mathbf{Q}$ - the set of all rationals, which obviously is not τ-closed.

To see where e-precontinuity stands among the other types of generalized continuity, recall that a function $f: (X, \tau) \to (Y, \sigma)$ is called α-*continuous* [24] if the preimage of every open subset of Y is an α-set in X. A set A is called an α-set [24] if $A \subseteq \text{Int}\overline{\text{Int}A}$ or equivalently if $A = U \backslash V$, where U is open and V is nowhere dense. It is well-known that the α-sets form a topology finer than the original

one and thus continuity always implies α-continuity. The following decomposition of α-continuity, which will be used later in the sequel, is probably known:

THEOREM 3.3. [27, Corollary 1] *A function* $f\colon (X,\tau) \to (Y,\sigma)$ *is* α-*continuous if and only if it is precontinuous and quasi-continuous.*

THEOREM 3.4. *Every* α-*continuous function is e-precontinuous.*

Proof. Every α-continuous function is precontinuous. For the second part, in the notion of Theorem 2.3 and Theorem 3.3 we have: $S_f = (X \setminus P_f) \cup Q_f = \emptyset \cup X = X$. $\diamondsuit$

By definition every e-precontinuous function is precontinuous. But the reverse is not always true as easily seen from Example 2.9.

THEOREM 3.5. *For a function* $f\colon (X,\tau) \to (Y,\sigma)$ *the following conditions are equivalent:*
(1) f *is* α-*continuous.*
(2) f *is quasi-continuous and e-precontinuous.*
(3) f *is mildly continuous and e-precontinuous.*

Proof. (1) $\Rightarrow$ (2) Every α-continuous function is quasi-continuous [27]. The second part was proved in Theorem 3.4 above.
(2) $\Rightarrow$ (3) is Corollary 2.4.
(3) $\Rightarrow$ (1) In the notion of Theorem 3.3 it is enough to show that f is quasi-continuous. By assumption $P_f = X$ and $\overline{S_f} = S_f$. Thus from Theorem 2.3 and due to mild continuity we have: $X = \overline{S_f} = S_f = (X \setminus P_f) \cup Q_f = \emptyset \cup Q_f = Q_f$. $\diamondsuit$

In the decomposition above e-precontinuity cannot be reduced to precontinuity. Recall again Example 2.9. Also, it is easy to find an example of a function, which is both mildly continuous and e-precontinuous but failing to be continuous. Consider for example the identity function $f\colon (X,\tau) \to (Y,\sigma)$, where $X = Y = \{a,b,c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{a,b\}, Y\}$.

In 1993, Przemski [25] introduced the concept of $D(c,\alpha)$-continuity and proved that a function is continuous if and only if it is

α-continuous and $D(c, \alpha)$-continuous.

THEOREM 3.6. *For a function $f: (X, \tau) \to (Y, \sigma)$ the following conditions are equivalent:*
(1) *f is continuous.*
(2) *f is mildly continuous, e-precontinuous and $D(c, \alpha)$-continuous.*

A *nodec space* is a topological space in which all nowhere dense subsets are closed. It can be easily proved that all nowhere dense sets are closed if and only if every α-set is open. Nodec spaces need not always satisfy high separation axioms. However every nodec space is semi-pre-$T_{\frac{1}{2}}$ [8]. Now we have:

THEOREM 3.7. *Let (X, τ) be nodec. For a function $f: (X, \tau) \to (Y, \sigma)$ the following conditions are equivalent:*
(1) *f is continuous.*
(2) *f is mildly continuous and e-precontinuous.*

A natural problem is to find a property of functions, weaker than quasi-continuity, which together with mild continuity would imply quasi-continuity. For that, consider the following definition:

DEFINITION 5. Let $f: (X, \tau) \to (Y, \sigma)$ be a function and set $PQ_f = P_f \cup Q_f = \{x \in X : x \in P_f$ or $x \in Q_f\}$, i.e. PQ_f contains all points of X in which f is precontinuous or quasi-continuous. If $PQ_f = X$ and S_f is a closed subset of X, then we say that f is *extremally β-continuous* ($=$ e-β-continuous).

