REGULARITY AND DECOMPOSABILITY OF RADON MEASURES (*)

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I shall discuss the connection between the regularity and σ-finiteness of Radon measures, and present topological conditions under which complete Radon measures are decomposable. These are joint results of Richard J. Gardner and myself obtained in close collaboration from 1977 to 1984. The principal references are [3] and [4].

The main theme of my lectures is to show that Radon measures are well-behaved in metacompact spaces, while in metalindelöf spaces their behavior depends on special set-theoretic axioms and is undecidable within the Zermelo–Fraenkel set theory including the axiom of choice. Typically, a Radon measure in a metalindelöf space behaves well when Martin's axiom and the negation of the continuum hypothesis are assumed, and the opposite is true when the continuum hypothesis is assumed.

I shall not always present the most general results available; rather, I shall strive to illuminate the ideas and techniques used in this area of mathematics. In particular, my attention will concentrate on three specific questions, which arise naturally in topological measure theory.

- Is every σ-finite Radon measure regular?
- Is every diffused regular Radon measure σ-finite?
- Is every Radon measure saturated?

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I show that each of them has an affirmative answer in metacompact spaces, while in metalindelöf spaces, none can be answered within the usual axioms of set theory.

These lectures are directed to the audience familiar with the rudiments of measure theory, general topology, and naive set theory. No a priori knowledge of axiomatic set theory is required.

1. First observations.

If E is a set, then $|E|$ and $\mathcal{P}(E)$ denote, respectively, the cardinality of E and the family of all subsets of E. For sets A and B, we denote by B^A the family of all maps $f : A \to B$.

Finite or countably infinite sets are called countable. An ordinal is identified with the set of all smaller ordinals. Thus for ordinals α and β, we have

$$\alpha \in \beta \Leftrightarrow \alpha < \beta \qquad \text{and} \qquad \alpha \subset \beta \Leftrightarrow \alpha \leq \beta \, .$$

Cardinals are the initial ordinals, denoted by $0, 1, \ldots, \omega, \omega_1, \ldots, \omega_\alpha, \ldots$ As usual, ω and ω_1 denote the first infinite and the first uncountable cardinals, respectively.

All topological spaces we shall consider are Hausdorff. The closure and interior of a subset E of a topological space is denoted by E^- and E°, respectively. Throughout these lectures, we shall assume that

- X is a nonempty set,
- $\mathcal{G}$ is a Hausdorff topology in X,
- $\mathcal{M}$ is a σ-algebra in X containing $\mathcal{G}$.

Note that $\mathcal{M}$ contains the *Borel σ-algebra* $\mathcal{B}$, i.e., the smallest σ-algebra in X containing the topology $\mathcal{G}$. As usual, the elements of $\mathcal{G}$, $\mathcal{M}$, and $\mathcal{B}$ are called the *open*, *measurable*, and *Borel subets* of X, respectively. The family of all compact subsets of X is denoted by $\mathcal{K}$.

On $\mathcal{M}$ we shall consider a fixed *Radon measure* μ, i.e., a σ-additive function

$$\mu : \mathcal{M} \to [0, +\infty]$$

such that each $x \in X$ has a neighborhood $U \in \mathcal{M}$ with $\mu(U) < +\infty$, and

$$\mu(A) = \sup\{\mu(K) : K \in \mathcal{K} \text{ and } K \subset A\}$$

for each $A \in \mathcal{M}$. A set $A \in \mathcal{M}$ is called *regular* whenever

$$\mu(A) = \inf\{\mu(G) : G \in \mathcal{G} \text{ and } A \subset G\} .$$

If every set in $\mathcal{M}$ is regular, we say that μ is *regular*. Clearly, such is the case when $\mu(X) < +\infty$. Since each compact subset of X is contained in an open set of finite measure, it follows that all compact subsets of X are regular. Nonetheless, μ need not be regular, as the following example shows.

EXAMPLE 1.1. Give ω_1 the discrete topology, and the unit interval $[0, 1]$ its usual topology. The set $Y = \omega_1 \times [0, 1]$ equipped with the product topology is a locally compact metrizable space. If B is a Borel subset of Y and $\alpha \in \omega_1$, then $B_\alpha = \{t \in [0, 1] : (\alpha, t) \in B\}$ is a Borel subset of $[0, 1]$. Thus the function

$$\nu : B \mapsto \sum_{\alpha\in\omega_1} \lambda(B_\alpha) ,$$

where λ is the Lebesgue measure in $[0, 1]$, is defined for each Borel set $B \subset Y$. It is easy to verify that ν is a Radon measure on the Borel σ-algebra in Y that is not regular. Indeed, the set $E = \omega_1 \times \{0\}$ is closed, $\nu(E) = 0$, and $\nu(G) = +\infty$ for every open set $G \subset Y$ containing E.

LEMMA 1.2. *If $A_1, A_2, \ldots$ are regular sets from $\mathcal{M}$, then so is $A = \bigcup_{n=1}^\infty A_n$.*

Proof. We may assume that $\mu(A) < +\infty$. Choose an $\varepsilon > 0$ and open sets G_n so that $A_n \subset G_n$ and $\mu(G_n - A_n) < \varepsilon/2^n$ for $n = 1, 2, \ldots$. Then A is contained in the open set $G = \bigcup_{n=1}^\infty G_n$ and

$$\mu(G - A) \leq \mu\left[\bigcup_{n=1}^\infty (G_n - A_n)\right] \leq \sum_{n=1}^\infty \mu(G_n - A_n) < \varepsilon .$$

Definition 1.3. The measure μ is called *σ-finite* if $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathcal{M}$ and $\mu(X_n) < +\infty$ for $n = 1, 2, \ldots$. If, in addition, each X_n is an open set, we say that μ is *moderated*.

Proposition 1.4. *Let μ be σ-finite. Then μ is moderated if and only if it is regular.*

Proof. As the converse is obvious, assume that μ is moderated, and choose an $A \in \mathcal{M}$. By our assumption, X is the union of open sets $X_1, X_2, \ldots$ whose measures are finite. The measure μ restricted to $\mathcal{M}_n = \{B \in \mathcal{M} : B \subset X_n\}$ is a finite Radon measure, and hence regular. Thus $A \cap X_n$ is a regular subset of X_n, and hence of X, as X_n is open in X. The regularity of A follows from Lemma 1.2.

