MINIMAL STRUCTURES FOR T_{FA} (*)

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Sommario. - *Dato il reticolo di tutte le topologie definibili per un insieme infinito X, quelle che sono minime rispetto alla proprietà T_{FA} possono essere identificate. L'argomento qui presentato offre un approccio teorico ricorrendo ad una relazione di pre-ordine indotta su X dalla topologia data. Conseguentemente viene illustrata la tecnica per stabilire il minimo in questione fornendo una descrizione alternativa della struttura topologica minima. Più precisamente, la struttura minima può essere descritta in rapporto al comportamento della relazione binaria venutasi a creare ed alla topologia intrinseca indotta su di un insieme parzialmente ordinato.*

Summary. - *Given the lattice of all topologies definable for an infinite set X, those which are minimal with respect to the property T_{FA} are identified. The argument presented offers an approach which may readily be interpreted order-theoretically, by invoking the specialization preorder induced on X by the given topology. Accordingly, the potential of the technique developed to establish minimality is illustrated in providing an alternative description of the topologically established minimal structure. Specifically, the minimal structure may be described in terms of the behaviour of the naturally occurring specialization order and the intrinsic topology on the resulting partially ordered set.*

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Introduction.

Given an arbitrary infinite set X, we identify those topologies on X which minimally satisfy the property T_{FA}. A topological space $(X, \mathcal{T})$ is said to be

- T_{SA} if and only if for each $x \in X$, either $\{x\}$ is $\mathcal{T}$-closed or $\{x\}$ is $\mathcal{T}$-open or $\overline{\{x\}} \setminus \{x\} = \{y\}$ where $\{y\}$ is $\mathcal{T}$-closed
- T_{SD} if and only if for each $x \in X$, either $\{x\}$ is $\mathcal{T}$-closed or $\overline{\{x\}} \setminus \{x\} = \{y\}$ where $\{y\}$ is $\mathcal{T}$-closed
- T_A if and only if for all $x \in X$, either $\{x\}$ is $\mathcal{T}$-closed or $\{x\}$ is $\mathcal{T}$-open or $\overline{\{x\}} \setminus \{x\}$ is a point-closure ([7])
- T_F if and only if for each $x \in X$, either $\{x\}$ is $\mathcal{T}$-kernelled, as defined below, or $\mathcal{T}$-closed (see [1], [2] and [3])
- T_{FA} if and only if $\mathcal{T}$ is T_F and T_A (equivalently, if and only if $\mathcal{T}$ is T_F and T_{SA})
- T_D if and only if for each $x \in X$, $\overline{\{x\}} \setminus \{x\}$ is $\mathcal{T}$-closed (see [1], [2], [5] and [11])
- T_{ES} if and only if for each $x \in X$, either $\{x\}$ is $\mathcal{T}$-open or $\{x\}$ is $\mathcal{T}$-closed (see [6] and [10]).

The property T_{FA} occupies a special position in the logical hierarchy of topological invariants. In a sense, it bridges the 'gap' between T_{SA} and T_{SD} (where T_{SD} implies T_{FA} which in turn implies T_{SA}) in that it is both implied by T_{ES} *and* implies T_F. This special nature of T_{FA} is particularly apparent in our investigations into its minimal structure where we identify some special cases of minimal T_{FA}-topologies. It transpires that for such cases we may draw upon some previously established minimality results concerning T_{ES} and T_{SD}. Such structures however represent only a partial solution and we develop some techniques with which to establish the complete solution.

We proceed by a development of a purely topological approach to the question of minimality, but indicate how a recognition of the underlying order structure of any topological space affords us new and valuable insight into the problem. By invoking the specialization pre-order induced on X by the given topology, we may adopt an order-theoretic approach which lends a welcome visual aspect to the discussion (see [7]). We reserve an order-theoretic interpretation of the established results for the final section of this work.

We begin with some definitions. Note that throughout this work, X shall denote an arbitrary infinite set and $LT(X)$ the lattice of all topologies for X.

DEFINITION 1. Given $\mathcal{T} \in LT(X)$ and $x \in X$, the intersection of all $\mathcal{T}$-open subsets of X which contain x is called the *$\mathcal{T}$-kernel of* $\{x\}$ and is denoted by $\widehat{\{x\}}$ (assuming no danger of ambiguity). We often refer to $\widehat{\{x\}}$ as a *point-kernel* and if $\widehat{\{x\}} = \{x\}$, we say that $\{x\}$ is *$\mathcal{T}$-kernelled.*

As usual, $\overline{\{x\}}$ denotes the $\mathcal{T}$-closure of $\{x\}$ and we similarly refer to it as a *point-closure*. Further, the $\mathcal{T}$-derived set of $\{x\}$ is $\overline{\{x\}} \setminus \{x\}$ which we often refer to as a *point-derived set*.

