LIMITS OF DIRICHLET PROBLEMS IN PERFORATED DOMAINS: A NEW FORMULATION (*)

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SOMMARIO. - *Sia A un operatore ellittico lineare del secondo ordine con coefficienti misurabili e limitati su un aperto limitato Ω di $\mathbf{R}^n$, sia*

$$K^* = \{w^* \in H^1_0(\Omega) : A^* w^* \leq 1 \ in \ \mathcal{D}'(\Omega) \ ,$$

$$e \ w^* \geq 0 \text{ a.e. } in \ \Omega\} \ ,$$

e sia Ω_h un'arbitraria successione di sottoinsiemi aperti di Ω. Dimostriamo il seguente risultato di compattezza: esistono una sottosuccessione, che indichiamo ancora con Ω_h, ed una funzione $w^* \in K^*$ tali che, per ogni $f \in L^\infty(\Omega)$, le soluzioni $u_h \in H^1_0(\Omega_h)$ delle equazioni $Au_h = f$ in Ω_h, estese a zero su $\Omega \backslash \Omega_h$, convergano debolmente in $H^1_0(\Omega)$ all'unica soluzione u del problema

$$(*) \quad \begin{cases} u \in H^1_0(\Omega) \cap L^\infty(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \quad \forall\, \varphi \in C^\infty_0(\Omega). \end{cases}$$

Studiamo inoltre in maniera sistematica le proprietà delle soluzioni di tale equazione. Dimostriamo infine il seguente risultato di densità: per ogni $w^* \in K^*$ esiste una successione Ω_h di sottoinsiemi aperti di Ω tali che per ogni $f \in L^\infty(\Omega)$ le soluzioni $u_h \in H^1_0(\Omega_h)$ dell'equazione $Au_h = f$ in Ω_h, estese a zero $\Omega \backslash \Omega_h$, convergano debolmente in $H^1_0(\Omega)$ alla soluzione di $(*)$.

SUMMARY. - *Let A be a linear elliptic operator of the second order with bounded measurable coefficients on a bounded open set Ω of $\mathbf{R}^n$, let*

$$K^* = \{w^* \in H^1_0(\Omega) : A^* w^* \leq 1 \ in \ \mathcal{D}'(\Omega) \ ,$$

$$and \ w^* \geq 0 \text{ a.e. } in \ \Omega\} \ ,$$

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and let Ω_h be an arbitrary sequence of open subsets of Ω. We prove the following compactness result: there exist a subsequence, still denoted by Ω_h, and a function $w^* \in K^*$ such that, for every $f \in L^\infty(\Omega)$, the solutions $u_h \in H_0^1(\Omega_h)$ of the equation $Au_h = f$ in Ω_h, extended by zero on $\Omega \setminus \Omega_h$, converge weakly in $H_0^1(\Omega)$ to the unique solution u of the problem

$$(*) \quad \begin{cases} u \in H_0^1(\Omega) \cap L^\infty(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \quad \forall\, \varphi \in C_0^\infty(\Omega). \end{cases}$$

We provide a self-contained study of the properties of the solutions of $(*)$. We prove also the following density result: for any $w^* \in K^*$ there exists a sequence Ω_h of open subsets of Ω such that for every $f \in L^\infty(\Omega)$ the solutions $u_h \in H_0^1(\Omega_h)$ of the equation $Au_h = f$ in Ω_h, extended by zero on $\Omega \setminus \Omega_h$, converge weakly in $H_0^1(\Omega)$ to the solution of $(*)$.

Introduction.

The purpose of this paper is the study of the asymptotic behaviour of solutions of linear elliptic equations with Dirichlet boundary conditions in varying domains. This problem has been considered by many authors under different assumptions on the domains and on the operators. A wide bibliography on this subject can be found in [5]. Here we are interested in a compactness result without any hypothesis on the domains.