The next example shows that, in general, a σ-finite Radon measure is not regular.

Example 1.5. For nonnegative integers k and n, let

$$q_{k,n} = (k2^{-n}, 2^{-n}), \qquad Q_n = \{q_{k,n} : k = 0, \ldots, 2^n\},$$

and $Q = \bigcup_{n=1}^{\infty} Q_n$. In $Y = [0,1] \cup Q$, we define a locally compact Hausdorff topology as follows: the points of Q are isolated, and a neighborhood base at $t \in [0,1]$ is given by the sets

$$U(t,\varepsilon) = \{t\} \cup \{q_{k,n} \in Q : 2|k2^{-n} - t| < 2^{-n} < \varepsilon\}$$

where $\varepsilon > 0$. Thus $U(t,\varepsilon)$ consists of t and the points of Q that lie inside the open wedge in $[0,1]^2$ with the vertex $(t,0)$, height ε, and the slopes of the sides equal to ± 2.

For a set $A \subset Y$, we let

$$\nu(A) = \sum_{n=1}^{\infty} 2^{-n} |A \cap Q_n|.$$

Since $|U(t,\varepsilon) \cap Q_n| \leq 1$ for $n = 1, 2, \ldots$, it is easy to see that ν is a σ-finite Radon measure on $\mathcal{P}(Y)$. We show, however, that the closed set $[0,1]$ is not regular.

To this end, choose an open set $G \subset Y$ containing $[0,1]$, and for each $t \in [0,1]$ select an $\varepsilon_t > 0$ with $U(t,\varepsilon_t) \subset G$. By the Baire category theorem (Theorem 3.1 below) applied to the interval $[0,1]$ with its usual topology, there is an open interval $J \subset [0,1]$ and an $\varepsilon > 0$ such that $\{t \in J : \varepsilon_t > \varepsilon\}$ is a dense subset of J. Thus $Q \cap (J \times [0,\varepsilon]) \subset G$. Choose an integer $N \geq 1$ with $|Q_N \cap (J \times [0,\varepsilon])| \geq 2$, and observe that $|Q_{N+n} \cap (J \times [0,\varepsilon])| > 2^n$ for $n = 0, 1, \ldots$. It follows that

$$\nu(G) \geq \sum_{n=0}^{\infty} 2^{-(N+n)} \cdot 2^n = +\infty .$$

Since $\nu([0,1]) = 0$, this establishes our claim.

The *support* of μ is the closed set

$$S = Y - \bigcup\{G \in \mathcal{G} : \mu(G) = 0\} .$$

Since μ is a Radon measure, it is easy to verify that $\mu(X - S) = 0$. From the definition of S it follows that $\mu(G \cap S) > 0$ for each open set G with $G \cap S \neq \emptyset$.

2. Covering properties.

Let $\mathcal{E}$ be a family of subsets of a topological space Z. If for each $z \in Z$, the collection $\{E \in \mathcal{E} : z \in E\}$ is finite or countable, then $\mathcal{E}$ is called *point-finite* or *point-countable*, respectively. If each $z \in Z$ has a neighborhood U such that the collection $\{E \in \mathcal{E} : E \cap U \neq \emptyset\}$ is finite or countable, then $\mathcal{E}$ is called *locally finite* or *locally contable*, respectively. A family $\mathcal{C}$ of sets *refines* $\mathcal{E}$ whenever each $C \in \mathcal{C}$ is contained in an $E \in \mathcal{E}$. If $Z = \bigcup \mathcal{E}$, we say that $\mathcal{E}$ is a *cover* of Z.

A space Z is called *metacompact* or *paracompact* or *metalindelöf* or *paralindelöf* if each open cover of Z is refined by an open cover that is, respectively, point-finite or locally finite or point-countable or locally countable. Clearly, each paracompact space is metacompact, and each paralindelöf space is metalindelöf. We shall not use paralindelöf spaces; their definition has been included only for completeness.

THEOREM 2.1. *Let X be metacompact. If μ is σ-finite, then it is regular.*

The proof of Theorem 2.1 is based on the following measure-theoretic lemma, which is independent of the topology of X.

LEMMA 2.2. *Let μ be σ-finite, and let $\mathcal{E}$ be a point-finite (in particular, disjoint) family of measurable sets. Then the collection $\mathcal{E}_+ = \{E \in \mathcal{E} : \mu(E) > 0\}$ is countable.*

Proof. By our assumption, $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathcal{M}$ and $\mu(X_n) < +\infty$ for $n = 1, 2, \ldots$. Suppose that $\mathcal{E}_+$ is uncountable. Since

$$\mathcal{E}_+ = \bigcup_{n,p=1}^{\infty} \left\{ E \in \mathcal{E} : \mu(E \cap X_n) > \frac{1}{p} \right\},$$

there are positive integers N and P and distinct sets $E_k \in \mathcal{E}$ such that $\mu(E_k \cap X_N) > 1/P$ for $k = 1, 2, \ldots$. Now let

$$A = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k ,$$

and observe that $\mu(A) \geq 1/P$. In particular, $A \neq \emptyset$, contrary to the point-finiteness of the family $\mathcal{E}$.

Proof of Theorem 2.1. For each $x \in X$ let V_x be an open neighborhood of x with $\mu(V_x) < +\infty$, and let $\mathcal{U}$ be a point-finite open cover of X that refines the cover $\{V_x : x \in X\}$. Let

$$\mathcal{U}_+ = \{U \in \mathcal{U} : \mu(U) > 0\} \qquad \text{and} \qquad U_0 = \bigcup (\mathcal{U} - \mathcal{U}_+) .$$

By Lemma 2.2, the family $\mathcal{U}_+$ is countable. As μ is a Radon measure, $\mu(U_0) = 0$, and we see that μ is moderated. An application of Proposition 1.4 completes the argument.

It follows from Theorem 2.1 that the space Y of Example 1.5 is not metacompact. We show by a topological argument that it is not metalindelöf.