THEOREM 3.8. *For a function $f: (X, \tau) \to (Y, \sigma)$ the following conditions are equivalent:*
(1) *f is quasi-continuous.*
(2) *f is mildly continuous and e-β-continuous.*

Proof. (1) $\Rightarrow$ (2) It is proved above that quasi-continuity implies mild continuity. Moreover, since f is quasi-continuous, then the set S_f is closed. It coincides with X. Since every point of X belongs to Q_f, then f is e-β-continuous.
(2) $\Rightarrow$ (1) Since S_f is closed and dense in X, then $S_f = X = (X \setminus$

$P_f) \cup Q_f$. If we assume that some point x of X is not a point of quasi-continuity, then it would not be a point of precontinuity. This obviously contradicts with the assumption that f is e-β-continuous and hence f is quasi-continuous. $\Diamond$

REMARK 3.9. The Dirichlet function shows that an e-β-continuous function need not be quasi-continuous. On the other hand the function from Example 2.9 shows that β-continuity is strictly weaker than e-β-continuity. Recall that a function $f\colon (X,\tau) \to (Y,\sigma)$ is called β-continuous [1] if the preimage of every open set in Y is β-open in X. A set A is called β-open if $A \subseteq \overline{\mathrm{Int}\overline{A}}$.

REMARK 3.10. In the decomposition of quasi-continuity given above e-β-continuity can not be reduced to β-continuity. Note again that the function from Example 2.9 is β-continuous and mildly continuous but not quasi-continuous.

Recall that a an *extremally disconnected* (= ED) space is a topological space in which open sets have open closures or equivalently a space in which all semi-open sets are α-sets [15]. Thus as a consequence of Theorem 3.8 we have the following result:

COROLLARY 3.11. *Let* (X,τ) *be nodec and ED. For a function* $f\colon (X,\tau) \to (Y,\sigma)$ *the following conditions are equivalent:*
 (1) f *is continuous.*
 (2) f *is mildly continuous and e-β-continuous.*

A subset A of a topological space (X,τ) is called *interior-closed* (= ic-set) [12] if $\mathrm{Int}A$ is closed in A. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called *ic-continuous* [12] if the inverse image under f of each open set of Y is an ic-set.

LEMMA 3.12. [12, Theorem 2] *A function* $f\colon (X,\tau) \to (Y,\sigma)$ *is continuous if and only if* f *is quasi-continuous and ic-continuous.*

As another consequence of Theorem 3.8 we have the following decomposition of continuity:

COROLLARY 3.13. *For a function* $f\colon (X,\tau) \to (Y,\sigma)$ *the following conditions are equivalent:*

(1) *f is continuous.*

(2) *f is ic-continuous, mildly continuous and e-β-continuous.*

The relations between the types of continuity mentioned in this paper are given in the diagram below. Note that none of the implications is reversible.
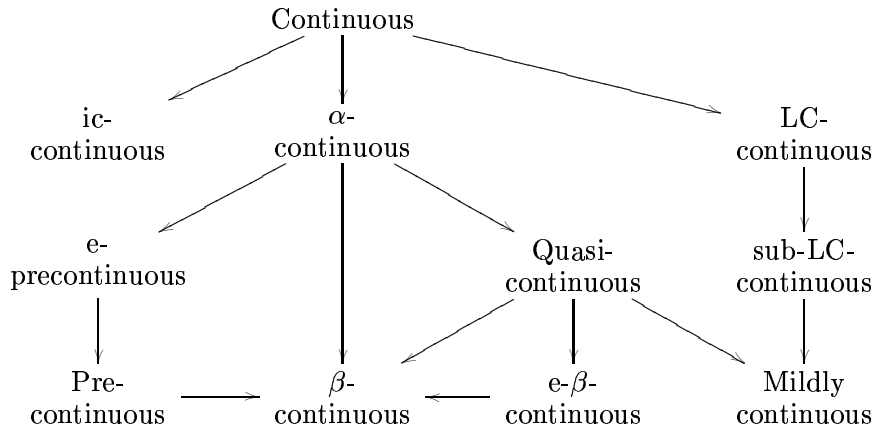
REFERENCES

[1] ABD EL-MONSEF M. E, EL-DEEB S. N. and MAHMOUD R.A., *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ., **12** (1983), 77–90.

[2] ARENAS F. G., DONTCHEV J. and GANSTER M., *On λ-sets and the dual of generalized continuity*, preprint.

[3] BAIRE R., *Sur les fonctions des variables réelles*, Ann. Mat. Pura. Appl., **3** (1899), 1-122.

[4] BLUMBERG H., *New properties of all real functions*, Trans. Amer. Math. Soc., **24** (1922), 113–128.