Of course, given x, $y \in X$, $x \in \overline{\{y\}}$ if and only if $y \in \widehat{\{x\}}$. We adopt the notation of [2] by writing

$$\begin{aligned}
N_D(\mathcal{T}) &= \Big\{x \in X : \{x\} = \overline{\{x\}}\Big\} \\
N_S(\mathcal{T}) &= \Big\{x \in X : \{x\} = \widehat{\{x\}}\Big\} \\
N_0(\mathcal{T}) &= \{x \in X : \{x\} \in \mathcal{T}\} \\
N_H(\mathcal{T}) &= \Big\{x \in X : \overline{\{x\}} = \{x, y\} \text{ where } y \in N_D(\mathcal{T}),\, y \neq x\Big\}.
\end{aligned}$$

Given $\mathcal{T}_1$, $\mathcal{T}_2 \in LT(X)$, $\mathcal{T}_1$ is said to be *stronger* or *finer* than $\mathcal{T}_2$ (or $\mathcal{T}_2$ to be *weaker* or *coarser* than $\mathcal{T}_1$) if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$ in $LT(X)$.

Finally, given subsets A and B of X, we denote by $|A|$ the cardinality of A and write $|A| < \omega$ if A is finite; we write $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$.

DEFINITION 2. Given $x \in X$ and $Y \subseteq X$, we define the following members of $LT(X)$:

$\mathcal{D}$	The discrete member of $LT(X)$
$\mathcal{I}(Y)$	$\{G \subseteq X : Y \subseteq G\} \cup \{\emptyset\}$
$\mathcal{E}(Y)$	$\mathcal{P}(X \backslash Y) \cup \{X\}$
$\mathcal{I}(x)$	$\{G \subseteq X : x \in G\} \cup \{\emptyset\}$, 'included point' member of$LT(X)$
$\mathcal{E}(x)$	$\mathcal{P}(X \backslash \{x\}) \cup \{X\}$, 'excluded point' member of $LT(X)$
$\mathcal{D}(Y)$	$\{G \subseteq X :$ $G \subseteq Y$ and $Y \backslash G$ is finite or $Y \subseteq G$ and $X \backslash G$ is finite $\} \cup \{\emptyset\}$
$\mathcal{C}$	The cofinite (or minimum T_1) member of$LT(X)$

DEFINITION 3. Given a subset K of X and a non-empty family $\mathcal{P}$ of subsets of X, $\mathcal{P}$ is said to be

(i) *associated* with K if and only if for each $P \in \mathcal{P}$, $P \cap K \neq \emptyset$

(ii) *simply associated* with K if and only if for each $P \in \mathcal{P}$, $P \cap K$ is a singleton.

DEFINITION 4. Given non-empty disjoint subsets Q and K of X and a partition $\mathcal{P}$ of $Q \cup K$ such that $\mathcal{P}$ is simply associated with Q and associated with K, we define $\mathcal{S}(\mathcal{P})$ to be the topology whose closed sets are generated by the family

$$\{\{y, x\} : \{y, x\} \subseteq P,\ y \neq x;\ \{x\} = P \cap Q \text{ for some } P \in \mathcal{P}\} \cup \{\emptyset, X\}.$$

LEMMA 5. *Let $\mathcal{T} \in LT(X)$, $A \subseteq X$, $x \neq y$ in X and $\mathcal{T}^* = \mathcal{T} \cap (\mathcal{I}(y) \cup \mathcal{E}(x))$. Then the $\mathcal{T}^*$-closure of A is described by*

$$\bar{A}^* = \begin{cases} \bar{A} & , \text{if } y \notin \bar{A} \\ \bar{A} \cup \overline{\{x\}} & , \text{if } y \in \bar{A}. \end{cases}$$

Proof. Clearly $\bar{A} \subseteq \bar{A}^*$. Now *either* $y \notin \bar{A}$ so that $\bar{A}$ is $\mathcal{T}^*$-closed whence $\bar{A}^* \subseteq \bar{A}$, *or* $y \in \bar{A}$ in which case $y \in \bar{A}^*$ and hence $x \in \bar{A}^*$. Thus $\overline{\{x\}} \subseteq \bar{A}^*$ and since $\bar{A} \cup \overline{\{x\}}$ is $\mathcal{T}^*$-closed, the result is immediate. ◊

LEMMA 6. *Let $\mathcal{T} \in LT(X)$, let x, $y \in X$ with $y \in N_S(\mathcal{T})$, $x \in N_D(\mathcal{T})$, $y \neq x$ and let $\mathcal{T}^* = \mathcal{T} \cap (\mathcal{I}(y) \cup \mathcal{E}(x))$. If $\mathcal{T}$ is T_F, then $\mathcal{T}^*$ is T_F.*

Proof. By Lemma 5, given $z \in X$, we have

$$\overline{\{z\}}^* = \begin{cases} \overline{\{z\}} & , z \neq y \\ \overline{\{y\}} \cup \overline{\{x\}} & , z = y. \end{cases}$$