Our problem can be described in the following way. Let A be a linear elliptic operator of the second order with bounded measurable coefficients on a bounded open set Ω of $\mathbf{R}^n$. Let $f \in H^{-1}(\Omega)$ and let Ω_h be an arbitrary sequence of open subsets of Ω. For each $h \in \mathbf{N}$ let us consider the solutions u_h of the Dirichlet problem

$$(0.1) \qquad u_h \in H_0^1(\Omega_h), \qquad Au_h = f \ \text{ in } \Omega_h$$

and study the behaviour of u_h when h tends to infinity. Using the variational method it is easy to prove that the sequence u_h has a subsequence that converges weakly in $H_0^1(\Omega)$ to some function u. We want to find the equation satisfied by the limit function u. Since $L^\infty(\Omega)$ is dense in $H^{-1}(\Omega)$, in our investigation it will be sufficient to consider $f \in L^\infty(\Omega)$. Let w_h^* be the solution of the Dirichlet problem

$$(0.2)_* \qquad w_h^* \in H_0^1(\Omega_h), \qquad A^*w_h^* = 1 \ \text{ in } \Omega_h\,.$$

As we did for u_h, we can prove that w_h^* has a weak limit w^* in $H_0^1(\Omega)$. By taking now suitable test-functions in $(0.2)_*$ and (0.1) and passing to the limit we find that u is a solution of the following problem:

$$(0.3) \qquad \begin{cases} u \in H_0^1(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \\ \forall\, \varphi \in C_0^\infty(\Omega). \end{cases}$$

The aim of this paper is precisely to provide a self-contained presentation of the properties of the solutions of equation (0.3) and to use this equation to give an elementary description of the asymptotic behaviour of the solutions of (0.1). Let us notice that the solution u_h of (0.1) satisfies an equation of the form (0.3) with w^* replaced by the solution w_h^* of $(0.2)_*$. Under suitable assumptions we shall prove some existence and uniqueness results for (0.3) and the continuous dependence of u on w^*. Let us denote by K^* the set of all functions w^* such that

$$(0.4) \quad w^* \in H_0^1(\Omega)\,,\ A^*w^* \le 1 \ \text{ in } \mathcal{D}'(\Omega), \text{ and } w^* \ge 0 \text{ a.e. in } \Omega.$$

Note that the solutions w_h^* of $(0.2)_*$ belong to K^* (Proposition 1.1) and so does their limit. For any $w^* \in K^*$ and any $f \in L^\infty(\Omega)$ we shall prove that there is one and only one solution of (0.3) in $H_0^1(\Omega) \cap L^\infty(\Omega)$. In order to do this we shall prove first the existence and the uniqueness of the solution for a more regular w^* and then approximate our problem by similar problems with smooth data. This method will allow us to prove the existence of a solution for (0.3) even in the case $f \in H^{-1}(\Omega)$ (Theorem 2.7). The estimates one can prove for this solution give its continuous dependence on w^* in the weak topology of $H_0^1(\Omega)$. This shows, in particular, that the family of problems of type (0.3) with $w^* \in K^*$ is closed under the weak convergence of the solutions.

Problem (0.3) was introduced by Dal Maso and Garroni in [5] where it is studied in an equivalent formulation. They proved that for every $w^* \in K^*$ there exists a positive Borel measure μ on Ω, depending on w^* and vanishing on the subsets of Ω of capacity zero, such that for every $f \in L^\infty(\Omega)$ the solution u of (0.3) coincides with the solution of the problem

$$(0.5) \qquad \begin{cases} u \in H_0^1(\Omega) \cap L^2_\mu(\Omega) \\ \langle Au, v\rangle + \int_\Omega uv\, d\mu = \langle f, v\rangle \qquad \forall\, v \in H_0^1(\Omega) \cap L^2_\mu(\Omega). \end{cases}$$

Note that, in many interesting cases, the measure μ can be very singular and this fact introduces a lot of technical difficulties.

One of the advantages of studying limits of Dirichlet problems by using directly (0.3) and not (0.5) is that some of the proofs can be made independently and in a rather elementary way. We do not have to use singular measures nor fine properties from capacity theory. The degeneracy of the equation (0.3), that follows from the fact that w^* can be zero on sets of positive measure, represents a difficulty of the problem but not a major one since it still allows us to prove some existence and uniqueness results.

The third part of the paper is devoted to the proof of the following density result. We shall show that for any $w^* \in K^*$ there exists a sequence Ω_h of open subsets of Ω such that for every $f \in L^\infty(\Omega)$ the solutions u_h of (0.1) converge weakly in $H_0^1(\Omega)$ to the solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (0.3). This means that the family of problems of type (0.3) with $w^* \in K^*$ can be considered as the closure of the Dirichlet problems (0.1) with respect to the weak convergence in $H_0^1(\Omega)$. By the theorems proved in the previous sections it is enough to prove the existence of a sequence Ω_h such that the solutions w_h^* of $(0.2)_*$ converge weakly in $H_0^1(\Omega)$ to w^*. This will be done by using the method of Cioranescu and Murat [4] following a simplified version of [6].