PROPOSITION 2.3. *A separable metalindelöf space Z is Lindelöf.*

Proof. Choose an open cover $\mathcal{U}$ of Z, and find a point-countable cover $\mathcal{V}$ of Z that refines $\mathcal{U}$. If D is a countable dense subset of Z, then $\mathcal{V} = \{V \in \mathcal{V} : D \cap V \neq \emptyset\}$, and we conclude that $\mathcal{V}$ is countable. Now for each $V \in \mathcal{V}$, select a $U_V \in \mathcal{U}$ with $V \subset U_V$, and observe that $\{U_V : V \in \mathcal{V}\}$ is a countable subcover of $\mathcal{U}$.

COROLLARY 2.4. *The space Y of Example 1.5 is not metalindelöf.*

Proof. Suppose that Y is metalindelöf. Since Q is a countable dense subset of Y, it follows from Proposition 2.3 that Y is Lindelöf, and so is the closed subspace $[0, 1]$. As $[0, 1]$ in the topology of Y is an uncountable discrete space, we have a contradiction.

In view of Corollary 2.4, it is natural to pose the following question, which will be shown undecidable within the usual axioms of set theory.

QUESTION 2.5. *If X is metalindelöf, is a σ-finite Radon measure μ regular?*

3. The continuum hypothesis and Martin's axiom.

Throughout these lectures we work in the *Zermelo–Fraenkel set theory* including the *axiom of choice* (abbreviated as ZFC). The Zermelo-Fraenkel axioms reflect the most fundamental properties of sets agreed upon by the vast majority of mathematicians. Their actual statements are immaterial for our considerations: no errors will be made by approaching sets from the naive point of view. We must keep in mind, however, that ZFC is a certain set of axioms that codify the mathematical universe in which we work. At places, we shall restrict this universe by assuming the *continuum hypothesis* (abbreviated as CH) or *Martin's axiom* (abbreviated as MA) and the *negation* of the continuum hypothesis. While CH is the familiar statement CH: $|2^\omega| = \omega_1$, some motivation is needed before introducing MA. We begin by recalling the Baire category theorem, whose proof can be found in [1, 3.9.4].

THEOREM 3.1. *In a nonempty compact space the intersection of countably many open dense sets is nonempty.*

A stronger version of Theorem 3.1 is a topological equivalent of CH.

PROPOSITION 3.2. *In a nonempty compact space the intersection of fewer than $|2^\omega|$ open dense sets is nonempty if and only if CH holds.*

Proof. Let $\omega_1 < |2^\omega|$. Giving $\omega_1 + 1$ the order topology, and $Z = (\omega_1 + 1)^\omega$ the product topology, we see that Z is a compact space. Let A be the set of all isolated ordinals in ω_1, and for each $\alpha \in A$ set

$$G_\alpha = \bigcup_{n \in \omega} \{z \in Z : z(n) = \alpha\}.$$

Clearly, each G_α is an open and dense subset of Z. If $z \in \bigcap_{\alpha \in A} G_\alpha$, then $z : \omega \to A$ is a surjective map. As $|A| = \omega_1$, this is impossible, and we conclude that $\bigcap_{\alpha \in A} G_\alpha = \emptyset$.

Conversely, if CH holds, then fewer than $|2^\omega|$ means countably many, and an application of Theorem 3.1 completes the proof.

We say that a space Z satisfies the *countable chain condition* (abbreviated as ccc) when each disjoint family of open sets is countable. The following are examples of spaces that satisfy the ccc.

- *A separable space.* Indeed, in each nonempty open set G of the disjoint family $\mathcal{E}$ we can select a point z_G from a countable dense set D. Since the map $G \mapsto z_G$ is injective, $\mathcal{E}$ is countable.

- *A space that is a support of a σ-finite Radon measure.* This follows directly from Lemma 2.2.

The space Z from the proof of Proposition 3.2 does not satisfy the ccc; indeed, the sets $H_\alpha = \{z \in Z : z(0) = \alpha\}$, where $\alpha \in A$, are open and disjoint.

Usually, MA is formulated in terms of partially ordered sets (see [5, Section 23]), however, for our purposes, it will be more convenient to use an equivalent topological formulation:

MA: *In a nonempty compact space that satisfies the ccc, the intersection of fewer than* $|2^\omega|$ *open dense sets is nonempty.*

It follows immediately from Proposition 3.2 that CH implies MA. Moreover, the assumptions that CH or MA + $\neg$CH holds are consistent with ZFC; more precisely, if ZFC is consistent, then so are ZFC+CH and ZFC+MA + $\neg$CH (see [5]).

We shall need a consequence of MA + $\neg$CH due to F.D. Tall ([10]).

LEMMA 3.3. *Let* Z *be a space that satisfies the ccc, and let* $\{H_\alpha : \alpha \in \omega_1\}$ *be a family of open sets such that* $H_\beta \subset H_\alpha$ *for each* $\alpha < \beta < \omega_1$. *Then there is a* $\gamma \in \omega_1$ *such that* $H_\alpha^- = H_\gamma^-$ *whenever* $\gamma < \alpha < \omega_1$.

Proof. If the lemma does not hold, then for each $\alpha \in \omega_1$ there is a $\beta(\alpha) \in \omega_1$ with $\beta(\alpha) > \alpha$ and $H_\alpha^- - H_{\beta(\alpha)}^- \neq \emptyset$. We let $\alpha_0 = \beta(0)$, and proceed by transfinite induction. Assuming that $\alpha_\gamma \in \omega_1$ has been defined for each ordinal $\gamma < \kappa < \omega_1$, let $\alpha_\kappa = \beta(\tau)$ where $\tau = \sup\{\alpha_\gamma : \gamma < \kappa\}$. Then $\{H_{\alpha_\gamma} - H_{\beta(\alpha_\gamma)}^-\}$ is an uncountable disjoint family of open sets, a contradiction.

PROPOSITION 3.4. (MA + $\neg$CH) Let Z be a compact space satisfying the ccc. Then each point-countable family of open sets is countable.