Further, $y \in N_S(\mathcal{T})$ implies that $y \in N_S(\mathcal{T}^*)$, and for any $z \neq y$ with $z \in N_D(\mathcal{T})$, $z \in N_D(\mathcal{T}^*)$. Finally, if $z \neq y$ and $z \notin N_D(\mathcal{T})$, then $z \in N_S(\mathcal{T})$ (since $\mathcal{T}$ is T_F), $z \neq x$ and $\{z\} = \bigcap\{G\setminus\overline{\{x\}} : G \in \mathcal{T}$ and $z \in G\}$ so that $z \in N_S(\mathcal{T}^*)$. Hence $\mathcal{T}^*$ is T_F. ◊

LEMMA 7. *If $(X, \mathcal{T})$ is T_{FA}, $x \in N_D(\mathcal{T})$ and $y \in N_0(\mathcal{T})$ where $x \neq y$, then $\mathcal{T}^* = \mathcal{T} \cap (\mathcal{I}(y) \cup \mathcal{E}(x))$* is T_{FA}.

Proof. By Lemma 6, $\mathcal{T}^*$ is T_F. Further, again by Lemma 5, we observe that $\{y\}$ is $\mathcal{T}^*$-open, $\overline{\{x\}}$ is $\mathcal{T}^*$-closed and for any $z \in X\setminus\{x, y\}$, *either* $\{z\}$ is $\mathcal{T}$-closed and therefore $\mathcal{T}^*$-closed, *or* $\{z\}$ is $\mathcal{T}$-open and hence $\mathcal{T}^*$-open, *or* $\overline{\{z\}}\setminus\{z\} = \overline{\{t\}}$ where $t \neq y$ so that $\overline{\{z\}}^*\setminus\{z\} = \overline{\{t\}}^*$. That is, $\mathcal{T}^*$ is T_A and hence T_{FA}. ◊

LEMMA 8. *If $(X, \mathcal{T})$ is minimal T_{FA}, then*

(i) $y \in N_0(\mathcal{T})$ *implies* $\overline{\{y\}} = \{y\} \cup N_D(\mathcal{T})$

(ii) $N_0(\mathcal{T}) \cap N_D(\mathcal{T}) = \emptyset$

(iii) $N_0(\mathcal{T}) \cap N_H(\mathcal{T}) = \emptyset$

(iv) $N_S(\mathcal{T}) = N_H(\mathcal{T}) \cup N_0(\mathcal{T})$ *(equivalently, $N_S(\mathcal{T}) \cap N_D(\mathcal{T}) = \emptyset$)*

(v) $N_0(\mathcal{T}) \neq \emptyset$ *implies $N_D(\mathcal{T})$ is $\mathcal{T}$-closed.*

Proof. *(i)* Since $\mathcal{T}$ is T_F, $\overline{\{y\}} \subseteq \{y\} \cup N_D(\mathcal{T})$ for any $y \in X$. Conversely, let $x \in N_D(\mathcal{T})$ and suppose that $x \notin \overline{\{y\}}$ where $y \in N_0(\mathcal{T})$. If $\mathcal{T}^* = \mathcal{T} \cap (\mathcal{I}(y) \cup \mathcal{E}(x))$ then, since $x \notin \overline{\{y\}}$, $\mathcal{T}^*$ is strictly weaker than $\mathcal{T}$ and by Lemma 7, $\mathcal{T}^*$ is T_{FA} — clearly a contradiction. We conclude that $x \in \overline{\{y\}}$ so that $\overline{\{y\}} = \{y\} \cup N_D(\mathcal{T})$.

(ii) If $t \in N_0(\mathcal{T}) \cap N_D(\mathcal{T})$, then $\overline{\{t\}} = \{t\} = \{t\} \cup N_D(\mathcal{T})$ by *(i)* above so that $N_D(\mathcal{T}) = \{t\}$, whence $N_S(\mathcal{T}) = X$. But this implies that $X = N_D(\mathcal{T})$ — an obvious contradiction. Thus $N_0(\mathcal{T}) \cap N_D(\mathcal{T}) = \emptyset$.

(iii) If $t \in N_0(\mathcal{T}) \cap N_H(\mathcal{T})$ then, again by *(i)* above, $\overline{\{t\}} = \{t\} \cup N_D(\mathcal{T}) = \{t, x\}$ for some $x \in N_D(\mathcal{T})$. Hence $N_D(\mathcal{T}) = \{x\}$ so that, by *(i)*, $\overline{\{z\}} = \{z, x\}$ for all $z \in X$ and $\mathcal{C} \cap \mathcal{E}(x) \subseteq \mathcal{T}$ in $LT(X)$. Thus $\mathcal{T} = \mathcal{C} \cap \mathcal{E}(x)$, since $\mathcal{C} \cap \mathcal{E}(x)$ is T_{FA}, and $N_0(\mathcal{T}) = \emptyset$ — clearly a contradiction. Hence $N_0(\mathcal{T}) \cap N_H(\mathcal{T}) = \emptyset$.