[5] BORSÌK J. and DOBOŠ J., *On certain decompositions of continuity*, Rend. Istit. Mat. Univ. Trieste, **20** (1988), no. 2, 275–282.

[6] BORSÌK J. and DOBOŠ J., *On decompositions of quasi continuity*, Real Anal. Exchange, **16** (1990/1991), 292–305.

[7] BOURBAKI N., *Éléments de mathématique,* Algèbre commutative, Chap. 2., Hermann, Paris, 1961.

[8] DONTCHEV J., *On generalizing semi-preopen sets,* Mem. Fac. Sci. Kochi Univ. Ser. A Math., **16** (1995), 35–48.

[9] DONTCHEV J. and PRZEMSKI M., *On the various decompositions of continuous and some weakly continuous functions,* Acta Math. Hungar., **71** (1-2) (1996), 81–92.

[10] GANSTER M. and REILLY I. L., *Locally closed sets and LC-continuous functions,* Internat. J. Math. Math. Sci., **12** (3) (1989), 417–424.

[11] GANSTER M. and REILLY I., *A decomposition of continuity,* Acta Math. Hungar., **56** (1990), 299–301.

[12] GANSTER M. and REILLY I., *Another decomposition of continuity,* Annals of the New York Academy of Sciences, Vol. **704** (1993), 135–141.

[13] GANSTER M., GRESSL F. and REILLY I., *On a decomposition of continuity,* Collection: General topology and applications (Staten Island, NY, 1989), 67–72, Lecture Notes in Pure and Appl. Math., **134**, Dekker, New York, 1991.

[14] HUSAIN T., *Almost continuous mappings,* Prace Mat., **10** (1966), 1–7.

[15] JANKOVIĆ D. S., *A note on mappings of extremally disconnected spaces,* Acta Math. Hungar., **46** (1985), 83–92.

[16] JELIĆ M., *A decomposition of pairwise continuity,* J. Inst. Math. Comput. Sci. Math. Ser., **3** (1990), no. 1, 25–29.

[17] JELIĆ M., *On a decomposition of p-continuity,* J. Inst. Math. Comput. Sci. Math. Ser., **4** (1991), no. 1, 57–60.

[18] KEMPISTY S., *Sur les fonctions quasicontinues,* Fund. Math., **19** (1932), 184–197.

[19] KLEE V. L. and UTZ W. R., *Some remarks on continuous transformations,* Proc. Amer. Math. Soc., **5** (1954), 182–184.

[20] LEVINE N., *A decomposition of continuity in topological spaces,* Amer. Math. Monthly, **68** (1961), 44–46.

[21] MASHHOUR A. S., ABD EL-MONSEF M. E. and EL-DEEB S. N., *On pre-continuous and weak pre-continuous mappings,* Proc. Math. Phys. Soc. Egypt, **53** (1982), 47–53.

[22] VAN MILL J. and MILLS C. F., *A boojum and other snarks,* Nederl. Acad. Wetensch. Proc. Ser. A, **83** (1980), 419–424.

[23] NAIMPALLY S. A., *Analysis the unknown,* preprint.

[24] NJÅSTAD O., *On some classes of nearly open sets,* Pacific J. Math., **15** (1965), 961–970.

[25] PRZEMSKI M., *A decomposition of continuity and α-continuity,* Acta Math. Hungar., **61** (1-2) (1993), 93–98.

[26] PTAK V., *Completeness and open mapping theorem,* Bull. Soc. Math. France, **86** (1958), 41–74.

[27] REILLY I. L. and VAMANAMURTHY M. K., *On α-continuity in topological spaces*, Acta Math. Hungar., **45** (1-2) (1985), 27–32.

[28] ROSE D., *On Levine's decomposition of continuity*, Canad. Math. Bull., **21** (1978), no. 4, 477–481.

[29] ROSE D., *A note on Levine's decomposition of continuity*, Indian J. Pure Appl. Math., **21** (1990), no. 11, 985–987.

[30] TONG J., *A decomposition of continuity*,Acta Math. Hungar.,**48**(1986), 11–15.

[31] TONG J., *A decomposition of fuzzy continuity*, Fuzzy Math., **7** (1987), no. 3-4, 97–98.

[32] TONG J., *On decomposition of continuity in topological spaces*, Acta Math. Hungar., **54** (1989), 51–55.

[33] YALVAÇ T. H., *Decomposition of continuity*, Acta Math. Hungar., **64** (3) (1994), 309–313.