

A COURSE ON YOUNG MEASURES (*)

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0. Introduction.

The development of Young measures Theory has a long story. Obviously it goes back to L.C. Young, specially [Y1]. The first aim was to give a description of limits of minimizing sequences in the Calculus of Variations and further in the Optimal control Theory

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(see L.C. Young [Y4], J. Warga [W] and A. Ghouila-Houri [GH]; see the 20th problem of Hilbert quoted in [Y2, p.84] and in I. Ekeland & R. Temam's book [ET, Comments to chapters 9 and 10]). More recently H. Berliocchi & J.M. Lasry [BL] extended the theory so as to make it work without compactness. Then E.J. Balder [Bd2, 4, 12] gave the parametric version of the Prohorov theorem and lower semi-continuity theorems which make the theory very efficient and applicable; he also developed many applications (note among them the one to the Fatou lemma in several dimensions, see [Bd8, Bd15] and his paper with C. Hess [BH]). For the use of Young measures in PDE and Mechanics, see L.C. Evans [Ev], M. Chipot & D. Kinderlehrer [CK] and D. Kinderlehrer & P. Pedregal [KP1–2].

In Section 1 we present briefly and elementarily Young measures and show that some frightening notions of Measure Theory (images, weak convergence, disintegration) used in the theory of Young measures are rather natural. We give some examples of Young measures, specially of limit Young measures which are not associated to functions.

In Section 2 we develop the basic topological results of Young measures Theory. In 1989 we gave a first course [Va6] on Young Measures. Some technicalities of this course are avoided here and some results not in it are given here. Sometimes we refer to [Va6], but many results come from one of Balder's numerous papers [Bd1–18]. The exposition is limited to the \mathbb{R}^d case for simplicity.

In Section 3 we expound the Visintin-Balder theorem ([Vi],[Bd5]), which gives sufficient conditions under which weak convergence in L^1 implies strong convergence. Most of the properties of weakly convergent sequences in L^1 which are not strongly convergent must have been understood by C. Olech [O2] and L. Tartar [T1–2] many years ago. Particularly L. Tartar showed the usefulness of Young measures in this question. But the Visintin theorem [Vi] brought a new result and its proof using Young measures, due to E.J. Balder [Bd5], allows many extensions.

In Section 4 we show that the Young measures Theory permits to give a direct proof of the weak-strong lower semi-continuity theorem

which is fundamental in the Calculus of Variations.

1. Presentation.

INTRODUCTION.

Let r^n be the n -th Rademacher function on $\Omega := [0, 1]$, that is $r^n(x) = +1$ if $x \in [\frac{k}{2^n}, \frac{k+1}{2^n}[$ for any even k , $r^n(x) = -1$ otherwise (or $r^n(x) = \text{sign}[\sin(2^{n+1}\pi x)]$). Let w^n be the primitive of r^n null at 0. The functions w^n belong to the Sobolev space $W^{1,p}([0, 1])$ (for any $p \in [1, +\infty]$). Since w^n tends uniformly to $w^\infty \equiv 0$ and ∇w^n ($\nabla w^n := (w^n)'$) tends weakly¹ to $\nabla w^\infty \equiv 0$, one can say “ w^n tends to $w^\infty \equiv 0$.” But this is not satisfactory if one wants to retain something of the behavior of the gradients. It may happen that $(w^n)_n$ is a minimizing sequence of an optimal control problem and that w^∞ is not an optimal solution. A good limit would be w^∞ but with another gradient. In this line L.C. Young introduced generalized curves in 1937 [Y1] (see also [McS] Def.2.8 pp.515–516) and generalized surfaces in 1942 [Y2]. We give an example. Minimize

$$\int_0^1 [X(t)^2 + (1 - u(t)^2)] dt$$

where u is a measurable function from $[0, 1]$ to $[-1, 1]$ and X satisfies the differential equation $\frac{d}{dt}X(t) = u(t)$ with the initial condition $X(0) = 0$. The infimum is 0 but cannot be reached. The Rademacher functions form a minimizing sequence $(r^n)_n$. Another formulation of this problem is: minimize

$$\int_0^1 f(t, X(t), X'(t)) dt$$

with² $f(t, x, v) = x^2 + 1 - v^2 + \delta(v|_{[-1,1]})$, $X(0) = 0$. The non existence of an optimal solution is connected to the non convexity

1 Even for $\sigma(L^\infty, L^1)$, the strongest of all the weak topologies.

2 $\delta(\cdot|A)$ is the *indicator function* of A which takes the value 0 on A , $+\infty$ outside.

of f with respect to the velocity v (see A. Ioffe [I] and C. Olech [O3]). There exists an optimal “generalized control”: $t \mapsto \frac{1}{2}(\delta_1 + \delta_{-1})$ (note that the function $w^\infty \equiv 0$ of the beginning has, at every x , 0 as gradient and that 0 is the barycenter of $\frac{1}{2}(\delta_1 + \delta_{-1})$). The same phenomenon appears in Mechanics when the energy function is not quasi-convex in the sense of C.B. Morrey [Mo]. The material may appear in two phases (or more). Papers by J.L. Ericksen [Er], M. Chipot & D. Kinderlehrer [CK], I. Fonseca [F] treat crystals.

DEFINITIONS.

We turn now to a precise definition of Young measures. We forget the functions w^n and consider only integrable functions u^n (which can be gradients). Let Ω be an open (or a Borel) subset of \mathbb{R}^N , μ the Lebesgue measure on Ω . We assume $\mu(\Omega) < +\infty$. We will denote by $\mathcal{B}(\Omega)_\mu$ the σ -algebra of all Lebesgue measurable subsets of Ω .

DEFINITION. A *Young measure* on $\Omega \times \mathbb{R}^d$ is a positive measure τ on $\Omega \times \mathbb{R}^d$ such that for any Borel set $A \subset \Omega$, $\tau(A \times \mathbb{R}^d) = \mu(A)$. The set of all Young measures is denoted by $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$.

All results extend to abstract measured spaces in place of Ω and to rather general metric spaces in place of \mathbb{R}^d . About this second point, one must note that a large part of the results do not involve the linear structure of \mathbb{R}^d .

The formula $\tau(A \times \mathbb{R}^d) = \mu(A)$ means that μ is the image of τ by the projection map $(x, \xi) \mapsto x$. (Recall that if λ is a measure on (X, \mathcal{F}) and $\varphi : (X, \mathcal{F}) \rightarrow (X', \mathcal{F}')$ is a measurable map, the image of λ by φ is the measure on \mathcal{F}' , $\lambda \circ \varphi^{-1}$, that is $A' \mapsto \lambda(\varphi^{-1}(A'))$.)

DEFINITION. For any measurable function $u : \Omega \rightarrow \mathbb{R}^d$, the *Young measure ν associated to u* is the (unique) Young measure carried by the graph of u . Another definition of ν is: it is the image of μ by the map $x \mapsto (x, u(x))$, that is, for any Borel subsets A, B of respectively Ω and \mathbb{R}^d , $\nu(A \times B) = \mu(A \cap u^{-1}(B))$. And, for any $\psi : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$

measurable and ≥ 0 or ν -integrable, one has

$$\int_{\Omega \times \mathbb{R}^d} \psi \, d\nu = \int_{\Omega} \psi(x, u(x)) \, \mu(dx).$$

A natural property holds: $\nu^1 = \nu^2 \iff u^1 = u^2$ μ -a.e. (reference [Va6, p.155]).

The Young measure ν associated to u represents the amount of chalk (or ink) laid down when drawing the graph of u , but proportionally to the abscissa: $\nu(A \times \mathbb{R}^d) = \mu(A)$; see Figure 1. Maybe a better analogy is the following: imagine a black television screen and that white light appears along the graph of u with intensity ν obeying to $\nu(A \times \mathbb{R}^d) = \mu(A)$.

Figure 1 – Thickness according to $\nu(A \times \mathbb{R}^d) = \mu(A)$.

There exist Young measures non associated to functions. For example

$$\tau = \mu \otimes \left[\frac{1}{2}(\delta_1 + \delta_{-1}) \right]$$

on $[0, 1] \times \mathbb{R}$, which is also $\frac{1}{2}(\nu^1 + \nu^{-1})$ where ν^1 (resp. ν^{-1}) is associated to the constant function $u^1 \equiv 1$ (resp. $u^{-1} \equiv -1$). (For any Borel subset C of $[0, 1] \times \mathbb{R}$, $\tau(C)$ is half of the sum of the one dimensional measures of the intersections $C \cap ([0, 1] \times \{1\})$ and $C \cap ([0, 1] \times \{-1\})$.) It will be proved that τ is the limit (see a first definition of convergence below) of the Young measures associated to the Rademacher functions. Note that although $r^n(x) \in \{-1, 1\}$, there does not exist an a.e. convergent subsequence (indeed if such

a subsequence $(r^{n_k})_k$ would exist, it would be L^1 -convergent, hence L^1 -Cauchy, but $n \neq m \Rightarrow \|r^n - r^m\|_{L^1} = 1$.

The notion of weak convergence of Young measures is essential. It is called the narrow convergence. If $\Omega \times \mathbb{R}^d$ was the rectangle $[a, b] \times [c, d]$, this notion would be nothing else but convergence of images (recall that a sequence $(\lambda^n)_n$ of measures on a compact metric space K converges weakly to λ if, for any real continuous function f on K , $\int_K f d\lambda^n \rightarrow \int_K f d\lambda$). In the general setting (see Section 2), Carathéodory integrands are used.

ABOUT DISINTEGRATION.

It is very useful to describe a Young measure τ by its *disintegration* which is a family, $(\tau_x)_{x \in \Omega}$, of probabilities on \mathbb{R}^d , characterized by $\forall \psi : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ measurable and ≥ 0 or τ -integrable,

$$\int_{\Omega \times \mathbb{R}^d} \psi d\tau = \int_{\Omega} \left[\int_{\mathbb{R}^d} \psi(x, \xi) \tau_x(d\xi) \right] \mu(dx).$$

When ν is associated to u , one has $\nu_x = \delta_{u(x)}$, $\delta_{u(x)}$ denoting the Dirac mass at $u(x)$. Conversely when $(\tau_x)_{x \in \Omega}$ is a given family of probabilities on \mathbb{R}^d , a measure τ on $\Omega \times \mathbb{R}^d$ is defined by the formula (consider $C = A \times B$ is sufficient):

$$\tau(C) = \int_{\Omega} \tau_x(C_x) \mu(dx), \text{ where } C_x = \{\xi : (x, \xi) \in C\}$$

(this can also be written $\tau = \int_{\Omega} [\delta_x \otimes \tau_x] \mu(dx)$; for the integration of Radon measures see [Bol]). Then $(\tau_x)_{x \in \Omega}$ is the disintegration of τ .