(iv) If $t \in N_S(\mathcal{T}) \cap N_D(\mathcal{T})$, then $N_0(\mathcal{T}) = \emptyset$ (otherwise, by Lemma 7, we may construct a strictly weaker T_{FA}-topology!) so that $\mathcal{T}$ is T_{SD} and therefore minimally T_{SD} (since T_{SD} implies T_{FA}). Then, $N_S(\mathcal{T}) = N_H(\mathcal{T})$ so that $N_S(\mathcal{T}) \cap N_D(\mathcal{T}) = \emptyset$! Hence, we must have $N_S(\mathcal{T}) \cap N_D(\mathcal{T}) = \emptyset$.

(v) If $y \in N_0(\mathcal{T})$ then, by *(i)* and *(ii)*, $\overline{\{y\}} \setminus \{y\} = N_D(\mathcal{T})$ and since $\mathcal{T}$ is T_D, the result follows. ♢

We quote without proof the following results from [6] and [9]:

THEOREM 9. *$(X, \mathcal{T})$ is minimal T_{ES} if and only if either $\mathcal{T} = \mathcal{C}$ or $\mathcal{T} = \mathcal{E}(X \backslash Y) \cup (\mathcal{C} \cap \mathcal{I}(Y))$ for some non-empty proper subset Y of X ([6]).*

THEOREM 10. *$(X, \mathcal{T})$ is minimal T_{SD} if and only if $\mathcal{T} = \mathcal{S}(\mathcal{P}) \cup (\mathcal{C} \cap \mathcal{I}(K))$ for some non-empty proper subset K of X and partition $\mathcal{P}$ of X such that $\mathcal{P}$ is simply associated with $X \backslash K$ and associated with K* ([9]).

Observe that $\mathcal{T} \subseteq \mathcal{C}$ in $LT(X)$ (so that $N_0(\mathcal{T}) = \emptyset$) and that $K = N_H(\mathcal{T}) = N_S(\mathcal{T})$.

THEOREM 11. *Given $\mathcal{T} \in LT(X)$,*

(i) $\mathcal{T}$ is minimal T_{FA} and $N_H(\mathcal{T}) = \emptyset$ if and only if $\mathcal{T} = \mathcal{E}(X \backslash Y) \cup (\mathcal{C} \cap \mathcal{I}(Y))$ for some non-empty proper subset Y of X such that

$X\backslash Y$ is non-singleton
(equivalently, if and only if $\mathcal{T}$ is minimal T_{ES}, $N_0(\mathcal{T}) \neq \emptyset$ and $|N_D(\mathcal{T})| > 1$).

(ii) $\mathcal{T}$ is minimal T_{FA} and $N_0(\mathcal{T}) = \emptyset$ if and only if $\mathcal{T} = \mathcal{S}(\mathcal{P}) \cup (\mathcal{C} \cap \mathcal{I}(K))$ for some non-empty proper subset K of X and partition $\mathcal{P}$ of X such that $\mathcal{P}$ is simply associated with $X\backslash K$ and associated with K
(equivalently, if and only if $\mathcal{T}$ is minimal T_{SD}).

Proof. *(i)* $\Rightarrow$: By hypothesis, $\mathcal{T}$ is T_{ES} and therefore minimal T_{ES} (since T_{ES} implies T_{FA}) so that $\mathcal{T} = \mathcal{E}(X\backslash Y) \cup (\mathcal{C} \cap \mathcal{I}(Y))$ (see [6]) where Y is a non-empty proper subset of X. (Observe that $\mathcal{T} \neq \mathcal{C}$ since $\mathcal{C} \cap \mathcal{E}(x)$ is T_{FA} for all $x \in X$). Further, $X\backslash Y$ is non-singleton (otherwise $X\backslash Y = \{x\}$ implies that $y \in N_H(\mathcal{T})$ for each $y \in Y = N_0(\mathcal{T})$, contradicting Lemma 8 (iii)), whence result.

$\Leftarrow$: Conversely, with $\mathcal{T}$ as described, $\mathcal{T}$ is minimal T_{ES} (again, see [6]) and $N_H(\mathcal{T}) = \emptyset$. Let $\mathcal{T}^* \subseteq \mathcal{T}$ in $LT(X)$ where $\mathcal{T}^*$ is T_{FA}. Then $N_0(\mathcal{T}^*) \subseteq N_0(\mathcal{T}) = Y$, $N_D(\mathcal{T}^*) \subseteq N_D(\mathcal{T}) = X\backslash Y$ and $N_H(\mathcal{T}^*) \subseteq N_D(\mathcal{T}) \cup N_H(\mathcal{T}) = X\backslash Y$. If $x \in N_H(\mathcal{T}^*)$, then $x \notin Y$ and $x \in \overline{\{y\}}^*$ for each $y \in Y$ (since $\overline{\{y\}} = \{y\} \cup X\backslash Y \subseteq \overline{\{y\}}^*$ for each $y \in Y$) so that $x \notin N_D(\mathcal{T}^*) \cup N_S(\mathcal{T}^*) = X$!