Proof. Let $\{G_\alpha : \alpha \in \omega_1\}$ be an enumeration of an uncountable point-countable family of open sets, and let $H_\alpha = \bigcup_{\alpha \leq \beta < \omega_1} G_\beta$ for each countable ordinal α. By Lemma 3.3, there is a $\gamma \in \omega_1$ such that $H_\alpha^- = H_\gamma^-$ whenever $\gamma \leq \alpha < \omega_1$. In particular, the sets H_α with $\gamma < \alpha < \omega_1$ are open dense subsets of a compact set H_γ^-. Since H_γ is an open subset of Z, its closure H_γ^- satisfies the ccc. Thus by MA + $\neg$CH, the intersection $D = \bigcap_{\gamma<\alpha<\omega_1} H_\alpha$ is nonempty. As each $z \in D$ is contained in uncountably many sets G_α, this is a contradiction.

Proposition 3.4 cannot be proved in ZFC alone; indeed, we have the following example of K. Kunen ([6]).

Example 3.5. (CH) Recall that a space is called *zero dimensional* if each of its points has a neighborhood base consisting of *clopen* sets, i.e., the sets that are simultaneosly closed and open.

There is a zero dimensional compact hereditarily Lindelöf space Z without isolated points, which is the support of a finite Radon measure χ on the Borel σ-algebra in Z such that $\chi(B) = 0$ for each nowhere dense Borel set $B \subset Z$. In particular, Z satisfies the ccc, and $\chi(B) = \chi(B^-) = \chi(B^\circ)$ for each Borel set $B \subset Z$.

We omit the intricate proof of the above statement; the interested reader is referred to [4, 5.10]. Instead, we derive from it some important consequences.

1. *If $C \subset Z$ is countable, then $\chi(C) = 0$. In particular, Z is not separable.* Indeed, as Z has no isolated points, $\chi(\{z\}) = 0$ for each $z \in Z$.

2. $|Z| \leq \omega_1$. Observe that Z is first countable, since it is compact and hereditarily Lindelöf ([1, 3.1.F(a)]). By [1, 3.1.30] and CH, we have $|Z| \leq |2^\omega| = \omega_1$. Since Z is not separable, the claim follows.

3. *There is an uncountable point-countable family of open subsets of Z.* To see this, let $\{z_\alpha : \alpha \in \omega_1\}$ be an enumeration of Z, and for each $\beta \in \omega_1$ let $H_\beta = Z - \{z_\alpha : \alpha \in \beta\}^-$. Then $H_\beta \subset H_\alpha$ whenever $\alpha < \beta < \omega_1$, and $\bigcap_{\alpha\in\omega_1} H_\alpha = \emptyset$. Since Z is not separable, no H_α is empty. It follows that $\mathcal{H} = \{H_\alpha : \alpha \in \omega_1\}$ is the desired family.

The space Z is an example of a compact *L-space*, i.e., a hereditarily Lindelöf space which is not separable (cf. with an S-space defined prior to Proposition 5.6).

4. Metalindelöf spaces.

This section is devoted to the proof of the following theorem, which is a consequence of Theorem 4.4 and Example 4.5 below.

THEOREM 4.1. *The answer to Question 2.5 is affirmative under* MA + ¬CH *and negative under CH. In particular, the question is undecidable within ZFC.*

Since the space X is neither compact nor does it satisfy the ccc, a direct application of MA is not possible. Nonetheless, the following lemma enables us to apply Proposition 3.4.

LEMMA 4.2. *There is a disjoint family $\mathcal{D}$ of nonempty compact subsets of X such that*

1. $\mu(D \cap G) > 0$ *for each* $D \in \mathcal{D}$ *and each* $G \in \mathcal{G}$ *with* $D \cap G \neq \emptyset$*;*
2. $\mu(A) = \sum_{D \in \mathcal{D}} \mu(A \cap D)$ *for each* $A \in \mathcal{M}$.

Proof. By Zorn's lemma there is a maximal family $\mathcal{D}$ of disjoint nonempty compact subsets of X satisfying the first condition of the lemma. In three steps, we show that $\mathcal{D}$ satisfies also the second condition.

Let $A \in \mathcal{M}$ be such that $A \cap \bigcup \mathcal{D} = \emptyset$, and suppose that $\mu(A) > 0$. Then there is a compact set $K \subset A$ with $\mu(K) > 0$, and we denote by S the support of μ restricted to the family $\{B \in \mathcal{M} : B \subset K\}$. Adding S to the family $\mathcal{D}$ contradicts to the maximality of $\mathcal{D}$, and consequently $\mu(A) = 0$.

If $K \subset X$ is a compact set, find an open set $G \subset X$ such that $K \subset G$ and $\mu(G) < +\infty$. By Lemma 2.2, the family $\mathcal{D}_0 = \{D \in \mathcal{D} : D \cap G \neq \emptyset\}$ is countable. Since $(K - \bigcup \mathcal{D}_0) \cap \bigcup \mathcal{D} = \emptyset$, we have

$$\mu(K) = \mu(K \cap \bigcup \mathcal{D}_0) = \sum_{D \in \mathcal{D}_0} \mu(K \cap D) = \sum_{D \in \mathcal{D}} \mu(K \cap D) .$$

Finally, if $A \subset X$ is an arbitrary measurable set, then

$$\mu(K) = \sum_{D \in \mathcal{D}} \mu(K \cap D) \leq \sum_{D \in \mathcal{D}} \mu(A \cap D)$$

for each compact set $K \subset A$. Thus $\mu(A) \leq \sum_{D \in \mathcal{D}} \mu(A \cap D)$, and as $\mathcal{D}$ is a disjoint family, the reverse inequality is obvious.

Any family $\mathcal{D}$ that satisfies the conditions of Lemma 4.2 is called a *concassage* of μ. As each member of the concassage has a positive measure, the next proposition follows from Lemma 2.2.