A Young measure τ could represent a black and white photograph. Above any x there is a conditional distribution τ_x (which is a probability measure on \mathbb{R}^d). In some sense this corresponds to the scanning of the image before TV transmission (exchange vertical and horizontal). Then the television set builds the image line after line (here vertical line after vertical line) in accordance to the formula:

$$\tau = \int_{\Omega} [\delta_x \otimes \tau_x] \mu(dx) !$$

The measure τ and the family $(\tau_x)_{x \in \Omega}$ are two ways of description of the same image. The second way does not imply the existence of stochastic events or of a player having a random strategy. In Figure 2, above x the image is dark in (x, ξ) because there is somebody, and in (x, ξ') it is clear because there is the sky.

Figure 2

Y.G. Reshetnyak [Re] calls $(\tau_x)_{x \in \Omega}$ a “layerwise decomposition” of τ . For a short proof of the existence of disintegration see L.C. Evans book [Ev]. (See also references [Bo2], [C2], [CV, pp.216–218], [DM], [Du], [Ed], [HJ], [N], [Sc], [Va1–2], [Va6, Cor.A5 p.181].)

EXAMPLES.

In examples 1 and 2 below, $\Omega = [0, 1]$ and μ is the Lebesgue measure. They are particular cases of the following (see Th.4). Let u be a measurable periodic function on \mathbb{R} with period 1 and $u^n(x) = u(nx)$. Then the Young measures ν^n converge to a limit τ whose “disintegration” τ_x is constant. It is the image by u , λ , of the Lebesgue

measure μ on $[0, 1]$: for any real bounded Borel function f on \mathbb{R}^d

$$\int_{\mathbb{R}^d} f d\tau_x = \int_{\mathbb{R}^d} f d\lambda = \int_{[0,1]} f(u(x)) dx .$$

EXAMPLE 1. Let $\Omega = [0, 1]$, $d = 1$ and $u^n(x) = \sin nx$. Then ν^n converges to τ where τ is carried by $\Omega \times]-1, 1[$ and has the density (see Figure 3)

$$(x, \xi) \mapsto \frac{1}{\pi \sqrt{1 - \xi^2}} .$$

Figure 3 – Limit of $\sin nx$.

EXAMPLE 2. The Young measures associated to the Rademacher functions on $\Omega = [0, 1]$ converge to a limit τ which has not a density. Its disintegration is:

$$\tau_x = \frac{1}{2}(\delta_1 + \delta_{-1}) .$$

Figure 4 – Rademacher functions (here $n = 3$) and their limit.
The two dotted lines converge to grey ones.

ABOUT RELAXATION.

At the beginning of this Section we have considered the control problem: minimize

$$\int_0^1 [X(t)^2 + (1 - u(t)^2)] dt$$

where u is a measurable function from $[0, 1]$ to $[-1, 1]$ and X satisfies the differential equation $\frac{d}{dt}X(t) = u(t)$ with the initial condition $X(0) = 0$. J. Warga [W], A. Ghouila-Houri [GH] and some others (see the rather “à la Bourbaki” definition of P. Michel [Mi]) introduced generalized controls which are nothing else but Young measures. Here there exists an optimal generalized control: $t \mapsto \frac{1}{2}(\delta_1 + \delta_{-1})$. Relaxation in the Calculus of Variations and the Optimal Control Theory have a very long story (see references [Bd2, 6–7], [BL], [Bu], [Da1–2], [Ek], [ET], [Gm], [GH], [ITi], [McS], [Va3–4], [W], [Y1, Y4]). In my opinion, in most problems, the direct study of the integral representation of the lower semi-continuous hull of the original functional is the best (as in G. Buttazzo’s book [Bu]). But the efficiency of Young Measures in the proof of the fundamental theorem of the Calculus of Variations will appear in Section 4

(Th.12) (see also the proof by C. Castaing of the weak compactness of $\{v \in L^1(\Omega, \mu; E) : v(x) \in K \text{ a.e.}\}$, where E is a Banach space and K a convex weak compact set [CV, Th.V.2 pp.126–127 and Coroll.V.4 p.130]). And they are used in problems where the lacking property is not convexity but quasi-convexity (see Comment 2 before the proof of Th.12 and Comment 1 after its proof).

2. Topological Properties.

THE NARROW TOPOLOGY.

DEFINITION. The *narrow topology* on $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$ is the weakest topology for which the maps $\tau \mapsto \int_{\Omega \times \mathbb{R}^d} \varphi d\tau$ are continuous, where φ runs through the set $\mathcal{C}th^b(\Omega; \mathbb{R}^d)$ of all bounded Carathéodory integrands³ on $\Omega \times \mathbb{R}^d$.

COMMENT. This topology is Hausdorff [Va6, 2b page 180]. In [BL] H. Berliocchi & J.M. Lasry used (when Ω is a locally compact space) integrands which are continuous functions on $\Omega \times \mathbb{R}^d$. Then they went to Carathéodory integrands using the Scorza-Dragoni theorem. The fact that Carathéodory integrands are more directly applicable appears in the following proposition.

PROPOSITION 1. *If ν^n and ν^∞ are the Young measures associated to the measurable functions u^n and u^∞ , then*

$$\nu^n \rightarrow \nu^\infty \text{ narrowly} \iff u^n \rightarrow u^\infty \text{ in measure.}$$

Proof. 1) First suppose $u^n \rightarrow u^\infty$ in measure and $\nu^n \not\rightarrow \nu^\infty$. Then there exist $\varepsilon > 0$, a bounded Carathéodory integrand φ and infinitely many n such that

$$\left| \int_{\Omega \times \mathbb{R}^d} \varphi d\nu^n - \int_{\Omega \times \mathbb{R}^d} \varphi d\nu^\infty \right| > \varepsilon.$$

³ A Carathéodory integrand is a function which is $\mathcal{B}(\Omega)_\mu \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable in (x, ξ) and continuous in ξ .

Extracting a subsequence one may assume that this holds for every n and that $u^n \rightarrow u^\infty$ almost everywhere. Then, since

$$\int_{\Omega \times \mathbb{R}^d} \varphi \, d\nu^n = \int_{\Omega} \varphi(x, u^n(x)) \, \mu(dx),$$

the Lebesgue dominated convergence theorem gives a contradiction.

2) For the converse implication it is sufficient to use the integrand

$$\varphi(x, \xi) := \min(1, \|\xi - u^\infty(x)\|).$$

For any $\varepsilon \in]0, 1[$, $\mu(\{x \in \Omega : \|u^n(x) - u^\infty(x)\| \geq \varepsilon\})$ is $\leq (\varepsilon)^{-1} \int_{\Omega \times \mathbb{R}^d} \varphi \, d\nu^n$ which tends to 0. ◇

LEMMA 2. Let \mathcal{T} be a topology on $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$ and \mathcal{G} a class of integrands ⁴ with ≥ 0 values such that $\forall \psi \in \mathcal{G}, \tau \mapsto \int_{\Omega \times \mathbb{R}^d} \psi \, d\tau$ is l.s.c. for \mathcal{T} . Let \mathcal{V} be a linear space of integrands such that $\forall \psi \in \mathcal{V}, \exists \alpha \in \mathcal{L}_+^1(\Omega, \mu)$ such that $(x, \xi) \mapsto \psi(x, \xi) + \alpha(x)$ belongs to \mathcal{G} . Then $\forall \psi \in \mathcal{V}, \tau \mapsto \int_{\Omega \times \mathbb{R}^d} \psi \, d\tau$ is finite valued and \mathcal{T} -continuous.

Proof. Let $\psi \in \mathcal{V}$ and $\alpha \in \mathcal{L}_+^1(\Omega, \mu)$ such that $(x, \xi) \mapsto \psi(x, \xi) + \alpha(x)$ belongs to \mathcal{G} . Then $\int_{\Omega \times \mathbb{R}^d} \psi \, d\tau = \int_{\Omega \times \mathbb{R}^d} (\psi + \alpha) \, d\tau - \int_{\Omega} \alpha \, d\mu$ is a \mathcal{T} -l.s.c. and $]-\infty, +\infty[$ -valued function of τ . But, since \mathcal{V} is a vector space, the same is true for $-\psi$, hence $\int_{\Omega \times \mathbb{R}^d} \psi \, d\tau$ is also a \mathcal{T} -u.s.c. and $[-\infty, +\infty[$ -valued function of τ . ◇

REMARK. The version without parameter is: let \mathcal{T} be a topology on $\text{Prob}(\mathbb{R}^d)$ and \mathcal{I} a class of Borel ≥ 0 functions such that $\forall f \in \mathcal{I}, \lambda \mapsto \int \mathbb{R}^d f \, d\lambda$ is l.s.c. for \mathcal{T} . Let \mathcal{U} be a linear space of functions such that $\forall g \in \mathcal{U}, \exists \alpha \in [0, +\infty[$ such that $g + \alpha$ belongs to \mathcal{I} . Then $\forall g \in \mathcal{U}, \lambda \mapsto \int_{\mathbb{R}^d} g \, d\lambda$ is finite valued and \mathcal{T} -continuous.

THEOREM 3. 1) Let \mathcal{A} be an algebra of subsets of Ω generating $\mathcal{B}(\Omega)$. In the definition of the narrow topology on the set $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$ of Young measures, the bounded Carathéodory integrands can be re-

⁴ For us integrands are always $\mathcal{B}(\Omega)_\mu \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable functions of the couple (x, ξ) .

placed without changing the topology by the integrands:

$$\varphi(x, \xi) = \sum_{i=1}^n 1_{A_i}(x) f_i(\xi) \quad (1)$$

where f_i is the restriction to \mathbb{R}^d of a function of $\mathcal{C}(\widehat{\mathbb{R}^d})$ ⁵ and (A_1, \dots, A_n) is an \mathcal{A} -partition of Ω .

2) When Ω is an open subset of \mathbb{R}^N (more generally a locally compact Polish space), the topology remains the same replacing $\mathcal{C}th^b(\Omega; \mathbb{R}^d)$ by $\mathcal{C}_c(\Omega \times \mathbb{R}^d)$, the space of continuous functions on $\Omega \times \mathbb{R}^d$ with compact supports.

COMMENT. Such an algebra \mathcal{A} will be useful in the proofs of Th.4 and Prop.8. The space $\mathcal{C}(\widehat{\mathbb{R}^d})$ is useful to embed $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$ in the dual of $L^1(\Omega, \mu; \mathcal{C}(\widehat{\mathbb{R}^d}))$. This is used again in the proof of Th.7 (Prohorov's theorem).