Hence $N_H(\mathcal{T}^*) = \emptyset$ so that $\mathcal{T}^*$ is T_{ES}, implying $\mathcal{T} = \mathcal{T}^*$. That is, $\mathcal{T}$ is minimal T_{FA}.

(ii) $\Rightarrow$: By hypothesis, $\mathcal{T}$ is T_{SD} and therefore minimal T_{SD}.

$\Leftarrow$: Conversely, since $\mathcal{T}$ is minimal T_{SD}, then $N_0(\mathcal{T}) = \emptyset$ and one can easily show that $\mathcal{T}$ is minimally T_{FA}. $\diamondsuit$

LEMMA 12. *If $(X, \mathcal{T})$ is minimal T_{FA} with $N_0(\mathcal{T}) \neq \emptyset$ and $N_H(\mathcal{T}) \neq \emptyset$, then*

(i) $N_H(\mathcal{T}) \cup N_0(\mathcal{T})$ is infinite

(ii) $|N_D(\mathcal{T})| > 1$.

Proof. *(i)* Suppose that $2 \leq |N_H(\mathcal{T}) \cup N_0(\mathcal{T})| < \omega$; then, by Lemma 8 (iii), $y \in N_H(\mathcal{T})$ implies $X\backslash \{y\} = N_0(\mathcal{T}) \cup N_D(\mathcal{T}) \cup (N_H(\mathcal{T})\backslash \{y\})$ so that by Lemma 8 (i), $X\backslash \{y\} = \bigcup\left\{\overline{\{x\}} : x \in N_0(\mathcal{T})\right\}$

$\cup \bigcup \left\{ \overline{\{z\}} : z \in N_H(\mathcal{T}),\, z \neq y \right\}$. That is, $X \setminus \{y\}$ is $\mathcal{T}$-closed, being a finite union of $\mathcal{T}$-closed sets, so that $y \in N_0(\mathcal{T})$ contradicting Lemma 8 (iii). It follows therefore that $N_H(\mathcal{T}) = \emptyset$, thus contradicting the given hypothesis. Hence $N_H(\mathcal{T}) \cup N_0(\mathcal{T})$ must be infinite.

(ii) If $N_D(\mathcal{T}) = \{x\}$, then $N_0(\mathcal{T}) = \emptyset$ (otherwise by Lemma 8 (i), $N_0(\mathcal{T}) \cap N_H(\mathcal{T}) \neq \emptyset$ contradicting Lemma 8 (iii)), contradicting the hypothesis.

Thus $|N_D(\mathcal{T})| > 1$. ◇

LEMMA 13. *Let $(X, \mathcal{T})$ be T_{FA} with*

(i) $N_0(\mathcal{T}) \cap N_D(\mathcal{T}) = \emptyset$

(ii) $N_0(\mathcal{T}) \cap N_H(\mathcal{T}) = \emptyset$

(iii) $N_S(\mathcal{T}) \cap N_D(\mathcal{T}) = \emptyset$.

If $\mathcal{T}^* \subseteq \mathcal{T}$ in $LT(X)$ where $\mathcal{T}^*$ is T_{FA}, then $N_0(\mathcal{T}^*) = N_0(\mathcal{T})$, $N_D(\mathcal{T}^*) = N_D(\mathcal{T})$ and $N_H(\mathcal{T}^*) = N_H(\mathcal{T})$.

Proof. Since $\mathcal{T}^* \subseteq \mathcal{T}$ in $LT(X)$, immediately $N_0(\mathcal{T}^*) \subseteq N_0(\mathcal{T})$, $N_S(\mathcal{T}^*) \subseteq N_S(\mathcal{T})$ and $N_D(\mathcal{T}^*) \subseteq N_D(\mathcal{T})$. Further, $N_H(\mathcal{T}^*) \subseteq N_H(\mathcal{T}) \cup N_D(\mathcal{T})$ so that $N_H(\mathcal{T}^*) \cup N_D(\mathcal{T}^*) \subseteq N_H(\mathcal{T}) \cup N_D(\mathcal{T})$.

By hypothesis, therefore, $y \in N_0(\mathcal{T})$ implies $y \notin N_H(\mathcal{T}^*) \cup N_D(\mathcal{T}^*)$ so that $y \in N_0(\mathcal{T}^*)$. That is, $N_0(\mathcal{T}) = N_0(\mathcal{T}^*)$.

It follows immediately that $N_H(\mathcal{T}^*) \cup N_D(\mathcal{T}^*) = N_H(\mathcal{T}) \cup N_D(\mathcal{T})$. Now $y \in N_H(\mathcal{T})$ implies that $y \notin N_D(\mathcal{T})$ so that $y \notin N_D(\mathcal{T}^*)$, whence $y \in N_H(\mathcal{T}^*)$. That is, $N_H(\mathcal{T}) \subseteq N_H(\mathcal{T}^*)$.