PROPOSITION 4.3. *The measure μ is σ-finite if and only if it has a countable concassage; in which case, each concassage of μ is countable.*

THEOREM 4.4. (MA + $\neg$CH) *Let X be metalindelöf. If μ is σ-finite, then it is regular.*

Proof. For each $x \in X$ let V_x be an open neighborhood of x with $\mu(V_x) < +\infty$, and let $\mathcal{U}$ be a point-countable open cover of X that refines the cover $\{V_x : x \in X\}$. By Proposition 4.3, there is a countable concassage $\mathcal{D}$ of μ. Let

$$\mathcal{U}_+ = \{U \in \mathcal{U} : U \cap \bigcup \mathcal{D} \neq \emptyset\} \quad \text{and} \quad U_0 = \bigcup (\mathcal{U} - \mathcal{U}_+) .$$

According to Lemma 4.2, we have $\mu(U_0) = 0$. Now each $D \in \mathcal{D}$ is compact and satisfies the ccc; for D is the support of μ restricted to $\{A \in \mathcal{M} : A \subset D\}$. As $\mathcal{U}$ is a point countable family of open sets, Proposition 3.4 implies that the collection $\{U \in \mathcal{U} : U \cap D \neq \emptyset\}$ is countable for each $D \in \mathcal{D}$. Thus

$$\mathcal{U}_+ = \bigcup_{D \in \mathcal{D}} \{U \in \mathcal{U} : U \cap D \neq \emptyset\}$$

is a countable family, and we see that μ is moderated. The theorem follows from Proposition 1.4.

REMARK 4.5. Without any set-theoretic assumptions, F.D. Tall ([10]) showed that in a compact space satisfying the ccc, every point-finite family of open sets is countable. In view of this, the proof of Theorem 4.4 provides an alternative way of proving Theorem 2.1, which does not depend on Lemma 2.2.

EXAMPLE 4.6. (CH) The main idea of this example is to destroy the separability of the space Y of Example 1.5 by replacing the points of Q with copies of the space Z introduced in Example 3.5.

By Example 3.5, 3, there is a point-countable family $\mathcal{H}$ of open subsets of Z with $|\mathcal{H}| = \omega_1$. Since Z is a zero dimensional compact space, we may assume that $\mathcal{H}$ consists of compact sets. Indeed, choosing a nonempty clopen set $H^* \subset H$ for each $H \in \mathcal{H}$, the family $\mathcal{H}^* = \{H^* : H \in \mathcal{H}\}$ is point-countable and $|\mathcal{H}^*| = \omega_1$; for the map $H \mapsto H^*$ is countable-to-one. We may also assume that $\chi(H) \geq r$ for an $r > 0$ and each $H \in \mathcal{H}$. By CH, there is a bijection $t \mapsto H^t$ between $[0,1]$ and $\mathcal{H}$.

Adhering to the notation of Example 1.5, let $X = [0,1] \cup (Q \times Z)$. Give Q the discrete topology, and let the neighborhoods of $x \in Q \times Z$ be determined by the product topology of $Q \times Z$. A neighborhood base at $t \in [0,1]$ is given by the sets

$$V(t,\varepsilon) = \{t\} \cup \Big([U(t,\varepsilon) - \{t\}] \times H^t\Big)$$

where $\varepsilon > 0$. It follows immediately from Examples 1.5 and metalindelöf.

Since $Q \times Z$ is σ-compact, it is Lindelöf. Thus each open cover of X is refined by an open cover $\mathcal{V} = \mathcal{U} \cup \{V(t, \varepsilon_t) : t \in [0,1]\}$ where $\mathcal{U}$ is a countable open cover of $Q \times Z$. If a point $(q,z) \in Q \times Z$ belongs to uncountably many $V(t, \varepsilon_t)$, then z belongs to uncountably many H^t, contrary to the choice of $\mathcal{H}$. Consequently, the cover $\mathcal{V}$ is point-countable.

If B is a Borel subset of X, then $B_q = \{z \in Z : (q,z) \in B\}$ is a Borel subset of Z for each $q \in Q$. Thus the function

$$\mu : B \mapsto \sum_{n=1}^{\infty} 2^{-n} \sum_{q \in Q_n} \chi(B_q)$$

is defined for each Borel set $B \subset X$. As χ is a finite Radon measure on Borel subsets of Z, it is easy to verify that μ is a σ-finite Radon measure on Borel subsets of X; we only need to observe that $|\{q \in Q_n : V(t,\varepsilon)_q \neq \emptyset\}| \leq 1$ for $n = 1, 2, \ldots$. Since $\chi(H^t) \geq r > 0$ for all $t \in [0,1]$, the same argument we used in Example 1.5 reveals that the closed set $[0,1]$ is not regular.

5. Regular Radon measures.

We call a Radon measure *diffused* if all singletons have measure zero, and pose a question that is a nontrivial converse of Question 2.5.

QUESTION 5.1. *Is a diffused regular Radon measure σ-finite?*

The next example shows that for nondiffused measures, the answer to Question 5.1 is trivially negative.

EXAMPLE 5.2. Give ω_1 the discrete topology, and for each $A \subset \omega_1$, let
$$\nu(A) = \begin{cases} |A| & \text{if } A \text{ is finite,} \\ +\infty & \text{if } A \text{ is infinite.} \end{cases}$$
A moment's reflection shows that ν is a regular Radon measure on $\mathcal{P}(\omega_1)$, which is not σ-finite.

A set $E \subset X$ is called *locally countable* if each $x \in X$ has a neighborhood U with $|E \cap U| \leq \omega$. The word "ample" in the next definition refers to the size of the Borel σ-algebra.

DEFINITION 5.3. We say that X is *ample* if each uncountable locally countable set $E \subset X$ contains an uncountable Borel subset of X.

THEOREM 5.4. *In an ample space, the answer to Question 5.1 is affirmative.*

Proof. Let X be ample, and let μ be regular and diffused. Proceeding towards a contradiction, assume that μ has an uncountable concassage $\mathcal{D}$. In each $D \in \mathcal{D}$ select a point x_D, and observe that $E = \{x_D : D \in \mathcal{D}\}$ is an uncountable locally countable set. Thus E contains an uncountable Borel set B, and we obtain a contradiction by calculating $\mu(B)$. Since compact subsets of B are countable and μ is a diffused Radon measure, $\mu(B) = 0$. If $G \subset X$ is an open subset containing B, then $\{G \cap D : D \in \mathcal{D}\}$ is an uncountable disjoint family. It follows from Lemmas 4.2 and 2.2 that $\mu(G) = +\infty$. Since μ is regular, this is impossible. Now the theorem is a consequence of Proposition 4.3.