Proof. 1) First recall how one can prove that the narrow topology on the set $\text{Prob}(\mathbb{R}^d)$ of all probabilities on \mathbb{R}^d coincide with the topology \mathcal{T} defined as the weakest topology making the maps (g belonging to $\mathcal{C}(\widehat{\mathbb{R}^d})$)

$$\lambda \mapsto \int_{\mathbb{R}^d} [g|_{\mathbb{R}^d}] d\lambda$$

continuous (here $g|_{\mathbb{R}^d}$ denotes the restriction of g to \mathbb{R}^d). See Bourbaki [Bo3] or Dellacherie-Meyer [DM] for the general case of completely regular spaces. In our case, or more generally for metrizable spaces, this relies on the following. If $f \in \mathcal{C}^b(\mathbb{R}^d)$ is ≥ 0 , set for $\xi \in \widehat{\mathbb{R}^d}$,

$$g_n(\xi) = \inf \{ f(\zeta) + n d(\zeta, \xi) : \zeta \in \mathbb{R}^d \}$$

(here d is a compatible metric on $\widehat{\mathbb{R}^d}$, hence d is bounded). Then g_n is ≥ 0 , n -lipschitzian and bounded on $\widehat{\mathbb{R}^d}$. The sequence $(g_n)_n$ is increasing in n and pointwise convergent to f on \mathbb{R}^d . The maps

$$\lambda \mapsto \int_{\mathbb{R}^d} [g_n|_{\mathbb{R}^d}] d\lambda$$

⁵ Here $\widehat{\mathbb{R}^d}$ denotes the Alexandrov one point compactification of \mathbb{R}^d ; $\mathcal{C}(\widehat{\mathbb{R}^d})$ denotes the Banach space of real continuous functions on $\widehat{\mathbb{R}^d}$.

are \mathcal{T} -continuous. By monotone convergence

$$\lambda \mapsto \int_{\mathbb{R}^d} f \, d\lambda$$

is \mathcal{T} -l.s.c. Thus Lemma 2 (its version without parameter) implies its \mathcal{T} -continuity for any $f \in \mathcal{C}^b(\mathbb{R}^d)$.

This extends to $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$ and the restrictions to $\Omega \times \mathbb{R}^d$ of bounded Carathéodory integrands on $\Omega \times \widehat{\mathbb{R}^d}$, thanks to the following formula (here ψ is a ≥ 0 bounded Carathéodory integrand on $\Omega \times \mathbb{R}^d$):

$$\psi_n(x, \xi) = \inf \{ \psi(x, \zeta) + n \, d(\zeta, \xi) : \zeta \in \mathbb{R}^d \}.$$

Then ψ_n is ≥ 0 , n -lipschitzean in ξ and bounded on $\Omega \times \widehat{\mathbb{R}^d}$. The sequence $(\psi_n)_n$ is increasing in n and pointwise convergent to ψ on $\Omega \times \mathbb{R}^d$. Moreover ψ_n is measurable because for any $\xi \in \widehat{\mathbb{R}^d}$ and any $\alpha \in \mathbb{R}$,

$$\{x \in \Omega : \psi_n(x, \xi) < \alpha\} = \text{pr}_\Omega \{ (x, \zeta) : \psi(x, \zeta) + n \, d(\zeta, \xi) < \alpha \}.$$

By the von-Neumann-Aumann-Sainte-Beuve projection theorem ([SB], [CV, Th.III. 23]) the latter set belongs to $\mathcal{B}(\Omega)_\mu$. Since ψ_n is separately measurable and continuous, by Lemma III.14 of [CV], it is globally measurable.

2) The Carathéodory integrands of $\mathcal{C}th^b(\Omega; \widehat{\mathbb{R}^d})$ may be identified with some elements of $L^1(\Omega, \mu; \mathcal{C}(\widehat{\mathbb{R}^d}))$ ⁶. Then $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$, which is a subset of $\mathcal{Y}(\Omega, \mu; \widehat{\mathbb{R}^d})$, may be identified with a subset of the dual of $L^1(\Omega, \mu; \mathcal{C}(\widehat{\mathbb{R}^d}))$ (for an integral representation of elements of this dual see [ITu]). The set of linear forms on $L^1(\Omega, \mu; \mathcal{C}(\widehat{\mathbb{R}^d}))$ defined by the elements τ of $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$:

$$\psi \mapsto \int_{\Omega \times \mathbb{R}^d} \psi \, d\tau,$$

is equicontinuous. The subset of elements of $L^1(\Omega, \mu; \mathcal{C}(\widehat{\mathbb{R}^d}))$ defined by (1) is dense in the L^1 -norm. So by a classical result the weakest

⁶ The elements of $\mathcal{L}^1(\Omega, \mu; \mathcal{C}(\widehat{\mathbb{R}^d}))$ correspond to “integrable” Carathéodory integrands, that is Carathéodory integrands ψ such that $\exists \alpha \in \mathcal{L}^1_+$ such that $\forall(x, \xi), |\psi(x, \xi)| \leq \alpha(x)$.

topology making the maps $\tau \mapsto \int_{\Omega \times \mathbb{R}^d} \psi d\tau$ continuous (ψ defined by (1)) coincides with the narrow topology.

3) Finally when μ is a measure on the locally compact Polish space Ω , the fact that the topology on $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$ defined by $\mathcal{C}_c(\Omega \times \mathbb{R}^d)$ coincides with the narrow topology has been proved by Berliocchi-Lasry [BL]. This can be explained as follows. From above the narrow topology is the weakest topology making the maps

$$\tau \mapsto \int_{\Omega \times \mathbb{R}^d} 1_A(x) f(\xi) \tau(d(x, \xi)) \quad (2)$$

continuous, where A is a Borel subset of Ω and $f \in \mathcal{C}^b(\mathbb{R}^d)$. By equicontinuity and denseness in $L^1(\Omega, \mu)$ of $\mathcal{C}_c(\Omega)$ (for the L^1 -norm), one may replace 1_A by φ , where $\varphi \in \mathcal{C}_c(\Omega)$, $\varphi \geq 0$: indeed there exists a sequence $(\varphi_n)_n$ convergent to 1_A . If the maps

$$\tau \mapsto \int_{\Omega \times \mathbb{R}^d} \varphi_n(x) f(\xi) \tau(d(x, \xi)) \quad (2n)$$

are continuous, (2) is continuous as the uniform limit of the (2n) (one has

$$\left| \int_{\Omega \times \mathbb{R}^d} \varphi_n f d\tau - \int_{\Omega \times \mathbb{R}^d} 1_A f d\tau \right| \leq \|f\|_\infty \int_{\Omega} |\varphi_n(x) - 1_A(x)| \mu(dx).$$

Now we prove that we may replace f by $g \in \mathcal{C}_c(\mathbb{R}^d)$. Let $\varphi \in \mathcal{C}_c(\Omega)$, $\varphi \geq 0$. Since f is minorized by a constant, we may assume $f \geq 0$. Let $(K_n)_n$ be a sequence of compact subsets of \mathbb{R}^d increasingly convergent to \mathbb{R}^d (for example the closed balls $\bar{B}(0, n)$). Set

$$g_n(\xi) = f(\xi)[1 - d(\xi, K_n)]^+.$$

Then g_n belongs to $\mathcal{C}_c(\mathbb{R}^d)$. If the maps

$$\tau \mapsto \int_{\Omega \times \mathbb{R}^d} \psi(x, \xi) \tau(d(x, \xi)),$$

where $\psi \in \mathcal{C}_c(\Omega \times \mathbb{R}^d)$, are continuous, then, as a consequence, so are the

$$\tau \mapsto \int_{\Omega \times \mathbb{R}^d} \varphi(x) g_n(\xi) \tau(d(x, \xi)),$$

and

$$\tau \mapsto \int_{\Omega \times \mathbb{R}^d} \varphi(x) f(\xi) \tau(d(x, \xi))$$

is l.s.c. Then, as in Lemma 2,

$$\tau \mapsto \int_{\Omega \times \mathbb{R}^d} \varphi(x) f(\xi) \tau(d(x, \xi))$$

is continuous for any $f \in \mathcal{C}^b(\mathbb{R}^d)$. ◇

PERIODIC FUNCTIONS.

THEOREM 4. *Let u be a measurable \mathbb{R}^d -valued periodic function on \mathbb{R} with period 1 and set $u^n(x) = u(nx)$. Then the Young measures ν^n (which belong to $\mathcal{Y}([0, 1], \mu; \mathbb{R}^d)$) associated to u^n converge to a limit τ whose disintegration τ_x is constant. This measure λ is the image by u of the Lebesgue measure μ on $[0, 1]$: for any real bounded Borel function f on \mathbb{R}^d*

$$\int_{\mathbb{R}^d} f d\lambda = \int_{[0,1]} f(u(x)) dx .$$

Hence $\tau = \mu \otimes \lambda$ and, as soon as u is not a.e. constant, λ is not a Dirac measure and τ is not associated to a function.

Proof. Let, for each $p \in \mathbb{N}$, \mathcal{A}_p denote the finite algebra of subsets of $[0, 1[$ generated by the intervals $\left[\frac{k}{2^p}, \frac{k+1}{2^p} \right[$ ($0 \leq k < 2^p$). The union of the \mathcal{A}_p generates the Borel tribe of $[0, 1[$. So, by Theorem 3, it is sufficient to prove the convergence

$$\int_{[0,1] \times \mathbb{R}^d} \varphi d\nu^n \rightarrow \int_{[0,1] \times \mathbb{R}^d} \varphi d\tau$$

when φ has the form $\varphi(x, \xi) = 1_A(x) f(\xi)$ where $A = \left[\frac{k}{2^p}, \frac{k+1}{2^p} \right[$ and f is bounded continuous on \mathbb{R}^d . Let $\llbracket r \rrbracket$ denote the integer part of $r \in [0, +\infty[$. When n is large, u^n has approximately $\llbracket (1/2^p)/(1/n) \rrbracket \simeq$

$n/2^p$ periods over A . Hence

$$\begin{aligned}
 \int_{[0,1] \times \mathbb{R}^d} \varphi \, d\nu^n &\simeq (n/2^p) \int_0^{1/n} f(u^n(x)) \, dx \\
 &= (n/2^p) \int_0^{1/n} f(u(nx)) \, dx \\
 &= (1/2^p) \int_0^1 f(u(s)) \, ds \\
 &= \mu(A) \int_0^1 f(u(s)) \, ds \\
 &= \mu(A) \int_{\mathbb{R}^d} f \, d\lambda \\
 &= \int_{[0,1] \times \mathbb{R}^d} \varphi \, d\tau.
 \end{aligned}$$

Finally $\tau = \mu \otimes \lambda$ is carried by a graph if and only if λ is a Dirac mass, say δ_a . But this happens if and only if $u(x) = a$ almost everywhere. \diamond

LOWER SEMI-CONTINUITY RESULTS.

LEMMA 5. *Let τ^n ($n \in \mathbb{N} \cup \{\infty\}$) be Young measures satisfying $\tau^n \rightarrow \tau^\infty$ narrowly. Let $\psi : \Omega \times \mathbb{R}^d \rightarrow [0, +\infty]$ be measurable in (x, ξ) and l.s.c. in ξ . Then*

$$\int_{\Omega \times \mathbb{R}^d} \psi \, d\tau^\infty \leq \liminf_{n \rightarrow \infty} \int_{\Omega \times \mathbb{R}^d} \psi \, d\tau^n.$$

More generally $\tau \mapsto \int_{\Omega \times \mathbb{R}^d} \psi \, d\tau$ is narrowly l.s.c.