On the other hand, suppose there exists $y \in N_H(\mathcal{T}^*)$ with $y \notin N_H(\mathcal{T})$; then $y \in N_D(\mathcal{T})$ and $y \notin N_D(\mathcal{T}^*)$ so that $y \in N_S(\mathcal{T}^*)$ (since $\mathcal{T}^*$ is T_F). But $N_S(\mathcal{T}^*) \subseteq N_S(\mathcal{T})$ so that $y \in N_D(\mathcal{T}) \cap N_S(\mathcal{T})$, contradicting *(iii)* of the hypothesis. We conclude that $N_H(\mathcal{T}^*) = N_H(\mathcal{T})$, from which it follows that $N_D(\mathcal{T}^*) = N_D(\mathcal{T})$. ◇

THEOREM 14. *$(X, \mathcal{T})$ is minimal T_{FA} with $N_0(\mathcal{T}) \neq \emptyset$ and $N_H(\mathcal{T}) \neq \emptyset$ if and only if $\mathcal{T} = \mathcal{E}(X \setminus B) \vee \mathcal{S}(\mathcal{P}) \vee \mathcal{D}(B \cup K)$ for some non-empty, disjoint subsets B, K and Q of X such that $B \cup K$ is infinite but has at least two elements in its complement, and partition $\mathcal{P}$ of $Q \cup K$ such that $\mathcal{P}$ is simply associated with Q and associated with K.*

(Moreover, the representation is canonical: $B = N_0(\mathcal{T})$, $K = N_H(\mathcal{T})$, $X\backslash(B \cup K) = N_D(\mathcal{T})$ *and* $Q = \{\overline{\{y\}}\backslash\{y\} : y \in N_H(\mathcal{T})\}$, *while* $\mathcal{P}$ *is the family of kernels of singletons in* Q.*)*

Proof. $\Leftarrow$: Given $z \in Q \cup K$, let P_z be the element of $\mathcal{P}$ which contains z; then observe that

$$\overline{\{z\}} = \begin{cases} \{z\} \cup (X\backslash(B \cup K)), & \text{if } z \in B \\ \{z, z_p\} & \text{, if } z \in K, \text{ where } \{z_p\} = P_z \cap (X\backslash K) \\ \{z\} & \text{, if } z \notin B \cup K. \end{cases}$$

It is readily verified that $B = N_0(\mathcal{T})$, $K = N_H(\mathcal{T})$, $K \cup B = N_S(\mathcal{T})$ and $X\backslash(B \cup K) = N_D(\mathcal{T})$ so that $\mathcal{T}$ is T_{SA}. Moreover, $\mathcal{T}$ is T_F since $\mathcal{D}(B \cup K)$ is T_F and T_F is preserved under strengthening of topology, whence $\mathcal{T}$ is T_{FA}.

Let $\mathcal{T}^* \subseteq \mathcal{T}$ in $LT(X)$ where $\mathcal{T}^*$ is T_{FA}. Then appealing to Lemma 13, $N_0(\mathcal{T}^*) = B$, $N_D(\mathcal{T}^*) = X\backslash(B \cup K)$ and $N_H(\mathcal{T}^*) = K$. Moreover, $\overline{\{z\}}^* = \overline{\{z\}}$ for all $z \in X$ (since $z \notin B \cup K$ clearly implies $\overline{\{z\}} = \{z\} = \overline{\{z\}}^*$, $z \in B$ implies $\overline{\{z\}} \subseteq \overline{\{z\}}^* \subseteq \{z\} \cup N_D(\mathcal{T}^*) = \overline{\{z\}}$ and $z \in K$ implies $\overline{\{z\}} = \{z, z_p\} \subseteq \overline{\{z\}}^*$ where $\{z_p\} = P_z \cap (X\backslash K)$ (and $z_p \in N_D(\mathcal{T}^*)$) so that $\overline{\{z\}} = \overline{\{z\}}^*$). Hence, $\mathcal{E}(X\backslash B) \subseteq \mathcal{T}^*$, $\mathcal{C} \cap \mathcal{I}(K \cup B) \subseteq \mathcal{T}^*$ and $\mathcal{S}(\mathcal{P}) \subseteq \mathcal{T}^*$ in $LT(X)$.

Finally, given $F = F_1 \cup [X\backslash(B \cup K)]$ where F_1 is a finite subset of $B \cup K$, *either* $F_1 = \emptyset$ in which case F is $\mathcal{T}^*$-closed (since $B \neq \emptyset$ and $\mathcal{T}^*$ is T_D) *or* $F_1 \neq \emptyset$ so that $F = \bigcup \left\{ \overline{\{x\}} : x \in F_1 \right\} \cup [X\backslash(B \cup K)] = \bigcup \left\{ \overline{\{x\}}^* : x \in F_1 \right\} \cup [X\backslash(B \cup K)]$ which is $\mathcal{T}^*$-closed. That is, $\mathcal{D}(B \cup K) \subseteq \mathcal{T}^*$ in $LT(X)$ so that $\mathcal{T} = \mathcal{T}^*$ and the result follows.