The following result shows that ample spaces abound.

PROPOSITION 5.5. *If X is metalindelöf, then each locally countable set $E \subset X$ is Borel.*

Proof. For each $x \in X$ choose an open neighborhood V_x of x so that $|E \cap V_x| \le \omega$, and let $\mathcal{U}$ be a point-countable open cover of X that refines $\{V_x : x \in X\}$. Now the family $\mathcal{E} = \{E \cap U \neq \emptyset : U \in \mathcal{U}\}$ is a relatively open cover of E, which is *star-countable*, i.e., $\{B \in \mathcal{E} : A \cap B \neq \emptyset\}$ is countable for each $A \in \mathcal{E}$.

For $A, B \in \mathcal{E}$, write $A \sim B$ if there are sets $C_0, \ldots, C_n$ in $\mathcal{E}$ such that $C_0 = A$, $C_n = B$, and $C_{i-1} \cap C_i \neq \emptyset$ for $i = 1, \ldots, n$. Clearly, $\sim$ is an equivalence relation on $\mathcal{E}$, and we denote by $\{\mathcal{E}_t : t \in T\}$ the disjoint family of the corresponding equivalence classes. If $\mathcal{E}_t$ contains a set $A \in \mathcal{E}$, then $\mathcal{E}_t = \bigcup_{n \in \omega} \mathcal{A}_n$ where

$$\mathcal{A}_0 = \{B \in \mathcal{E} : A \cap B \neq \emptyset\} \text{ and } \mathcal{A}_n = \left\{B \in \mathcal{E} : B \cap \bigcup \mathcal{A}_{n-1} \neq \emptyset\right\}$$

for $n = 1, 2, \ldots$. Since $\mathcal{E}$ is star countable, we see that $\mathcal{E}_t$ is countable, and hence so is $E^t = \bigcup \mathcal{E}_t$. Now $\{E^t : t \in T\}$ is a relatively open cover of E, which is disjoint by the choice of $\sim$. As the set E^t are nonempty, for each $t \in T$, we can define a map $n \mapsto x_n^t$ from ω onto E^t. Then the sets $E_n = \{x_n^t : t \in T\}$ are discrete, and $E = \bigcup_{n \in \omega} E_n$. We complete the proof by showing that each discrete set $D \subset X$ is Borel.

Indeed, for each $x \in D$ there is an open set $G_x \subset X$ such that $D \cap G_x = \{x\}$. Thus D is a relatively closed subset of the open set $G = \bigcup_{x \in D} G_x$, which means that it is a Borel subset of X.

Ample spaces are closely connected with so called *S-spaces*, i.e., hereditarily separable spaces which are not Lindelöf (cf. with L-spaces defined in Example 3.5).

PROPOSITION 5.6. *If X is not ample, then it contains an S-space.*

Proof. By definition, there is a uncountable locally contable set $E \subset X$ which contains no Borel subset of X. Clearly, E is not Lindelöf, and it suffices to show that E is separable; for then it is clear that E is hereditarily separable. By Zorn's lemma, there is a maximal disjoint family $\mathcal{U}$ of relatively open nonempty countable subsets of E. As E is locally countable, the maximality of $\mathcal{U}$ implies that $\bigcup \mathcal{E}$ is dense in E. In each $U \in \mathcal{U}$ select a point x_u, and observe that the set $B = \{x_u : U \in \mathcal{U}\}$ is discrete, and hence a Borel subset of X. According to our assumption B is countable. It follows that $\bigcup \mathcal{U}$ is countable, and the separability of E is established.

THEOREM 5.7. *It is consistent with ZFC that in a regular space the answer to Question 5.1 is affirmative.*

Proof. S. Todorčevič proved that it is consistent with ZFC to assume that no regular S-spaces exist (see [8, 7.2.1]). In view of this, the theorem follows from Proposition 5.6 and Theorem 5.4.

REMARK 5.8. Question 5.1 is actually undecidable in ZFC; the highly technical example which shows this has been constructed in [3, 13.14].

6. Large Radon measures.

For the sake of brevity, we say that a Radon measure is *large* if it is not σ-finite. To facilitate the study of large Radon measures, we let

$$\mathcal{M}_f = \{A \in \mathcal{M} : \mu(A) < +\infty\} \quad \text{and} \quad \mathcal{M}_0 = \{A \subset \mathcal{M} : \mu(A) = 0\},$$

and recall that the measure μ is

- *complete* if $A \in \mathcal{M}$ whenever $A \subset B$ for a $B \in \mathcal{M}_0$;
- *saturated* if $A \in \mathcal{M}$ whenever $A \subset X$ and $A \cap B \in \mathcal{M}$ for each $B \in \mathcal{M}_f$.

Definition 6.1. A *decomposition* of μ is a disjoint family $\mathcal{H} \subset \mathcal{M}_f$ such that

1. $A \in \mathcal{M}$ whenever $A \subset X$ and $A \cap H \in \mathcal{M}$ for each $H \in \mathcal{H}$;
2. $\mu(A) = \sum_{H \in \mathcal{H}} \mu(A \cap H)$ for each $A \in \mathcal{M}$.

If a decomposition of μ exists, the measure μ is called *decomposable*.

Although closely related, a decomposition and a concassage of μ are different concepts, which coincide only under special circumstances (see the proofs of Propositions 6.2 and 6.3 below). In general, a decomposition of μ (if it exists) provides more information about μ than a concassage; mainly because the union of a concassage need not be measurable. Impressive examples demonstrating pathological behavior of large Radon measures can be found in [2]. Our goal is to show that no pathologies occur when X is metacompact, or metalindelöf and $\mathrm{MA} + \neg\mathrm{CH}$ holds.

Proposition 6.2. *If μ is σ-finite, it is decomposable and hence saturated.*

Proof. If $\mathcal{D}$ is a concassage of μ, then it follows from Proposition 4.3 that $\mathcal{H} = \mathcal{D} \cup \{\bigcup \mathcal{D}\}$ is a decomposition of μ.