Proof. Set

$$\psi_n(x, \xi) = \min(n, \inf\{\psi(x, \zeta) + n \, d(\zeta, \xi) : \zeta \in \mathbb{R}^d\}),$$

where d is a compatible metric on \mathbb{R}^d , for example the euclidean metric. Then ψ_n is bounded measurable in (x, ξ) (see the argument at the end of Part 1 of the proof of Th.3), continuous in ξ (even n -lipschitzean) and increasingly convergent to ψ . Since each map

$$\tau \mapsto \int_{\Omega \times \mathbb{R}^d} \psi_n \, d\tau$$

is narrowly continuous, the result follows from the monotone convergence theorem. \diamond

THEOREM 6. *Let $(u^n)_n$ be a sequence of measurable functions from Ω to \mathbb{R}^d , ν^n the associated Young measures and suppose that $\nu^n \rightarrow \tau$ narrowly where τ is some Young measure. Let $\psi : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ be measurable in (x, ξ) and l.s.c. in ξ . Assume that the sequence of negative parts $(\psi(\cdot, u^n(\cdot)))^-_n$ is uniformly integrable. Then*

$$\int_{\Omega \times \mathbb{R}^d} \psi \, d\tau \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \psi(x, u^n(x)) \, \mu(dx).$$

Moreover the right-hand side member belongs to $[-\infty, +\infty]$ and

$$\liminf_{n \rightarrow \infty} \int_{\Omega} \psi(x, u^n(x)) \, \mu(dx) < +\infty \implies \int_{\Omega \times \mathbb{R}^d} \psi^+ \, d\tau < +\infty.$$

COMMENTS. 1) In applications $(u^n)_n$ is often a weakly convergent sequence of functions in $L^1(\Omega, \mu; \mathbb{R}^d)$. Then the negative parts $(\psi(\cdot, u^n(\cdot)))^-$ are UI as soon as a minoration hypothesis such as

$$\psi(x, \xi) \geq \alpha(x) - b\|\xi\|, \text{ where } \alpha \in \mathcal{L}^1 \text{ and } b \in [0, +\infty],$$

is assumed. This holds when $\psi(x, \xi) = \langle p(x), \xi \rangle$ for $p \in \mathcal{L}^\infty(\Omega, \mu; \mathbb{R}^d)$.

2) Note that E.J. Balder has used in [Bd16] and some other papers the Komlós theorem [K] to get this type of results.

3) For examples showing the utility of the uniform integrability of the negative parts and that $\int_{\Omega \times \mathbb{R}^d} \psi^+ \, d\tau = \int_{\Omega \times \mathbb{R}^d} \psi^- \, d\tau = +\infty$ is possible, see after the proof.

4) Theorem 6 admits some nicely formulated mathematical consequences (see J.M. Ball [Ba], [Va6, Th.17 and Cor.18 pp.167–168]), which are in the spirit of Part 1a of the proof of Th.9:

THEOREM. *Let $(u^n)_n$ be a sequence of measurable functions from Ω to \mathbb{R}^d , ν^n the associated Young measures and suppose $\nu^n \rightarrow \tau$.*

1) *Let $\psi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable in (x, ξ) and continuous in ξ . Assume that the sequence of functions $(\psi(\cdot, u^n(\cdot)))_n$ is uniformly*

integrable. Then

$$\int_{\Omega} \psi(x, u^n(x)) \mu(dx) \rightarrow \int_{\Omega \times \mathbb{R}^d} \psi d\tau.$$

2) Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be continuous. Assume that the sequence $(\varphi \circ u^n)_n$ is uniformly integrable. Then

$$\int_{\Omega} \left[\int_{\mathbb{R}^d} |\varphi(\xi)| \tau_x(d\xi) \right] \mu(dx) < +\infty \text{ } \mu\text{-a.e.},$$

and $\varphi \circ u^n$ $\sigma(L^1, L^\infty)$ -converges to v , where

$$v(x) := \int_{\mathbb{R}^d} \varphi(\xi) \tau_x(d\xi) \text{ } \mu\text{-a.e.}$$

To illustrate Part 2, consider $u^n = r^n$ or $u^n(x) = \sin(nx)$ and $\varphi(\xi) = \xi^2$. More generally the images of a weakly convergent sequence of functions by a non-linear map do not converge weakly. For another example of the bad behavior of non-linear maps (even bilinear) see the comment after the statement of Lemma 11. When all the functions u^n take their values in a same compact subset of \mathbb{R}^d (as in B. Dacorogna [Da1, Th.6.1 p.52] and L. Tartar [Ta1]), uniform integrability is automatic.

Proof of Theorem 6. If ψ was ≥ 0 , the result would be a particular case of Lemma 5 applied to the Young measures associated to the u^n . A technical trick is the introduction, for $r \in [0, +\infty[$, of the positive integrand $\psi_r = \sup(-r, \psi) + r$. Let also ψ_r^0 denote $\sup(-r, \psi)$. Since ψ_r is ≥ 0 , Lemma 5 implies

$$\int_{\Omega \times \mathbb{R}^d} \psi_r d\tau \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \psi_r(x, u^n(x)) \mu(dx).$$

Subtracting $\int_{\Omega \times \mathbb{R}^d} r d\tau = \int_{\Omega} r d\mu$, one gets

$$\int_{\Omega \times \mathbb{R}^d} \psi_r^0 d\tau \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \psi_r^0(x, u^n(x)) \mu(dx).$$

Now let $A_{nr} = \{x : \psi(x, u^n(x)) < -r\}$. For any n ,

$$\int_{A_{nr}} \psi(x, u^n(x)) \mu(dx)$$

is ≤ 0 and, for r large enough, is $\geq -\varepsilon$. Note that for $r = 0$, $\int_{A_{nr}} \psi(x, u^n(x)) \mu(dx)$ is $\geq -M$, where M is a bound of the L^1 -norm of the negative parts $(\psi(\cdot, u^n(\cdot)))^-$. Then

$$\begin{aligned} \int_{\Omega} \psi(x, u^n(x)) \mu(dx) &= \int_{A_{nr}} \psi(x, u^n(x)) \mu(dx) + \int_{\Omega \setminus A_{nr}} \psi_r^0(x, u^n(x)) \mu(dx) \\ &= \int_{A_{nr}} \psi(x, u^n(x)) \mu(dx) + \int_{\Omega} \psi_r^0(x, u^n(x)) \mu(dx) - \\ &\quad - \int_{A_{nr}} \psi_r^0(x, u^n(x)) \mu(dx) \\ &\geq \int_{A_{nr}} \psi(x, u^n(x)) \mu(dx) + \int_{\Omega} \psi_r^0(x, u^n(x)) \mu(dx) \\ &\geq \int_{\Omega} \psi_r^0(x, u^n(x)) \mu(dx) - \varepsilon. \end{aligned}$$

Note that for $r = 0$, $\psi_r^0 = \psi^+$, hence

$$\int_{\Omega} \psi(x, u^n(x)) \mu(dx) \geq \int_{\Omega} \psi^+(x, u^n(x)) \mu(dx) - M,$$

and

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^d} \psi^+ d\tau &\leq \liminf_{n \rightarrow \infty} \int_{\Omega} \psi^+(x, u^n(x)) \mu(dx) \\ &\leq \liminf_{n \rightarrow \infty} \int_{\Omega} \psi(x, u^n(x)) \mu(dx) + M. \end{aligned}$$

This proves the last precision in the statement. Now return to r corresponding to $\varepsilon > 0$. One has

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega} \psi(x, u^n(x)) \mu(dx) &\geq \liminf_{n \rightarrow \infty} \int_{\Omega} \psi_r^0(x, u^n(x)) \mu(dx) - \varepsilon \\ &\geq \int_{\Omega \times \mathbb{R}^d} \psi_r^0 d\tau - \varepsilon. \end{aligned}$$

Since $\psi \leq \psi_r^0$, one has

$$\liminf_{n \rightarrow \infty} \int_{\Omega} \psi(x, u^n(x)) \mu(dx) \geq \int_{\Omega \times \mathbb{R}^d} \psi d\tau - \varepsilon,$$

and the statement follows from the fact that ε is arbitrarily small. \diamond

EXAMPLE 1. Even without parameter the negative parts must be controlled. Let $d = 1$, $f(\xi) = -\xi^2$ and, for $n \geq 1$, $\theta^n = (1 - \frac{1}{n})\delta_0 + \frac{1}{n}\delta_n$ which converges to $\theta^\infty := \delta_0$. Then

$$\int_{\mathbb{R}} f d\theta^n = -n, \text{ but } \int_{\mathbb{R}} f d\theta^\infty = 0.$$

EXAMPLE 2. Even if for all $n \in \mathbb{N}$, $\int_{\Omega} \psi(x, u^n(x)) \mu(dx) \in \mathbb{R}$ and if the negative parts $\psi(\cdot, u^n(\cdot))^-$ are uniformly integrable, one may have

$$\int_{\Omega \times \mathbb{R}^d} \psi^+ d\tau = \int_{\Omega \times \mathbb{R}^d} \psi^- d\tau = +\infty.$$

Let $\Omega = [0, 1]$, μ the Lebesgue measure, $d = 1$ and $\psi(x, \xi) = -\infty$ if $\xi \in [-\infty, -1]$, 0 if $\xi \in]-1, 0]$, $\frac{1}{\xi}$ if $\xi \in]0, +\infty[$. Let $u^n(x) = x$ on $[\frac{1}{n}, \frac{1}{2}]$, $= -1 + \frac{1}{n}$ elsewhere. Then u^n converges in measure to u^∞ where $u^\infty(x) = x$ on $[0, \frac{1}{2}]$, $= -1$ on $[\frac{1}{2}, 1]$. Then (note that the negative parts $(\psi(\cdot, u^n(\cdot)))^-$ are null)

$$\int_{[0,1]} \psi(x, u^n(x)) dx = \log n - \log 2 \rightarrow +\infty,$$

but, with $\tau = \nu^\infty$,

$$\int_{\Omega \times \mathbb{R}^d} \psi^+ d\tau = \int_{\Omega \times \mathbb{R}^d} \psi^- d\tau = +\infty.$$

PROHOROV'S THEOREM.

THEOREM 7. (Prohorov with parameter). *Let $(u^n)_n$ be a norm-bounded sequence in $L^1(\Omega, \mu; \mathbb{R}^d)$. There exist a strictly increasing sequence $(n_k)_k$ and a Young measure τ such that $\nu^{n_k} \rightarrow \tau$ narrowly.*

COMMENTS. 1) This compactness result extends to tight subsets of the set of Young measures. A subset \mathcal{H} of $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$ is *tight* if $\forall \varepsilon > 0, \exists K$ compact, $K \subset \mathbb{R}^d$, such that

$$\forall \tau \in \mathcal{H}, \tau(\Omega \times (\mathbb{R}^d \setminus K)) < \varepsilon.$$

2) The subset \mathcal{H} is tight if and only if [Va6, Prop.8 p.161] there exists an inf-compact function $h : \mathbb{R}^d \rightarrow [0, +\infty]$ such that

$$\sup_{\tau \in \mathcal{H}} \int_{\Omega \times \mathbb{R}^d} h(\xi) \tau(d(x, \xi)) < +\infty.$$

The converse holds: if \mathcal{H} is relatively narrowly compact, then \mathcal{H} is tight [Va6, Prop.10 p.162]. In [Bd2] E.J. Balder introduced an integrand in place of h , and A. Jawhar [J] proved that this is equivalent to the tightness notion extended to a multifunction in place of K (see the comments in [Va6, p.165]).