$\Rightarrow$: Let $K = N_H(\mathcal{T})$, $B = N_0(\mathcal{T})$ and $Q = \left\{ x \in N_D(\mathcal{T}) : x \in \overline{\{y\}} \text{ for some } y \in K \right\}$. Then by Lemma 8, $N_D(\mathcal{T}) = X\backslash(B \cup K)$ and B and K are disjoint while, by Lemma 12, $B \cup K$ is infinite with $|X\backslash(B \cup K)| > 1$. Further, since $K \neq \emptyset$, $Q \neq \emptyset$ and so for each $z \in Q$, write $P_z = \left\{ y \in K : z \in \overline{\{y\}} \right\} \cup \{z\}$. Then $\mathcal{P} = \{P_z : z \in Q\}$ defines a partition of $Q \cup K$ and it is readily verified that $\mathcal{P}$ has the stipulated associations with Q and K.

It follows routinely that $\mathcal{E}(X\backslash B) \vee \mathcal{S}(\mathcal{P}) \vee \mathcal{D}(B \cup K) \subseteq \mathcal{T}$ in $LT(X)$ and, since the former is T_{FA} by the proof of sufficiency, $\mathcal{T} = \mathcal{E}(X\backslash B) \vee \mathcal{S}(\mathcal{P}) \vee \mathcal{D}(B \cup K)$. ◇

Thus, the minimal T_{FA}-structure is completely identified by Theorems 11 and 14. We conclude the given approach with the following which is essentially a corollary to several previous results.

COROLLARY 15. *Given $\mathcal{T} \in LT(X)$, the following statements are equivalent:*

(i) $\mathcal{T}$ is minimal T_{FA} and $2 \leq |N_0(\mathcal{T}) \cup N_H(\mathcal{T})| < \omega$.

(ii) $\mathcal{T}$ is minimal T_{FA} and $N_H(\mathcal{T}) = \emptyset$ and $2 \leq |N_0(\mathcal{T})| < \omega$.

(iii) $\mathcal{T}$ is minimal T_{FA}, minimal T_{ES} and minimal T_F.

(iv) $\mathcal{T}$ is minimal T_F, and T_{ES}.

(v) $\mathcal{T} = \mathcal{D}(Y)$ where $Y \subseteq X$ is such that $2 \leq |Y| < \omega$.

Proof. *(i)* implies *(ii)*. By Lemma 12, either $N_0(\mathcal{T}) = \emptyset$ or $N_H(\mathcal{T}) = \emptyset$. Now if $N_0(\mathcal{T}) = \emptyset$, then $N_H(\mathcal{T})$ is finite so that by Lemma 11 (ii), X is finite! Hence $N_H(\mathcal{T}) = \emptyset$ and so $2 \leq |N_0(\mathcal{T})| < \omega$.

The converse *(ii)* implies *(i)* is immediate.

(ii) implies *(iii)*. By Lemma 11 (i), $\mathcal{T}$ is minimal T_{ES} with $\mathcal{T} = \mathcal{D}(Y)$ where $Y \subseteq X$ is such that $2 \leq |Y| < \omega$ so that $\mathcal{T}$ is minimal T_F (see [3]).

(iii) implies *(iv)*. This is immediate.

(iv) implies *(v)*. This is immediate (again, see [3]).

(v) implies *(ii)*. Since $\mathcal{D}(A) = \mathcal{E}(X \backslash A) \cup (\mathcal{C} \cap \mathcal{I}(A))$ for any non-empty finite subset A of X, then in particular $\mathcal{T} = \mathcal{D}(Y) = \mathcal{E}(X \backslash Y) \cup (\mathcal{C} \cap \mathcal{I}(Y))$. By Lemma 11 (i) then, $\mathcal{T}$ is minimal T_{FA}, $N_H(\mathcal{T}) = \emptyset$ and since $Y = N_0(\mathcal{T})$, the result is immediate. ◊

Order.

DEFINITION 16. A binary relation $\leq$ on X is said to be a *pre-order* (and $(X, \leq)$ is referred to as a *pre-ordered set*) if and only if $\leq$ is both reflexive and transitive. If, in addition, $\leq$ is anti-symmetric, then $\leq$ is said to be a *partial order* on X and $(X, \leq)$ is called a *partially ordered set* (or *poset*). Given x, $y \in X$, we write $x \leq y$ if and only if $(x, y) \in \leq$. If $x \leq y$ in X with $x \neq y$, we write $x < y$.

Given a poset $(X, \leq)$ with $\emptyset \subset Y \subseteq X$, then Y is said to be *diverse* if and only if x, $y \in Y$ and $x \leq y$ implies $x = y$. Y is said to be *linear*, or a *chain*, or *totally ordered* if and only if x, $y \in Y$ implies that either $x \leq y$ or $y \leq x$.

x is a *predecessor* for y if and only if $x < y$ and whenever $z < y$, $z \in X$, then $z \leq x$.

x is said to be *maximal* (*minimal*) if and only if $x \leq z$ ($z \leq x$), $z \in X$ implies that $x = z$.

x is said to be *ultramaximal* if and only if x is maximal and for any non-maximal element $z \in X$, $z \leq x$.