1. Notations and Preliminaries.

Let us fix an $n \times n$ matrix (a_{ij}) of functions of $L^\infty(\mathbf{R}^n)$ satisfying, for a suitable constant $\alpha > 0$, the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \geq \alpha|\xi|^2 \tag{1.1}$$

for a.e. $x \in \mathbf{R}^n$ and for every $\xi \in \mathbf{R}^n$.

For every open set U of $\mathbf{R}^n$ let $A : H^1(U) \to H^{-1}(U)$ and $A^* : H^1(U) \to H^{-1}(U)$ be the elliptic operators defined by

$$Au = -\sum_{i,j=1}^{n} D_i(a_{ij}D_ju) \qquad \text{and} \qquad A^*u = -\sum_{i,j=1}^{n} D_i(a_{ji}D_ju),$$

where $H^1(U)$ and $H_0^1(U)$ are the usual Sobolev spaces and $H^{-1}(U)$ is the dual of $H_0^1(U)$. It is well known that, on $H_0^1(U)$, A^* is the adjoint operator of A, that is: $\langle A^*u, v\rangle = \langle Av, u\rangle$ for every $u, v \in H_0^1(U)$, where $\langle .,.\rangle$ denotes the duality pairing between $H^{-1}(U)$ and $H_0^1(U)$.

Let Ω be a bounded open subset of $\mathbf{R}^n$. For any $u \in H_0^1(\Omega)$ we shall denote by u^+ and u^- the positive and the negative parts of u: $u^+ = u \vee 0$, $u^- = -(u \wedge 0)$. Then $u = u^+ - u^-$ and it can be easily proved that for any $u \in H_0^1(\Omega)$, u^+, $u^- \in H_0^1(\Omega)$. If U is an open subset of Ω, each function $u \in H_0^1(U)$ will always be extended to Ω by setting $u = 0$ in $\Omega \backslash U$.

Let K^* be the set of functions which satisfy (0.4); then it is easy to see that K^* is a closed convex subset of $H_0^1(\Omega)$. Moreover, for every $w^* \in K^*$

$$\alpha \int_\Omega |Dw^*|^2\,dx \;\le\; \langle A^*w^*, w^*\rangle \;\le\; \int_\Omega w^*\,dx,$$

and this estimate shows that K^* is bounded, and hence weakly compact in $H_0^1(\Omega)$.

Let w_0^* be the solution of the Dirichlet problem

$$A^*w_0^* = 1, \quad w_0^* \in H_0^1(\Omega).$$

By the maximum principle we have $w^* \le w_0^*$ a.e. in Ω for every $w^* \in K^*$. As $w_0^* \in L^\infty(\Omega)$ (see [9]), the set K^* is bounded in $L^\infty(\Omega)$.

Let us denote by H^* the set of all functions $w^* \in H_0^1(\Omega)$ with the property that there exists an open subset U of Ω such that w^* is the solution of the Dirichlet problem

$$\begin{cases} w^* \in H_0^1(U) \\ A^*w^* = 1 \text{ in } U. \end{cases}$$

We shall show that the closure $\bar{H}^*$ of H^* in the weak topology of $H_0^1(\Omega)$ coincides with K^*. We begin with the easier inclusion: $\bar{H}^* \subseteq K^*$. It is enough to prove that $H^* \subseteq K^*$.

PROPOSITION 1.1. *Let U be an open subset of Ω and let w^* be the solution of the Dirichlet problem*

$$A^*w^* = 1 \text{ in } U\,, \quad w^* \in H_0^1(U)\,. \tag{1.2}$$

Then $w^* \in K^*$.