Proposition 6.3. *If μ is complete, then it is decomposable if and only if it is saturated.*

Proof. As the converse is obvious, let μ be complete and saturated, and let $\mathcal{D}$ be a concassage of μ. We show that $\mathcal{D}$ is also a decomposition of μ. To this end, choose an $A \subset X$ such that $A \cap D \in \mathcal{M}$ for each $D \in \mathcal{D}$. If $B \in \mathcal{M}_f$, find a σ-compact set $C \subset B$ with $\mu(B - C) = 0$. By Lemma 2.2, the family $\mathcal{D}^* = \{D \in \mathcal{D} : C \cap D \neq \emptyset\}$ is countable. Since μ is complete and $\mu(C - \bigcup \mathcal{D}^*) = 0$, the set

$$A \cap C = \left[A \cap \left(C - \bigcup \mathcal{D}^*\right)\right] \cup \bigcup_{D \in \mathcal{D}^*} [(A \cap D) \cap C]$$

belongs to $\mathcal{M}$. Using the completeness of μ again, we see that $A \cap (B - C) \in \mathcal{M}$, and consequently $A \cap B \in \mathcal{M}$. As μ is saturated, $A \in \mathcal{M}$.

Example 6.6 below shows that a saturated Radon measure μ which is not complete need not be decomposable. On the other hand, recall that by a standard process of *completion* (see [9, 1.36]) the Radon measure μ on $\mathcal{M}$ can be extended to a complete Radon measure $\mu^{\sim}$ on the σ-algebra $\mathcal{M}^{\sim}$ containing $\mathcal{M}$. Moreover, if μ is saturated, then so is $\mu^{\sim}$. Thus up to the completion, the question of decomposability of μ is reduced to that whether μ is saturated. This, in turn, depends on how well we can disentangle a concassage of μ.

LEMMA 6.4. *Assume either that X is metacompact, or that X is metalindelöf and* MA + ¬CH *holds. Given a concassage $\mathcal{D}$ of μ, we can find a concassage $\mathcal{E}$ of μ which satisfies the following conditions.*

1. *Each $E \in \mathcal{E}$ is a relatively clopen subset of a $D \in \mathcal{D}$.*
2. *$\mathcal{E}$ is the union of a collection $\{\mathcal{E}_t : t \in T\}$ of countable families such that the sets $\bigcup \mathcal{E}_t$ form a disjoint relatively clopen cover of $\bigcup \mathcal{D}$.*

Proof. By the hypothesis, we can find a point-finite or point-countable open cover $\mathcal{U}$ of X such that $\mu(U) < +\infty$ for each $U \in \mathcal{U}$. In view of Lemma 2.2, the family $\mathcal{D}_U = \{D \in \mathcal{D} : D \cap U \neq \emptyset\}$ is countable for each $U \in \mathcal{U}$. Moreover, for every $D \in \mathcal{D}$, the family $\mathcal{U}_D = \{U \in \mathcal{U} : D \cap U \neq \emptyset\}$ is also countable. This follows from Lemma 2.2 or Remark 4.5 when $\mathcal{U}$ is point-finite, and from Proposition 3.4 when $\mathcal{U}$ is point-countable. If $Y = \bigcup \mathcal{D}$ and $U \in \mathcal{U}$, then $U \cap Y = U \cap \bigcup \mathcal{D}_U$. Thus

$$\begin{aligned}\{V \cap Y : V \in \mathcal{U} \text{ and } (V \cap Y) \cap U \neq \emptyset\} \subset \\ \left\{V \cap Y : V \in \mathcal{U} \text{ and } V \cap \bigcup \mathcal{D}_U \neq \emptyset\right\} \\ = \bigcup_{D \in \mathcal{D}_U} \{V \cap Y : V \in \mathcal{U}_D\}\end{aligned}$$

for every $U \in \mathcal{U}$. We infer that the family $\mathcal{U}^* = \{U \cap Y \neq \emptyset : U \in \mathcal{U}\}$, which is a relatively open cover of Y, is star-countable (see the proof of Proposition 5.5). Proceeding as in the proof of Proposition 5.5, it is easy to show that there is a disjoint collection $\{\mathcal{U}_t^* : t \in T\}$ of nonempty countable subfamilies of $\mathcal{U}^*$ such that the sets $Y_t = \bigcup \mathcal{U}_t^*$ form a disjoint relatively open, and hence clopen, cover of Y. Now set $\mathcal{E}_t = \{D \cap Y_t \neq \emptyset : D \in \mathcal{D}\}$ and $\mathcal{E} = \bigcup_{t \in T} \mathcal{E}_t$. As $Y_t = \bigcup \mathcal{E}_t$ for each $t \in T$, it suffices to demonstrate that the families $\mathcal{E}_t$ are countable.

To this end, fix a $t \in T$, and observe that $|\mathcal{E}_t| = |\{D \in \mathcal{D} : D \cap Y_t \neq \emptyset\}|$ and

$$\{D \in \mathcal{D} : D \cap Y_t \neq \emptyset\} = \bigcup_{U \cap Y \in \mathcal{U}_t^*} \{D \in \mathcal{D} : D \cap (U \cap Y)\}.$$

Since $\mathcal{U}_t^*$ is countable and $\{D \in \mathcal{D} : D \cap (U \cap Y)\} = \mathcal{D}_U$ for each $U \in \mathcal{U}$, the countability of $\mathcal{E}_t$ follows.

THEOREM 6.5. *Assume either that X is metacompact, or that X is metalindelöf and* MA + ¬CH *holds. Then μ is saturated, and each concassage $\mathcal{D}$ of μ satisfies the following conditions.*

1. *The space $Y = \bigcup \mathcal{D}$ is the union of disjoint relatively clopen σ-compact subsets; in particular, Y is paracompact whenever it is regular.*

2. *For every $\mathcal{D}^* \subset \mathcal{D}$, the set $\bigcup \mathcal{D}^*$ is a Borel subset of X; in particular, Y is a Borel subset of X and $\mu(X - Y) = 0$.*

Proof. Choose a concassage $\mathcal{E} = \bigcup_{t \in T} \mathcal{E}_t$ of μ according to Lemma 6.4, and let $Y_t = \bigcup \mathcal{E}_t$ for each $t \in T$. Since $\{Y_t : t \in T\}$ is a disjoint clopen cover of Y consisting of σ-compact sets, the first condition of the theorem follows from [1, 5.1.30].