DEFINITION 17. Given a poset $(X, \leq)$ with $x \in X$, we define

$$\uparrow\{x\} = \{y \in X : x \leq y\}.$$

$$\downarrow\{x\} = \{y \in X : y \leq x\}.$$

DEFINITION 18. Given a poset $(X, \leq)$ with x, $y \in X$, we define the dual partial order $\leq^*$ of $\leq$ by $x \leq^* y$ if and only $y \leq x$. Then x is said to be *connected* to y if and only if there is a finite sequence $x_0 = x$, $x_1, x_2, \ldots, x_n = y$ of elements of X such that $(x_i, x_{i+1}) \in \leq \cup \leq^*$, each $i \in n$.

$(X, \leq)$ is said to be *connected* if and only if x is connected to y for all x, $y \in X$.

The *components* of $(X, \leq)$ are the equivalence classes with respect to the relation: $x \approx y$ if and only if x is connected to y.

DEFINITION 19. Let $(X, \leq)$ be a poset with $Y \subseteq X$, and let $n \in \omega$. If C is a chain in X with $|C| = n$, then C is said to have *length* $n - 1$. If the least upper bound, l, of the lengths of all finite chains in Y exists, then we say that Y has *length* l.

Y is said to be a *semi-tree* if and only if for each $y \in Y$, $\{z \in Y : z \leq y\}$ is a chain. Y is said to be a *tree* if and only if Y is a semi-tree with minimum element.

DEFINITION 20. Given a poset $(X, \leq)$, we define the following intrinsic topologies for X:

- The *weak* topology, $\mathcal{W}$, whose closed sets are generated by the family $\{\emptyset, X, \downarrow\{x\} : x \in X\}$.

  Thus, $\mathcal{W}$ is the smallest topology on X in which all sets of the form $\downarrow\{x\}$ are closed. Note further that $\overline{\{x\}} = \downarrow\{x\}$ for all $x \in X$.

- The topology, denoted by $\mathcal{M}$, whose closed sets are generated by the family $\{\emptyset, X, \downarrow\{x\}, \downarrow\{x\} \setminus \{x\} : x \in X\}$.

- The topology, $\mathcal{L}$, which has as (open) base, the family $\mathcal{M} \cup \{\{x\} : x \text{ is ultramaximal}\}$.

- The *Alexandroff* topology, $\mathcal{A}$, whose open sets are generated by sets of the form $\uparrow\{x\}$. (It is easily seen that $\mathcal{A}$ is 'principal' in that arbitrary intersections of open sets are open.)

Note that $\mathcal{W} \subseteq \mathcal{M} \subseteq \mathcal{L} \subseteq \mathcal{A}$, and that for each of these topologies, $\overline{\{x\}} = \downarrow\{x\}$. Given a topological space $(X, \mathcal{T})$, its specialization order is defined by $x \leq y \Leftrightarrow x \in \overline{\{y\}}$. In fact, given a pre-order $\leq$ and a topology $\mathcal{T}$ for X, it is well-known that $\mathcal{T}$ will have $\leq$ as its specialization order if and only if $\mathcal{W} \subseteq \mathcal{T} \subseteq \mathcal{A}$. (See [7] or [1].) That is, $\mathcal{W}$ is the smallest and $\mathcal{A}$ is the largest of the topologies with a given specialization order and all such topologies have $\overline{\{x\}} = \downarrow\{x\}$ and $\widehat{\{x\}} = \uparrow\{x\}$ for all $x \in X$.

Order-theoretic minimality characterizations.

We now present an order-theoretic description of the previously established minimality results. For the sake of completeness, we include also the order-theoretic characterizations for minimal T_{ES} and minimal T_{SD}.

Let $\mathcal{T} \in LT(X)$ with induced order $\leq$.

THEOREM 21. *$(X, \mathcal{T})$ is* minimal T_{ES} *if and only if $(X, \leq)$ is a poset such that* either

(i) X is diverse and $\mathcal{T} = \mathcal{W}$ or

(ii) all maximal chains in X have unit length, every maximal element is ultramaximal and $\mathcal{T} = \mathcal{L}$.

THEOREM 22. *$(X, \mathcal{T})$ is* minimal T_{SD} *if and only if $(X, \leq)$ is a poset such that all components of $(X, \leq)$ are trees of length 1 and $\mathcal{T} = \mathcal{W}$.*

THEOREM 23. *$(X, \mathcal{T})$ is* minimal T_{FA} *if and only if $(X, \leq)$ is a poset such that all maximal chains in X have unit length and* either

(i) every component is a tree and $\mathcal{T} = \mathcal{W}$ or

(ii) there are at least two minimal elements, each maximal but non-ultramaximal element has a predecessor and $\mathcal{T} = \mathcal{L}$.

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