Proof. To prove that $w^* \in K^*$ we follow the argument of [2]. Let z be the solution of the variational inequality

$$z \in K_U\,,\ \langle A^* z - 1, v - z\rangle \geq 0 \qquad \forall\, v \in K_U\,,$$

where

$$K_U = \{v \in H^1_0(\Omega)\ :\ v \leq 0 \text{ a.e. on } \Omega\backslash U\}\,. \tag{1.3}$$

By the maximum principle we have $z \geq 0$ a.e. on Ω (see, e.g., [8], Chapter II, Theorem 6.4), so that $z = 0$ a.e. on $\Omega\backslash U$, hence $z \in H^1_0(U)$ (see, e.g., [1]). If $v \in H^1_0(U)$ and $v = 0$ a.e. on $\Omega\backslash U$, then $v \in K_U$. Therefore from the variational inequality we obtain easily that $z|_U$ is a solution of (1.2), hence $z = w^*$ a.e. in Ω. Since all solutions of variational inequalities with an obstacle condition of the form (1.3) are subsolutions of the corresponding equation (see, e.g., [8], Chapter II, remark after def. 6.3), we conclude that $A^* w^* \leq 1$ in Ω in the sense of distributions, hence $w^* \in K^*$. ◊

The inclusion $K^* \subseteq \bar{H}^*$ will be proved in the third section.

2. Some existence and uniqueness results for the limit problem.

As mentioned in the introduction, we want to prove that for any $w^* \in K^*$ and any $f \in L^\infty(\Omega)$ there exists one and only one solution of (0.3) in $H^1_0(\Omega) \cap L^\infty(\Omega)$. In order to do this we need to prove first some lemmas. Let us begin with the case of $w^* \in W^{1,\infty}(\Omega)$.

Lemma 2.1. *Let* w^* *be a function in* $W^{1,\infty}(\Omega)$ *such that* $A^* w^* \leq 1$ *in* $\mathcal{D}'(\Omega)$ *and* $w^* \geq \varepsilon$ *in* Ω*, for some constant* $\varepsilon > 0$*. Then there exists a unique solution of the problem*

$$\begin{cases} u \in H^1_0(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^* w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \\ \forall\, \varphi \in H^1_0(\Omega). \end{cases} \tag{2.1}$$

Proof. The existence of a unique solution of (2.1) is a consequence of the Lax-Milgram lemma. Indeed, let us consider the bilinear form on $H_0^1(\Omega)\times H_0^1(\Omega)$ defined by:

$$a(u,\varphi) = \langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle.$$

It can be easily seen that

$$a(u,\varphi) = \int_\Omega (\sum_{i,j=1}^n a_{ij} D_j u D_i \varphi) w^* \, dx -$$

$$- \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j w^* D_i \varphi) u \, dx + \int_\Omega u\varphi \, dx.$$

To show that a is coercive we estimate

$$a(u,u) = \int_\Omega (\sum_{i,j=1}^n a_{ij} D_j u D_i u) w^* \, dx -$$

$$- \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j w^* D_i u) u \, dx + \int_\Omega u^2 \, dx.$$

Since $w^* \in W^{1,\infty}(\Omega)$, the distribution A^*w^* belongs to $H^{-1,\infty}(\Omega)$. Then the inequality $A^*w^* \le 1$ in $\mathcal{D}'(\Omega)$ implies that $\langle 1 - A^*w^*, v\rangle \ge 0$ for every $v \in H_0^{1,1}(\Omega)$, $v \ge 0$. As $u \in H_0^1(\Omega)$, we have $u^2 \in H_0^{1,1}(\Omega)$ and $uD_iu = \frac{1}{2}D_i(u^2)$. Then

$$- \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j w^* D_i u) u \, dx + \frac{1}{2}\int_\Omega u^2 \, dx = \frac{1}{2}\langle 1 - A^*w^*, u^2\rangle \ge 0.$$

Since $w^* \ge \varepsilon$ a.e. in Ω, the ellipticity condition (1.1) implies that

$$\int_\Omega (\sum_{i,j=1}^n a_{ij} D_j u D_i u) w^* \, dx \ge \varepsilon\alpha \|Du\|^2_{L^2(\Omega)},$$

which, together with the previous inequality gives

$$a(u,u) \ge \varepsilon\alpha\|Du\|^2_{L^2(\Omega)} + \langle 1 - A^*w^*, u^2\rangle + \frac{1}{2}\int_\Omega u^2 \, dx \ge$$

$$\ge \varepsilon\alpha\|Du\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} \ge c_\varepsilon \|u\|^2_{H_0^1(\Omega)}$$

for some constant $c_\varepsilon > 0$, and this proves that a is coercive.