Both families $\mathcal{D}$ and $\mathcal{E}$ are disjoint, $\mathcal{E}$ refines $\mathcal{D}$, and $\bigcup \mathcal{D} = \bigcup \mathcal{E}$. Hence given a $\mathcal{D}^* \subset \mathcal{D}$ there is an $\mathcal{E}^* \subset \mathcal{E}$ with $\bigcup \mathcal{E}^* = \bigcup \mathcal{D}^*$. The set $E^* = \bigcup \mathcal{E}^*$ is the union of a disjoint collection $\{E^* \cap Y_t : t \in T\}$ consisting of relatively open subsets of E^*. Moreover, each set $E^* \cap Y_t = \bigcup(\mathcal{E}^* \cap \mathcal{E}_t)$ is σ-compact. Thus for each $t \in T$, we can find an open set $U_t \subset X$ and a collection $\{K_{t,n} : n \in \omega\}$ of compact subsets of X so that

$$E^* \cap Y_t = E^* \cap U_t = \bigcup_{n\in\omega} K_{t,n} \,.$$

Now it is easy to see that for each $n \in \omega$, the set $L_n = \bigcup_{t\in T} K_{t,n}$ is relatively closed in the open set $\bigcup_{t\in T} U_t$. Consequently, $E^* = \bigcup_{n\in\omega} L_n$ is a Borel subset of X. In particular, Y is a Borel subset of X, and by Lemma 4.2, we have $\mu(X - Y) = 0$. Consequently, in proving that μ is saturated, we may assume that $X = Y$.

Let $A \subset X$ be such that $A \cap B \in \mathcal{M}$ for every $B \in \mathcal{M}_f$. Then $A \cap Y_t \in \mathcal{M}$ for each $t \in T$. Since Y_t is σ-compact and open, it is a countable union of open sets of finite measure. Using this, construct G_δ sets G_t and H_t with $\mu(H_t) = 0$ so that

$$A \cap Y_t \subset G_t \subset Y_t \qquad \text{and} \qquad G_t - A \cap Y_t \subset H_t \subset Y_t \,.$$

As the sets Y_t are disjoint, the unions $G = \bigcup_{t\in T} G_t$ and $H = \bigcup_{t\in T} H_t$ are still G_δ subsets of X. Moreover, $A \subset G$, $G - A \subset H$, and

$$\mu(H) = \sum_{E\in\mathcal{E}} \mu(H \cap E) = \sum_{t\in T} \mu(H \cap Y_t) = \sum_{t\in T} \mu(H_t) = 0 \,.$$

By the assumption, the set $A \cap H$ is measurable. Hence so is

$$G - A = (G - A) \cap H = G \cap H - A \cap H \,,$$

and the measurability of A follows.

Example 6.6. Let Y and ν be as in Example 1.1. Being metrizable, Y is paracompact by [1, 5.1.3]. Theorem 6.5 thus implies that ν is a saturated Radon measure. Clearly, ν is not complete, and we show that it is not decomposable. Proceeding towards a contradiction, suppose that $\mathcal{H}$ is a decomposition of ν. For each $\alpha \in \omega_1$, the set $I^\alpha = \{\alpha\} \times [0,1]$ is a Borel subset of Y, and

$$\sum_{H\in\mathcal{H}} \nu(I^\alpha \cap H) = \nu(I^\alpha) = \lambda([0,1]) = 1 \,.$$

Since ν Radon, there is an $H \in \mathcal{H}$ and a compact set $K^\alpha \subset H \cap I^\alpha$ with $\nu(K^\alpha) > 0$. As ν is diffused, K^α is uncountable. According to [7, §30, XIV and §36, V, Corollary 2], K^α contains a Borel set B^α which is not of Borel class α (see [7, §30, II]). It follows that $B = \bigcup_{\alpha\in\omega_1} B^\alpha$ is not a Borel set. On the other hand, each $H \in \mathcal{H}$ contains only countably many sets K^α, and so $B \cap H$ is a Borel set for every $H \in \mathcal{H}$; a contradiction.

The next two examples show that the assumptions of Theorem 6.5 are essential for the validity of the first condition of this theorem.

EXAMPLE 6.7. Let Y and ν be as in Example 1.5. Setting

$$\mu(A) = \begin{cases} \nu(A) + |\, A \cap [0,1]\,| & \text{if } A \cap [0,1] \text{ is finite,} \\ +\infty & \text{if } A \cap [0,1] \text{ is infinite,} \end{cases}$$

defines a Radon measure μ on $\mathcal{P}(Y)$, and the family $\mathcal{D} = \{\, \{y\} : y \in Y\}$ is a concassage of μ. Since Y is not metalindelöf (Corollary 2.4), the union $\bigcup \mathcal{D} = Y$ is not paracompact.

EXAMPLE 6.8. (CH) Let X and μ be as in Example 4.5. Setting

$$\varphi(B) = \begin{cases} \mu(B) + |\, B \cap [0,1]\,| & \text{if } B \cap [0,1] \text{ is finite,} \\ +\infty & \text{if } B \cap [0,1] \text{ is infinite,} \end{cases}$$

for each Borel set $B \subset X$ defines a Radon measure φ on the Borel σ-algebra $\mathcal{B}$ in X, and the family

$$\mathcal{D} = \Big\{ \{q\} \times Z : q \in Q \Big\} \cup \Big\{ \{t\} : t \in [0,1] \Big\}$$

is a concassage of φ. The space X is metalindelöf but not metacompact. This follows from Theorem 2.1, since μ is a σ-finite Radon measure on $\mathcal{B}$ which is not regular. We conclude that the union $\bigcup \mathcal{D} = X$ is not paracompact.

Finally, very technical examples in [4, 4.5 and 4.6] show that under CH, there is an nonsaturated diffused Radon measure μ defined on the Borel σ-algebra in a metalindelöf space X; moreover, μ has a concassage whose union is not a Borel set. This implies, in particular, that our last question is undecidable in ZFC.

QUESTION 6.9. *If X is a metalindelöf space, is a Radon measure μ saturated?*

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