3) The classical Prohorov theorem says (without parameter): a set of probabilities on a completely regular topological space which is tight is relatively narrowly compact; the converse holds for Polish or locally compact space (Bourbaki [Bo3] or, for one implication, Dellacherie-Meyer [DM]) but not for \mathbb{Q} , see D. Preiss [P]. The tightness hypothesis avoid the loss of mass by escape to infinity.

Proof (ideas of the). Consider the set $\mathcal{Y}(\Omega, \mu; K)$ where K is a compact metric space. There the Carathéodory integrands may be identified with elements of $L^1(\Omega, \mu; \mathcal{C}(K))$ ($\mathcal{C}(K)$ denotes the Banach space of real continuous functions on K) and $\mathcal{Y}(\Omega, \mu; K)$ may be identified with a weak* closed bounded subset of the dual of $L^1(\Omega, \mu; \mathcal{C}(K))$ (recall that an integral representation of this dual is given in A. & C. Ionescu Tulcea's book [ITu]). So by the Alaoglu-Bourbaki theorem, $\mathcal{Y}(\Omega, \mu; K)$ is narrowly compact. Then the elements of $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$ can be considered as elements of $\mathcal{Y}(\Omega, \mu; K)$ where K is some compact metric over-space of \mathbb{R}^d ($K = \mathbb{R}^d$ is a possible choice). The boundedness of $(u^n)_n$ implies the tightness hypothesis (which prevents to go in $K \setminus \mathbb{R}^d$): there exists an inf-compact function $h : \mathbb{R}^d \rightarrow [0, +\infty]$ (here $h(\xi) = \|\xi\|$) and $M < +\infty$ such that

$$\forall n, \int_{\Omega \times \mathbb{R}^d} h(\xi) \nu^n(d(x, \xi)) \leq M.$$

Let \widehat{h} be the extension of h to K defined by $\widehat{h}(\xi) = +\infty$ if $\xi \notin \mathbb{R}^d$. If $\nu^{n_k} \rightarrow \tau$ in $\mathcal{Y}(\Omega, \mu; K)$, by Lemma 5, $\int_{\Omega \times K} \widehat{h}(\xi) \tau(d(x, \xi)) \leq M$, hence τ is carried by $\Omega \times \mathbb{R}^d$. By Theorem 3, τ is the narrow limit of $(\nu^{n_k})_k$ in $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$. \diamond

DENSENESS.

PROPOSITION 8. *The set of Young measures associated to functions is dense in the set of all Young measures $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$.*

COMMENT. This denseness result extends to the case when μ is an abstract nonatomic measure. For a first version, see L.C. Young [Y1, pp.226–228]). The following proof seems to be new.

Proof. We give the proof for $\Omega = [0, 1]$. Let $\tau \in \mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$, (ψ_1, \dots, ψ_n) be a finite sequence in $\mathcal{C}th^b(\Omega; \mathbb{R}^d)$ and $\varepsilon > 0$. We have to exhibit a measurable function u such that

$$\forall i \in \{1, \dots, n\}, \left| \int_{\Omega \times \mathbb{R}^d} \psi_i d\tau - \int_{\Omega \times \mathbb{R}^d} \psi_i d\nu \right| \leq \varepsilon. \quad (3)$$

Thanks to Theorem 3 we may suppose, as in the proof of Th.4, that for some p , each ψ_i does not depend on x on $\left[\frac{k}{2^p}, \frac{k+1}{2^p}\right] \times \mathbb{R}^d$, this for all k such that $0 \leq k < 2^p$. It is sufficient to work with the first interval $\left[0, \frac{1}{2^p}\right]$. Let $f_i \in \mathcal{C}^b(\mathbb{R}^d)$ be such that $\psi_i(x, \xi) = f_i(\xi)$ for every $x \in \left[0, \frac{1}{2^p}\right]$. Let $\bar{\tau}$ be the measure on \mathbb{R}^d of total mass $1/2^p$:

$$\bar{\tau} := \int_0^{1/2^p} \tau_x dx.$$

It is known (see P.R. Halmos [H, Th.C Sec.41 p.173] and R.M. Dudley [Du, Comment on §8.2 p.215]) that there exists a measurable function

$$u : \left[0, \frac{1}{2^p}\right] \rightarrow \mathbb{R}^d$$

for which the image of the Lebesgue measure of $\left[0, \frac{1}{2^p}\right]$ equals $\bar{\tau}$. Then one has

$$\forall i \in \{1, \dots, n\}, \int_{\mathbb{R}^d} f_i(\xi) \bar{\tau}(d\xi) = \int_0^{1/2^p} f_i(u(x)) dx.$$

So, repeating this for the intervals $\left[\frac{k}{2^p}, \frac{k+1}{2^p}\right]$ ($1 \leq k < 2^p$), and pasting the functions obtained to get a function u defined on $[0, 1]$, (3) is obtained with $\varepsilon = 0$. \diamond

3. Weak and Strong Convergence and Oscillations.

GENERAL OBSERVATIONS.

In $L^1(\Omega, \mu; \mathbb{R}^d)$, $u^n \rightarrow u^\infty$ strongly implies $u^n \rightharpoonup u^\infty$ weakly, that is

$$\forall p \in L^\infty, \int_{\Omega} \langle p, u^n - u^\infty \rangle d\mu \rightarrow 0.$$

But the converse does not hold. If $u^n \rightharpoonup u^\infty$ weakly, it is possible that u^n does not converge to u^∞ strongly.

EXAMPLES. Let $\Omega = [0, 1]$, $d = 1$ and $(r^n)_n$ be the sequence of Rademacher functions. One has $r^n \rightharpoonup 0$, but $\forall n$, $\|r^n\|_{L^1} = 1$. Another example is $u^n(x) = \sin(nx)$. Then u^n converges weakly to $u^\infty \equiv 0$, but not strongly since $\|u^n\|_{L^1} \rightarrow \frac{2}{\pi}$.

The following facts are useful. If $u^n \rightharpoonup u^\infty$, one can say:

1) $(u^n)_n$ is norm bounded in L^1 (this consequence of the Banach-Steinhaus theorem holds because the index set is \mathbb{N} , this is not valid for generalized sequences). (For an example of a strange generalized sequence, let H be an infinite dimensional Hilbert space, $(e_n)_{n \geq 1}$ an orthonormal sequence in H and $x_n = \sqrt{n} e_n$. Then there exist a directed ordered set I and a map $\varphi : I \rightarrow \mathbb{N}^*$ such that $(x_{\varphi(\alpha)})_{\alpha \in I}$ weakly converges to 0. But necessarily $\|x_{\varphi(\alpha)}\|$ tends to $+\infty$.)

2) since the set $\{u^n : n \in \mathbb{N} \cup \{\infty\}\}$ is weakly compact (this is not valid for generalized sequences), $(u^n)_n$ is uniformly integrable. This is stronger than 1). It is a consequence of the Dunford-Pettis theorem (see Dunford-Schwartz [DS, Th.IV.8.9 p.292]). For the extension of this part of Dunford-Pettis theorem to Banach spaces, see J.K. Brooks & N. Dinculeanu [BD] and J. Diestel [Di].

3) One knows (Lebesgue-Vitali's theorem [DS, Th.III.6.15 p.150]) that, if $(u^n)_n$ is uniformly integrable, its strong convergence is equivalent to its convergence in measure, that is (recall that we assume

$\mu(\Omega) < +\infty$):

$$\forall \varepsilon > 0, \mu(\{x \in \Omega : \|u^n(x) - u^\infty(x)\| \geq \varepsilon\}) \rightarrow 0.$$

PROPERTIES OF YOUNG MEASURES CONNECTED TO WEAK CONVERGENCE.

The following theorem gathers what Young Measures bring to the study of weak convergent sequences in $L^1(\Omega, \mu; \mathbb{R}^d)$. We enjoy particularly Part 2.

THEOREM 9. *Suppose $u^n \rightharpoonup u^\infty$ in $L^1(\Omega, \mu; \mathbb{R}^d)$.*

1) There exist a strictly increasing sequence $(n_k)_k$ and a Young measure τ such that $\nu^{n_k} \rightarrow \tau$. Then μ -a.e. the disintegration τ_x has a barycenter $\text{bar}(\tau_x)$, $u^\infty(x) = \text{bar}(\tau_x)$ μ -a.e. and

$$\|u^{n_k} - u^\infty\|_{L^1} \rightarrow \int_{\Omega \times \mathbb{R}^d} \|\xi - u^\infty(x)\| \tau(d(x, \xi)).$$

Moreover, if τ_x is a.e. a Dirac measure, then $\tau = \nu^\infty$ and $u^{n_k} \rightarrow u^\infty$ strongly.

2) If u^n does not converge strongly, there exist a sequence $(n_k)_k$ and a Young measure τ as in 1) above such that τ is not associated to a function.

3) $u^n \rightarrow u^\infty$ strongly $\iff \nu^n \rightarrow \nu^\infty$ narrowly.

REMARK. To see the necessity in Part 1 of the extraction of a subsequence, consider $u^n = r^n$ if n is even, $\equiv 0$ if n is odd. In spite of the weak convergence $u^n \rightharpoonup u^\infty$, the sequence $(\nu^n)_n$ may have several limit points.

Proof. 1) a) The sequence $(u^n)_n$ is bounded in $L^1(\Omega, \mu; \mathbb{R}^d)$. So by the Prohorov theorem (Th.7), there exist a strictly increasing sequence $(n_k)_k$ and a Young measure τ such that $\nu^{n_k} \rightarrow \tau$. Let M be a bound of the L^1 -norm of the u^n . Since $(x, \xi) \mapsto \|\xi\|$ is a ≥ 0

integrand l.s.c. in ξ , one has (Lemma 5 or Th.6)

$$\begin{aligned} \int_{\Omega} [\int_{\mathbb{R}^d} \|\xi\| \tau_x(d\xi)] \mu(dx) &= \int_{\Omega \times \mathbb{R}^d} \|\xi\| \tau(d(x, \xi)) \leq \\ &\leq \liminf_{k \rightarrow \infty} \int_{\Omega \times \mathbb{R}^d} \|\xi\| \nu^{n_k}(d(x, \xi)) = \liminf_{k \rightarrow \infty} \|u^{n_k}\|_{L^1} \leq M. \end{aligned}$$

Hence $\text{bar}(\tau_x)$ exists for almost every x . Now let $p \in L^\infty(\Omega, \mu; \mathbb{R}^d)$. Consider the integrand $\psi(x, \xi) = \langle p(x), \xi \rangle$. It is continuous in ξ . The negative parts

$$(\psi(x, u^n(x)))^- = \langle p(x), u^n(x) \rangle^-$$

are $\leq \|p\|_{L^\infty} \|u^n(x)\|$, hence uniformly integrable. By Theorem 6,

$$\int_{\Omega \times \mathbb{R}^d} \langle p(x), \xi \rangle \tau(d(x, \xi)) \leq \liminf_{k \rightarrow \infty} \int_{\Omega \times \mathbb{R}^d} \langle p(x), u^{n_k}(x) \rangle \mu(dx).$$