LEMMA 2.2. *Under the hypotheses of the previous lemma the solution u of (2.1) satisfies the estimate*

$$\|u\|_{H_0^1(\Omega)} \le c\|f\|_{H^{-1}(\Omega)},$$

where the constant $c > 0$ depends only on Ω and on the ellipticity constant α and does not depend on ε.

Proof. By taking $\varphi = \dfrac{u}{w^*}$ as test-function in (2.1) we obtain

$$\langle Au, u\rangle - \langle A^* w^*, \frac{u^2}{w^*}\rangle + \langle 1, \frac{u^2}{w^*}\rangle = \langle f, u\rangle.$$

Since $\dfrac{u^2}{w^*} \in H_0^{1,1}(\Omega)$ and $1 - A^*w^* \ge 0$ we have $\langle 1 - A^*w^*, \dfrac{u^2}{w^*}\rangle \ge 0$. By the ellipticity condition we get

$$\alpha\|Du\|^2_{L^2(\Omega)} \le \langle f, u\rangle \le \|f\|_{H^{-1}(\Omega)}\|u\|_{H_0^1(\Omega)}$$

and the Poincaré Inequality implies the conclusion of the lemma.

LEMMA 2.3. *Under the hypotheses of Lemma 2.1, if $f \ge 0$ in Ω then the solution u of (2.1) is positive.*

Proof. This can be easily seen by taking in (2.1) the test-function $\varphi = \dfrac{u^-}{w^*}$. Indeed, we have

$$\langle Au, u^-\rangle + \langle 1 - A^*w^*, \frac{uu^-}{w^*}\rangle = \langle f, u^-\rangle.$$

Since $uu^- = -(u^-)^2$ and $1 - A^*w^* \ge 0$, we have $\langle 1 - A^*w^*, \dfrac{uu^-}{w^*}\rangle \le 0$. As $f \ge 0$ in Ω and $u^- \ge 0$ a.e. in Ω, we have $\langle f, u^-\rangle \ge 0$, hence $\langle Au, u^-\rangle \ge 0$. The definition of u^- and the ellipticity condition (1.1) imply

$$0 \le \langle Au, u^-\rangle = -\langle Au^-, u^-\rangle \le -\alpha\|u^-\|^2_{H_0^1(\Omega)} \le 0$$

so that $u^- = 0$ a.e. in Ω. ◊

We shall use this lemma to compare the solution u of (2.1) with the solutions of the problems

$$\text{(2.2)} \qquad \begin{cases} w \in H_0^1(\Omega) \cap L^\infty(\Omega) \\ \langle Aw, w^*\varphi\rangle - \langle A^*w^*, w\varphi\rangle + \langle 1, w\varphi\rangle = \langle 1, w^*\varphi\rangle \\ \forall\, \varphi \in C_0^\infty(\Omega) \end{cases}$$

and

$$\text{(2.3)} \qquad w_0 \in H_0^1(\Omega) \qquad Aw_0 = 1 \qquad \text{in } \Omega.$$

LEMMA 2.4. *Under the hypotheses of Lemma 2.1 problem (2.2) has a unique solution w and $w \le w_0$ a.e. in Ω, where w_0 is the solution of (2.3).*

Proof. Lemma 2.1 gives the existence of a unique $w \in H_0^1(\Omega)$ that satisfies the equation in (2.2) for any $\varphi \in H_0^1(\Omega)$. By Lemma 2.3 we have $w \ge 0$. Then by taking the test-functions $\dfrac{(w-w_0)^+}{w^*}$ in (2.2) and $(w-w_0)^+$ in (2.3) and taking the difference of the two equalities we obtain $(w-w_0)^+ = 0$ a.e. in Ω, that is $w \le w_0$ a.e. in Ω. Since $w_0 \in L^\infty(\Omega)$ we get $w \in L^\infty(\Omega)$ and so w is a solution of (2.2). The uniqueness follows by density arguments. ◊

LEMMA 2.5. *Under the hypotheses of Lemma 2.1, if $f \in L^\infty(\Omega)$ then the solution u of (2.1) satisfies the estimate $|u| \le \|f\|_{L^\infty(\Omega)} w$ a.e. in Ω, where w is the solution of (2.2).*

Proof. Let $c = \|f\|_{L^\infty(\Omega)}$. Multiplying the equation in (2.2) by c and subtracting the equation (2.1) satisfied by u we obtain that $cw - u$ is the solution of the equation in (2.1) with f replaced by $c - f$. Applying now Lemma 2.3 we get $cw - u \ge 0$ a.e. in Ω, hence $u \le cw$ a.e. in Ω. The inequality $u \ge -cw$ is proved in a similar way. ◊

LEMMA 2.6. *Let $w^* \in K^*$ and let Ω' be a regular bounded open* *subset of* $\mathbf{R}^n$ *such that* $\Omega \subset\subset \Omega'$. *If* w^* *is extended by* 0 *on* $\Omega' \backslash \Omega$, *then* $A^* w^* \leq 1$ *in* $\mathcal{D}'(\Omega')$.

Proof. This property was proved in [3], Lemma A, in the case of the Laplace operator $-\Delta$. For the sake of completeness, we repeat the proof here for a general operator A.

Let us define the set $K^{w^*} = \{v \in H_0^1(\Omega') \, : \, v \leq w^* \text{ a.e. in } \Omega'\}$ and let z be the solution of

$$
\left\{ \begin{array}{l} z \in K^{w^*} \\ \langle A^* z - 1, v - z \rangle_{\Omega'} \geq 0 \qquad \forall \, v \in K^{w^*} , \end{array} \right. \tag{2.4}
$$

where $\langle \ , \ \rangle_{\Omega'}$ denotes the duality product between $H^{-1}(\Omega')$ and $H_0^1(\Omega')$.

Then $z \geq 0$ in Ω'. Indeed, z is the greatest subsolution of $A^* v = 1$ that belongs to K^{w^*}. (See, e.g., [8], Chapter II, Theorem 6.4.) As 0 is such a subsolution we have $z \geq 0$ in Ω'.

We claim that $z = w^*$. If we take $v = w^*$ in (2.4), we obtain

$$
\langle A^* z - 1, w^* - z \rangle_{\Omega'} \geq 0.
$$

Since $0 \leq z \leq w^*$ in Ω', we have $w^* - z = 0$ on $\Omega' \backslash \Omega$ hence

$$
\langle A^* z - 1, w^* - z \rangle_{\Omega} \geq 0 \, . \tag{2.5}
$$

As $1 - A^* w^* \geq 0$ in Ω, we get

$$
\langle A^* w^* - 1, w^* - z \rangle_{\Omega} \leq 0,
$$

and subtracting (2.5) we obtain

$$
\langle A^* (w^* - z), w^* - z \rangle_{\Omega} \leq 0
$$

and so, the ellipticity of A^* implies $w^* - z = 0$ in Ω. Since $0 \leq z \leq w^* = 0$ in $\Omega' \backslash \Omega$, we have shown that $w^* = z$ in Ω' and the conclusion follows from the inequality

$$
1 - A^* z \geq 0 \text{ in } \Omega' ,
$$

which holds for all solutions of variational inequalities with an obstacle of the form (2.4). (See, e.g., [8], Chapter II, remark after Def. 6.3.) $\diamondsuit$

THEOREM 2.7. *For any $w^* \in K^*$ and any $f \in H^{-1}(\Omega)$ there exists a solution u of the problem*

$$(0.3) \qquad \begin{cases} u \in H_0^1(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \\ \forall\, \varphi \in C_0^\infty(\Omega), \end{cases}$$

which satisfies the estimate $\|u\|_{H_0^1(\Omega)} \le c\|f\|_{H^{-1}(\Omega)}$, for a suitable constant c that depends only on Ω and on the ellipticity constant α and does not depend on w^*.

Proof. Let us consider a regular bounded open subset Ω' of $\mathbf{R}^n$ such that $\Omega \subset\subset \Omega'$ and let us extend w^* by 0 on $\Omega' \setminus \Omega$.

By Lemma 2.6 we have that $\nu^* = 1 - A^*w^* \ge 0$ in $\mathcal{D}'(\Omega')$, hence ν^* is a positive Radon measure. As $A^*w^* \in H^{-1}(\Omega')$, we have also that $\nu^* \in H^{-1}(\Omega')$.