Since this is also valid for $-p$, one has

$$\int_{\Omega \times \mathbb{R}^d} \langle p(x), \xi \rangle \tau(d(x, \xi)) = \lim_{k \rightarrow \infty} \int_{\Omega \times \mathbb{R}^d} \langle p(x), u^{n_k}(x) \rangle \mu(dx).$$

As

$$\begin{aligned} \int_{\Omega} [\int_{\mathbb{R}^d} \langle p(x), \xi \rangle \tau_x(d\xi)] \mu(dx) &= \int_{\Omega} \left\langle p(x), \int_{\mathbb{R}^d} \xi \tau_x(d\xi) \right\rangle \mu(dx) \\ &= \int_{\Omega} \langle p(x), \text{bar}(\tau_x) \rangle \mu(dx), \end{aligned}$$

one has $u^\infty(x) = \text{bar}(\tau_x)$ a.e.

b) Now consider the integrand continuous in ξ , $\psi(x, \xi) = \|\xi - u^\infty(x)\|$. The sequences of negative parts $(\psi(\cdot, u^n(\cdot)))^- \equiv 0$ and $(-\psi(\cdot, u^n(\cdot)))^- = \|u^n(\cdot) - u^\infty(\cdot)\|$ are still uniformly integrable. Hence by application of Theorem 6, we get the formula of the statement.

c) If τ_x is a.e. a Dirac measure, then $\tau_x = \delta_{u^\infty(x)}$ and $\tau = \nu^\infty$. Moreover the formula of the statement implies

$$\|u^{n_k} - u^\infty\|_{L^1} \rightarrow 0.$$

2) Suppose that u^n does not converge strongly and that for every convergent subsequence $(\nu^{n_p})_p$ of the sequence $(\nu^n)_n$, its limit is associated to a function. Then, up to the extraction of a subsequence

one may assume $\forall n, \|u^n - u^\infty\|_{L^1} > \varepsilon$ where ε is > 0 . By Part 1 there exists a convergent subsequence $(\nu^{n_k})_k$. Its limit τ is associated to a function v , i.e. τ_x is a.e. a Dirac measure. By Part 1c, we get the contradiction $u^{n_k} \rightarrow u^\infty$.

3) This results from Proposition 1 and the Lebesgue-Vitali theorem. \diamond

COMMENTS ABOUT OSCILLATIONS.

When $u^n \rightharpoonup u^\infty$ and does not converge strongly, a limit measure of a subsequence $(\nu^{n_k})_k$ contains information about the asymptotic oscillatory behaviour of the subsequence $(u^{n_k})_k$ (recall that in applications the functions u^n are the gradients of functions belonging to some Sobolev space). This is specially meaningful in Mechanics when the energy function is not quasi-convex in the sense of C.B. Morrey [Mo] (see the Introduction of Section 1). Similar phenomena were already studied in Control Theory (when some convexity is lacking) under the name of Relaxation and L.C. Young was the first to speak of oscillations and to introduce measures to describe them (see [Y1, p.231]: “The direction of a sailing ship, or that of a mountain road, is... constantly oscillating...”). From Part 2 of Th.9, if $u^n \rightharpoonup u^\infty$, $u^n \not\rightarrow u^\infty$ there exists $\nu^{n_k} \rightarrow \tau$ with τ non associated to a function and there exists a Borel set ⁷ A with $\mu(A) > 0$ such that $\forall x \in A$, τ_x is not a Dirac measure. Then, above A , the functions u^{n_k} oscillate with “frequencies tending to $+\infty$ ” (it would be valuable to give this a more precise meaning). See Figure 5.

⁷ The biggest is $A = \{x \in \Omega : \int_{\mathbb{R}^d} \|\xi - u^\infty(x)\| \tau_x(d\xi) > 0\}$. This formula proves that A can be chosen measurable.

Figure 5

THE VISINTIN-BALDER THEOREM.

DEFINITION. Let C be a convex subset of a linear space and $y \in C$. The point y is an *extreme point* of C if y is not barycenter in a non trivial way of two points of C .

Equivalently “there does not exist a linear segment S contained in C such that y is a relative interior point of S ” or “there does not exist $y_1 \neq y_2$ in C such that $y = \frac{1}{2}(y_1 + y_2)$.”

When E is a normed linear vector space, an extreme point is a boundary point but not all boundary points are extremal.

NOTATION. $\partial_{\text{ext}}C$ for the set of extreme points of C .

NOTATION. For a sequence $(y_n)_n$ in \mathbb{R}^d , $\text{Ls}(y_n)$ denotes the set of limit points of the sequence. It is the *Painlevé-Kuratowski limit sup* of the sequence of singletons $(\{y_n\})_n$ (see [At]).

THEOREM 10 (Visintin-Balder). *Let u^n ($n \in \mathbb{N} \cup \{\infty\}$) be a sequence of functions in $L^1(\Omega, \mu; \mathbb{R}^d)$ such that $u^n \rightharpoonup u^\infty$ weakly. Consider the hypotheses:*

(H1 - Visintin's hypothesis) for every x there exists a closed convex subset of \mathbb{R}^d , $\Gamma(x)$, such that $\forall n$, μ -a.e. $u^n(x) \in \Gamma(x)$ and μ -a.e. $u^\infty(x) \in \partial_{\text{ext}}\Gamma(x)$.

(H2) for every x there exists a closed convex subset of \mathbb{R}^d , $\Gamma(x)$, such that $d(u^n(x), \Gamma(x)) \rightarrow 0$ in measure and μ -a.e. $u^\infty(x) \in \partial_{\text{ext}}\Gamma(x)$.

(H3 - Balder's hypothesis) μ -a.e. $u^\infty(x) \in \partial_{\text{ext}}\overline{\text{co}}(Ls(u^n(x)))$. Then one has always μ -a.e. $u^\infty(x) \in \overline{\text{co}}(Ls(u^n(x)))$ and, under the hypothesis "there exist $\Gamma(x)$ closed convex subsets of \mathbb{R}^d , such that $\forall n$, μ -a.e. $u^n(x) \in \Gamma(x)$," one has μ -a.e. $u^\infty(x) \in \Gamma(x)$. Moreover

$$(H1) \implies (H3) \implies u^n \rightarrow u^\infty \text{ strongly}$$

and

$$(H2) \implies u^n \rightarrow u^\infty \text{ strongly.}$$

COMMENT. This theorem comes from Visintin and Balder works [Vi], [Bd5]. It gives only sufficient conditions under which the weak convergence implies the strong convergence (see the examples after the proof). For a necessary and sufficient condition see M. Girardi [Gi1-2], [Va8], Balder-Girardi-Jalby [BGJ], V. Jalby [Jb]. The following statement is proved in [Va8]:

THEOREM. Suppose $u^n \rightharpoonup u$ in $L^1(\Omega, \mu; \mathbb{R}^d)$. Then $u^n \rightarrow u$ strongly if and only if the following criterion is satisfied: $\forall \varepsilon > 0$, $\forall A \subset \Omega$ with $\mu(A) > 0$, $\exists N \in \mathbb{N}$, $\exists B \subset A$ with $\mu(B) > 0$, such that $\forall n \geq N$,

$$\frac{1}{\mu(B)} \int_B \left\| u^n(x) - \frac{1}{\mu(B)} \int_B u^n d\mu \right\| \mu(dx) < \varepsilon.$$

Proof of Theorem 10.

1) First we prove μ -a.e. $u^\infty(x) \in \overline{\text{co}}(Ls(u^n(x)))$. For any Young measure τ limit of a subsequence $(\nu^{n_k})_k$, the τ_x are carried by $Ls(u^n(x))$. Indeed, consider the integrands (recall that the indicator function of C is $\delta(\xi | C) = 0$ if $\xi \in C$, $= +\infty$ otherwise)

$$\psi_n(x, \xi) = \delta(\xi | \text{cl}\{u^p(x) : p \geq n\}).$$

For $k \geq n$, one has

$$\int_{\Omega \times \mathbb{R}^d} \psi_n d\nu^{n_k} = 0,$$

hence by the lower semi-continuity property (Lemma 5 or Th.6)

$$\int_{\Omega \times \mathbb{R}^d} \psi_n d\tau = 0.$$

Then $\tau_x(\text{cl}\{u^p(x) : p \geq n\}) = 1$ μ -a.e. So in the limit $\tau_x(\text{Ls}(u^n(x))) = 1$, that is τ_x is carried by $\text{Ls}(u^n(x))$. By Part 1 of Th.9, $u^\infty(x)$ equals $\text{bar}(\tau_x)$. Consequently $u^\infty(x) \in \overline{\text{co}}(\text{Ls}(u^n(x)))$. If moreover $\forall n$, μ -a.e. $u^n(x) \in \Gamma(x)$ one has $\text{Ls}(u^n(x)) \subset \Gamma(x)$, hence the inclusion

$$\overline{\text{co}}(\text{Ls}(u^n(x))) \subset \Gamma(x).$$

2) The implication (H1) \Rightarrow (H3) results from the elementary fact that if ξ ($:= u^\infty(x)$) is an extreme point of $\Gamma(x)$ and belongs to $\overline{\text{co}}(\text{Ls}(u^n(x)))$, then ξ is still an extreme point of the smaller set $\overline{\text{co}}(\text{Ls}(u^n(x)))$.

3) Now we prove (H3) $\Rightarrow u^n \rightarrow u^\infty$ strongly. Suppose that u^n does not converge strongly. Then, up to the extraction of a subsequence one may assume $\forall n, \|u^n - u^\infty\|_{L^1} > \varepsilon$ where ε is > 0 . Let $(\nu^{n_k})_k$ be a convergent subsequence of the sequence $(\nu^n)_n$, and τ its limit. Since $\text{bar}(\tau_x)$ ($= u^\infty(x)$) is an extreme point of $\overline{\text{co}}(\text{Ls}(u^n(x)))$, it is rather easy to prove (for details see [Va6, pp.172–173]; surely this has been proved by G. Choquet [Ch] in the case of convex compact sets) that $\tau_x = \delta_{u^\infty(x)}$. By Part 1 of Th.9 we get the contradiction $u^n \rightarrow u^\infty$ strongly.

4) Now we prove (H2) $\Rightarrow u^n \rightarrow u^\infty$ strongly. Suppose that u^n does not converge strongly. Then, up to the extraction of a subsequence one may assume that $\forall n, \|u^n - u^\infty\|_{L^1} > \varepsilon$ where ε is > 0 and that $d(u^n(x), \Gamma(x)) \rightarrow 0$ a.e. Then outside of a negligible set, one has $\text{Ls}(u^n(x)) \subset \Gamma(x)$ and (H3) holds for this subsequence. So, by 3) above, we get the contradiction $\|u^n - u^\infty\|_{L^1} \rightarrow 0$. \diamond

REMARK. The fact that, under the hypothesis “for every x there exists a closed convex subset of \mathbb{R}^d , $\Gamma(x)$, such that $\forall n$, μ -a.e. $u^n(x) \in \Gamma(x)$,” one has $u^\infty(x) \in \Gamma(x)$ μ -a.e. can be proved directly using the

strong closedness and the convexity of the set $\{v \in L^1(\Omega, \mu; \mathbb{R}^d) : v(x) \in \Gamma(x) \text{ a.e.}\}$. The relation

$$u^\infty(x) \in \overline{\text{co}}(\text{Ls}(u^n(x))) \text{ } \mu\text{-a.e.}$$

is easier to prove with Young measures, but a proof without Young measures has been given by Amrani-Castaing-Valadier [ACV2, Th.8]. (Recall that Z. Artstein obtained in [Ar, Prop. C] the formula $u^\infty(x) \in \text{co}(\text{Ls}(u^n(x)))$ μ -a.e.) The original proof of Visintin developed in [Va5] does not use Young Measures. For a quick presentation of the ideas, see [Va9, Th.3] and for a proof of Balder's result [Bd5] without Young measures see [ACV1-2] (and also T. Rzezuchowski [Rz2]).

EXAMPLE 1. (this example and the following come from [Va9]). Let $\Omega = [0, 1]$, $d = 1$ and for $k \in \mathbb{N}$, $p \in \{0, \dots, 2^k - 1\}$,

$$v^n = 1_{\left[\frac{p}{2^k}, \frac{p+1}{2^k}\right]} \quad \text{if } n = 2^k + p.$$

Then $\|v^n\|_{L^1} = 2^{-k}$ tends to 0 and it is easy to see, and classical (this is the most usual example of a sequence converging in measure but not a.e.), that for any x in Ω , $\text{Ls}(v^n(x)) = \{0, 1\}$. Then if $u^{2^n} = v^n$, $u^{2^{n+1}} = -v^n$, $\|u^n\|_{L^1} \rightarrow 0$ and $\text{Ls}(u^n(x)) = \{-1, 0, 1\}$. So, with $u^\infty \equiv 0$,

$$\forall x \in \Omega, u^\infty(x) \notin \partial_{\text{ext}} \overline{\text{co}}(\text{Ls}(u^n(x))).$$

Note that if $u^n \rightarrow u^\infty$ there exists a subsequence such that $u^{n_k}(x) \rightarrow u^\infty(x)$ μ -a.e. Then $\text{Ls}(u^{n_k}(x)) = \{u^\infty(x)\}$, hence Balder's condition is satisfied for such a subsequence: $u^\infty(x) \in \partial_{\text{ext}} \overline{\text{co}}(\text{Ls}(u^{n_k}(x)))$. But this condition is not necessary for the whole sequence.

EXAMPLE 2. Even for subsequences the Visintin condition is not a necessary condition. Let r^n be the Rademacher functions, $a_n \in]0, +\infty[$, $a_n \rightarrow 0$, and

$$u^n = a_n r^n.$$

Then $u^n \rightarrow 0$ strongly but, for any subsequence, $0 \in \text{int}[\overline{\text{co}}\{u^{n_k}(x) : k \in \mathbb{N}\}]$ because

$$\sup_{k \in \mathbb{N}} u^{n_k}(x) > 0 \text{ } \mu\text{-a.e.}$$

(consider the Lebesgue measure on $[0, 1]$ as a probability. The events $\{u^{n_k} \leq 0\}$ have probability $1/2$ and are independent so have a negligible intersection) and, symmetrically,

$$\inf_{k \in \mathbb{N}} u^{n_k}(x) < 0 \text{ } \mu\text{-a.e.}$$

A CONSEQUENCE OF THE VISINTIN THEOREM.

One can recover from the Visintin theorem in L^1 the following result (proved in [Vi] and, with more details, in [Va5]).

Suppose $p \in]1, +\infty[$, \mathbb{R}^d is equipped with a strictly convex norm, $u^n \rightharpoonup u^\infty$ in L^p and $\|u^\infty\|_{L^p} \geq \limsup \|u^n\|_{L^p}$, then $u^n \rightarrow u^\infty$ strongly in L^p . The proof does not use uniform convexity arguments. One of the arguments is: if $\varphi(x, \cdot)$ is finite valued and strictly convex on \mathbb{R}^d , then for any $\xi \in \mathbb{R}^d$, $(\xi, \varphi(x, \xi))$ is an extreme point of the epigraph of $\varphi(x, \cdot)$ (here the epigraph is $\{(\xi, r) \in \mathbb{R}^d \times \mathbb{R} : \varphi(x, \xi) \leq r\}$).

For other similar results and proofs see Y.G. Reshetnyak [Re, Th.3], A. Visintin [Vi], E. Balder [Bd16], H. Benabdellah [Bn1–3], H. Brezis [Br].

4. Applications to the Calculus of Variations.

A FIBER PRODUCT LEMMA.

First we give a kind of fiber product type limit result.

LEMMA 11. Let σ^n ($n \in \mathbb{N} \cup \{\infty\}$) denote Young measures on $\Omega \times \mathbb{R}^{d_1}$ and let τ^n ($n \in \mathbb{N} \cup \{\infty\}$) denote Young measures on $\Omega \times \mathbb{R}^{d_2}$ such that $\sigma^n \rightarrow \sigma^\infty$ and $\tau^n \rightarrow \tau^\infty$ narrowly. Let θ^n ($n \in \mathbb{N} \cup \{\infty\}$)

be the Young measures on $\Omega \times \mathbb{R}^{d_1+d_2}$ defined by

$$\theta_x^n = \sigma_x^n \otimes \tau_x^n.$$

If σ^∞ is associated to a function, then θ^n converges narrowly to θ^∞ .

COMMENT. In the next theorem the result is applied with σ^n and τ^n associated (for $n \in \mathbb{N}$) to functions u^n and v^n (then for $n \in \mathbb{N}$, $\theta_x^n = \delta_{(u^n(x), v^n(x))}$, that is θ^n is associated to the function $(u^n(\cdot), v^n(\cdot))$). Without the hypothesis “ σ^∞ is associated to a function,” the result is false: take σ^n and τ^n both associated to the Rademacher functions. Then θ^n converge to the Young measure whose disintegration is $\frac{1}{2}(\delta_{(1,1)} + \delta_{(-1,-1)})$, but

$$\theta_x^\infty = \frac{1}{4}(\delta_{(1,1)} + \delta_{(1,-1)} + \delta_{(-1,1)} + \delta_{(-1,-1)}).$$

Proof. 1) If $\varphi \in \mathcal{C}th^b(\Omega; \mathbb{R}^{d_1})$, one has

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi(x, \xi) \theta^n(d(x, \xi, \zeta)) &= \\ &= \int_{\Omega} \left[\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi(x, \xi) [\sigma_x^n \otimes \tau_x^n](d(\xi, \zeta)) \right] \mu(dx) \\ &= \int_{\Omega} \left[\int_{\mathbb{R}^{d_1}} \varphi(x, \xi) \sigma_x^n(d\xi) \right] \mu(dx) \\ &= \int_{\Omega \times \mathbb{R}^{d_1}} \varphi d\sigma^n. \end{aligned}$$

Similarly for $\varphi' \in \mathcal{C}th^b(\Omega; \mathbb{R}^{d_2})$ the following formula holds:

$$\int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi'(x, \zeta) \theta^n(d(x, \xi, \zeta)) = \int_{\Omega \times \mathbb{R}^{d_2}} \varphi' d\tau^n.$$

2) a) The set of points of a convergent sequence and its limit being a compact set, by the Prohorov theorem, there exists h_1 (resp. h_2) a positive inf-compact function on \mathbb{R}^{d_1} (resp. \mathbb{R}^{d_2}) such that

$$M_1 := \sup_{n \in N \cup \{\infty\}} \int_{\Omega \times \mathbb{R}^{d_1}} h_1(\xi) \sigma^n(d(x, \xi)) < +\infty$$

$$M_2 := \sup_{n \in \mathbb{N} \cup \{\infty\}} \int_{\Omega \times \mathbb{R}^{d_2}} h_2(\zeta) \tau^n(d(x, \zeta)) < +\infty .$$

(Note that in applications the Young measures σ^n and τ^n are associated to functions of L^1 which form bounded sequences, so one can choose $h_1(\xi) = \|\xi\|$ and $h_2(\zeta) = \|\zeta\|$.) Since $(\xi, \zeta) \mapsto h_1(\xi)$ and $(\xi, \zeta) \mapsto h_2(\zeta)$ are l.s.c., their sum $(h(\xi, \zeta) = h_1(\xi) + h_2(\zeta))$ is also l.s.c. Hence, for all $r \in [0, +\infty[$, $\{h \leq r\} := \{(\xi, \zeta) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} : h(\xi, \zeta) \leq r\}$ is closed. Since h_1 and h_2 are ≥ 0 , $\{h \leq r\}$ is a subset of the compact set $\{h_1 \leq r\} \times \{h_2 \leq r\}$. So the set $\{h \leq r\}$ is compact. By Part 1 above,

$$\sup_{n \in \mathbb{N} \cup \{\infty\}} \int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} h(\xi, \zeta) \theta^n(d(x, \xi, \zeta)) \leq M_1 + M_2 .$$

Hence the sequence $(\theta^n)_{n \in \mathbb{N}}$ is tight and, by the Prohorov theorem, admits a convergent subsequence.

3) Suppose that there exist $\psi_0 \in \mathcal{C}th^b(\Omega, \mu; \mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ and $\varepsilon > 0$ such that, for infinitely many n ,

$$\left| \int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \psi_0 d\theta^n - \int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \psi_0 d\theta^\infty \right| > \varepsilon . \tag{4}$$

Extracting a subsequence one may suppose that (4) holds for every n and that θ^n converges to $\lambda \in \mathcal{Y}(\Omega, \mu; \mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Let u be the function to which σ^∞ is associated. Set $\psi(x, \xi, \zeta) = \varphi(x, \xi) := \min(1, \|\xi - u(x)\|_{\mathbb{R}^{d_1}})$. By Part 1, one has

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \psi(x, \xi, \zeta) \theta^n(d(x, \xi, \zeta)) &= \int_{\Omega \times \mathbb{R}^{d_1}} \varphi d\sigma^n \\ &\rightarrow \int_{\Omega \times \mathbb{R}^{d_1}} \varphi d\sigma^\infty = 0 . \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \psi(x, \xi, \zeta) \lambda(d(x, \xi, \zeta)) = \\ &= \int_{\Omega} \left[\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi(x, \xi) \lambda_x(d(\xi, \zeta)) \right] \mu(dx) . \end{aligned}$$

Consequently, λ_x is μ -a.e. carried by

$$\{(\xi, \zeta) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} : \varphi(x, \xi) = 0\} = \{u(x)\} \times \mathbb{R}^{d_2} .$$

Hence λ_x has the form $\delta_{u(x)} \otimes \alpha_x$ with $\alpha_x \in \text{Prob}(\mathbb{R}^{d_2})$ (for any Borel subset B of \mathbb{R}^{d_2} , $\alpha_x(B) = \lambda_x(\{u(x)\} \times B) = \lambda_x(\mathbb{R}^{d_1} \times B)$). For $\varphi' \in \mathcal{C}th^b(\Omega; \mathbb{R}^{d_2})$

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi'(x, \zeta) \lambda(d(x, \xi, \zeta)) &= \\ &= \lim \int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi'(x, \zeta) \theta^n(d(x, \xi, \zeta)) \\ &= \lim \int_{\Omega \times \mathbb{R}^{d_2}} \varphi' d\tau^n \\ &= \int_{\Omega \times \mathbb{R}^{d_2}} \varphi' d\tau^\infty. \end{aligned}$$

Moreover by Part 1,

$$\int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \varphi'(x, \zeta) \lambda(d(x, \xi, \zeta)) = \int_{\Omega \times \mathbb{R}^{d_1}} \varphi' d\alpha.$$

This proves $\alpha = \tau^\infty$, hence $\lambda = \theta^\infty$ which contradicts (4). ◇

THE WEAK-STRONG LOWER SEMI-CONTINUITY THEOREM.

Now the weak-strong lower semi-continuity theorem which is fundamental in the Calculus of Variations follows easily.

THEOREM 12. *Let $\psi : \Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \rightarrow \overline{\mathbb{R}}$ measurable in (x, ξ, ζ) and l.s.c. in (ξ, ζ) . Let $(u^n)_n$ be a sequence of measurable functions from Ω to \mathbb{R}^{d_1} such that $u^n \rightarrow u^\infty$ in measure and $(v^n)_n$ a sequence in $L^1(\Omega, \mu; \mathbb{R}^{d_2})$ such that $v^n \rightharpoonup v^\infty$ weakly. Assume that $\forall x, \psi(x, u^\infty(x), \cdot)$ is convex and that the sequence of negative parts*

$$(x \mapsto \psi(x, u^n(x), v^n(x)))^-_n$$

is uniformly integrable. Then

$$\int_{\Omega} \psi(x, u^\infty(x), v^\infty(x)) \mu(dx) \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \psi(x, u^n(x), v^n(x)) \mu(dx).$$

COMMENTS. 1) This theorem holds for a metrizable Suslin space in place of \mathbb{R}^{d_1} and a Banach space in place of \mathbb{R}^{d_2} . See [Va7, Th.3 p.6] for a precise statement, a shorter proof than the one of C. Castaing & P. Clazure [CC], historical comments (specially references to some E.J. Balder's papers) and a correction to [Va6, Th.21 p.171].

2) Theorem 12 is often applied when Ω is an open subset of \mathbb{R}^N , $u^n \in W^{1,p}(\Omega; \mathbb{R})$ and $v^n = \nabla u^n$. Then $d_1 = 1, d_2 = N$. It still applies if $u^n \in W^{1,p}(\Omega; \mathbb{R}^N)$ and $v^n = \nabla u^n$, with $d_1 = N$ and $d_2 = N \times N$ ($v^n(x)$ is a $N \times N$ matrix). But then the convexity of $\psi(x, u^\infty(x), \cdot)$ is a too restrictive hypothesis, not appropriate to physical problems. The good one is quasi-convexity (see especially B. Dacorogna [Da2]).

3) Theorem 12 requires some strong convergence of u^n . For example, with $\Omega = [0, 1], d_1 = d_2 = 1, u^n = -r^n, v^n = r^n$ and $\psi(x, \xi, \zeta) = \xi\zeta$, one has

$$\int_{\Omega} \psi(x, u^\infty(x), u^\infty(x)) dx = 0$$

but, for all $n \in \mathbb{N}$,

$$\int_{\Omega} \psi(x, u^n(x), v^n(x)) dx = -1.$$

Proof. One can extract a subsequence such that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\Omega} \psi(x, u^{n_k}(x), v^{n_k}(x)) \mu(dx) &= \\ &= \liminf_{n \rightarrow \infty} \int_{\Omega} \psi(x, u^n(x), v^n(x)) \mu(dx). \end{aligned}$$

One may suppose that the Young measure τ^{n_k} associated to v^{n_k} converges to a Young measure (a priori non associated to a function) τ^∞ . With θ^n defined in Lemma 11, θ^{n_k} converges to θ^∞ . Moreover, $\forall n \in \mathbb{N}, \theta^n$ is associated to $(u^n(\cdot), v^n(\cdot))$. By the lower semi-continuity theorem (Th.6),

$$\int_{\Omega \times \mathbb{R}^{d_1 \times \mathbb{R}^{d_2}}} \psi d\theta^\infty \leq \liminf_{k \rightarrow \infty} \int_{\Omega} \psi(x, u^{n_k}(x), v^{n_k}(x)) \mu(dx).$$

But $v^\infty(x)$ is the barycenter of τ_x^∞ (Th.9). Then thanks to the Jensen inequality applied to the convex function $\psi(x, u^\infty(x), \cdot)$

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \psi \, d\theta^\infty &= \int_{\Omega} \left[\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \psi(x, \xi, \zeta) \theta_x^\infty(d(\xi, \zeta)) \right] \mu(dx) \\ &= \int_{\Omega} \left[\int_{\mathbb{R}^{d_2}} \psi(x, u^\infty(x), \zeta) \tau_x^\infty(d\zeta) \right] \mu(dx) \\ &\geq \int_{\Omega} \psi(x, u^\infty(x), v^\infty(x)) \mu(dx). \end{aligned}$$

◇

COMMENTS. 1) The case when ζ is a matrix and $\psi(x, u^\infty(x), \cdot)$ is quasi-convex is not so easy. The Jensen inequality does not hold for quasi-convex functions at any point ζ . The only hope is that for Young measures τ^∞ which are limits of gradients, the Jensen inequality is satisfied at $\text{bar}(\tau_x^\infty)$. Papers by D. Kinderlehrer & P. Pedregal [KP1–2] give some results in this line. Note that the Morrey-Acerbi-Fusco theorem ([AF], [Da2, Th.2.4 p.166]) needs control of negative and positive parts. For some strange behavior of vectorial problems see, in B. Dacorogna’s book [Da2, pp.158–159], the Tartar-Ball-Murat example.

2) Young measures have all the same projection measure on Ω , the given measure μ . So they do not allow variations of mass on Ω . When such variations have to be considered, it may be valuable to use the Y.G. Reshetnyak result [Re] (see also J. Jacod & J. Memin [JM]). Y. Reshetnyak associates to a vector measure $m \in \mathcal{M}(\Omega; \mathbb{R}^d)$ the positive measure ρ on $\Omega \times S^{d-1}$ defined as the image of $|m|$ by $x \mapsto (x, \frac{dm}{d|m|}(x))$. When $u \in L^1(\Omega, \mu; \mathbb{R}^d)$ and $m = u \mu$, the Young measure ν associated to u is connected to ρ by the following: if ψ is an integrand on $\Omega \times \mathbb{R}^d$ positively homogeneous in ξ , then

$$\int_{\Omega \times \mathbb{R}^d} \psi \, d\nu = \int_{\Omega} \psi(x, u(x)) \mu(dx) = \int_{\Omega \times S^{d-1}} \psi \, d\rho.$$

(It is well known since C. Goffman & J. Serrin’s paper [GS] that, if ψ is positively homogeneous in ξ , then, for any positive measure θ such that $|m| \ll \theta$, one has

$$\int_{\Omega} \psi(x, \frac{dm}{d|m|}(x)) |m|(dx) = \int_{\Omega} \psi(x, \frac{dm}{d\theta}(x)) \theta(dx).$$

In my opinion, Reshetnyak's result is specially well adapted to problems where the integrand is positively homogeneous. For its application to the Calculus of Variations see M. Giaquinta, G. Modica & J. Souček [GMS] and more recently, for a problem where positive homogeneity is a data, C. Castaing & V. Jalby [CJ].

BOUNDED SEQUENCES IN L^1 .

There are several possibilities for handling bounded sequences in L^1 . One may embed L^1 in its bidual $(L^\infty)'$ (see C. Castaing & M. Valadier [CV, chapter VIII], V.L. Levin [Le]). Another possibility (when Ω has some topological properties, for example is locally compact) is to embed L^1 in \mathcal{M}^1 , working with the $\sigma(L^1, \mathcal{C})$ or the $\sigma(\mathcal{M}^1, \mathcal{C})$ topology (see among many references C. Olech [O1] and G. Bouchitté & M. Valadier [BV]). The Komlós theorem [K] may also be useful. Here we state the following biting lemma (or its Slaby's formulation) and we will show in Theorem 14 how the biting lemma allows us to understand what is the barycenter $\text{bar}(\tau_x)$ of the limit τ of a convergent subsequence $(\nu^{n_k})_k$ when starting from a bounded, but not uniformly integrable, sequence $(u^n)_n$ in $L^1(\Omega, \mu; \mathbb{R}^d)$. For these sequences the Prohorov theorem (Th.7) still applies (the simplest example is surely $u^n = n 1_{[0, \frac{1}{n}]}$).

THEOREM 13 (biting lemma). *Let $(u^n)_n$ be a bounded sequence in $L^1(\Omega, \mu; \mathbb{R}^d)$. There exist a subsequence $(u^{n_k})_k$ and a sequence $(A_p)_p$ in $\mathcal{B}(\Omega)$ which decreases to \emptyset such that the sequence $(1_{\Omega \setminus A_k} u^{n_k})_k$ is uniformly integrable.*

We do not prove this technical result (recently used by K. Zhang [Z] and J.M. Ball & K. Zhang [BZ]).

REFERENCES. [Gp], [BC], [Eg], [Sb], [BM], [C8], [Va6, Th.23 p.173].

THEOREM 14. *With the notations of Theorem 13, for any subsequence $(\nu^{n_k(m)})_m$ which converges to a Young measure τ (here $k :$*

$\mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing map), then μ -almost everywhere τ_x has a barycenter $u(x)$, u is integrable and the sequence $(1_{\Omega \setminus A_{k(m)}} u^{n_{k(m)}})_m$ weakly converges to u .

Proof. Since the sequence $(1_{\Omega \setminus A_{k(m)}} u^{n_{k(m)}})_m$ is uniformly integrable, it is sufficient to prove, as in Part 1a of the proof of Th.9, that the Young measure τ^m associated to $1_{\Omega \setminus A_{k(m)}} u^{n_{k(m)}}$ converges to τ . Let $\psi \in \mathcal{C}^b(\Omega; \mathbb{R}^d)$ and M be a uniform majorant of $|\psi(x, \xi)|$. Then

$$\begin{aligned} \left| \int_{\Omega \times \mathbb{R}^d} \psi \, d\tau^m - \int_{\Omega \times \mathbb{R}^d} \psi \, d\nu^{n_{k(m)}} \right| &= \\ &= \left| \int_{A_{k(m)}} [\psi(x, 0) - \psi(x, u^{n_{k(m)}})] \mu(dx) \right| \\ &\leq 2 \int_{A_{k(m)}} M \mu(dx) \rightarrow 0. \end{aligned}$$

With $\nu^{n_{k(m)}} \rightarrow \tau$, this proves the result. \diamond

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