We can approximate (a_{ij}) by a sequence (a_{ij}^ε) of matrices of class C^∞ which converges a.e. to (a_{ij}) and satisfies the ellipticity and boundedness conditions with the same constants as (a_{ij}). We shall denote the corresponding operators by A_ε and A_ε^*. Let $\nu_\varepsilon^* \in C^\infty(\Omega')$, $\nu_\varepsilon^* \ge 0$, approximate ν^* strongly in $H^{-1}(\Omega')$ and let w_ε^* be the solution of the Dirichlet problem

$$\begin{cases} w_\varepsilon^* - \varepsilon \in H_0^1(\Omega'), \\ 1 - A_\varepsilon^* w_\varepsilon^* = \nu_\varepsilon^* \text{ in } H^{-1}(\Omega'). \end{cases}$$

From the regularity theory we deduce that $w_\varepsilon^* \in C^\infty(\Omega')$.

Let us prove that $w_\varepsilon^* - \varepsilon$ converges to w^* weakly in $H_0^1(\Omega')$. Since $w_\varepsilon^* - \varepsilon$ is bounded in $H_0^1(\Omega')$ it has a weak limit $v \in H_0^1(\Omega')$. We write the weak form of the equation:

$$\int_{\Omega'} \Big(\sum_{i,j=1}^n a_{ji}^\varepsilon D_i\varphi D_j w_\varepsilon^*\Big)\, dx = \int_{\Omega'} \varphi\, dx - \langle \nu_\varepsilon^*, \varphi\rangle \qquad \forall\, \varphi \in H_0^1(\Omega').$$

As (a_{ji}^ε) is bounded, we have $|a_{ji}^\varepsilon D_i\varphi| \le M|D_i\varphi| \in L^2(\Omega')$ and the pointwise convergence a.e. of $a_{ji}^\varepsilon D_i\varphi$ to $a_{ji} D_i\varphi$ implies, by the Lebesgue Dominated Convergence Theorem, the strong convergence in $L^2(\Omega')$. As $D_j w_\varepsilon^*$ converges weakly in $L^2(\Omega')$ to $D_j v$ we obtain that the left hand side of the equation converges to

$$\int_{\Omega'} \sum_{i,j=1}^n a_{ji} D_i\varphi D_j v\, dx\, .$$

Then, as $\nu^*_\varepsilon \to \nu^*$ in $H^{-1}(\Omega')$, $\langle \nu^*_\varepsilon, \varphi\rangle$ converges to $\langle \nu^*, \varphi\rangle$, so that v satisfies the same equation as w^*, i.e.

$$\int_{\Omega'} \Big(\sum_{i,j=1}^{n} a_{ji} D_j v D_i \varphi\Big)\, dx = \int_{\Omega'} \varphi\, dx - \langle \nu^*, \varphi\rangle \qquad \forall\, \varphi \in H^1_0(\Omega')\,,$$

hence $w^* = v$ a.e. in Ω'.

Let us prove now that $w^*_\varepsilon - \varepsilon \to w^*$ strongly in $H^1_0(\Omega')$. In the equations satisfied by w^* and w^*_ε we take as test-functions $w^* - w^*_\varepsilon + \varepsilon$ and obtain

$$\langle A^* w^*, w^* - w^*_\varepsilon\rangle + \langle \nu^*, w^* - w^*_\varepsilon + \varepsilon\rangle =$$

$$= \langle 1, w^* - w^*_\varepsilon + \varepsilon\rangle = \langle A^*_\varepsilon w^*_\varepsilon, w^* - w^*_\varepsilon\rangle + \langle \nu^*_\varepsilon, w^* - w^*_\varepsilon + \varepsilon\rangle.$$

The ellipticity condition for A^*_ε gives

$$\alpha \| D(w^* - w^*_\varepsilon)\|^2_{L^2(\Omega')} \le \langle A^*_\varepsilon (w^* - w^*_\varepsilon), w^* - w^*_\varepsilon\rangle$$

and using the previous equality we substitute $\langle A^*_\varepsilon w^*_\varepsilon, w^* - w^*_\varepsilon\rangle$ and obtain

$$\alpha \| D(w^* - w^*_\varepsilon)\|^2_{L^2(\Omega')} \le \langle A^*_\varepsilon w^* - A^* w^*, w^* - w^*_\varepsilon\rangle + \langle \nu^*_\varepsilon - \nu^*, w^* - w^*_\varepsilon + \varepsilon\rangle.$$

As $\nu^*_\varepsilon \to \nu^*$ strongly in $H^{-1}(\Omega')$ and $w^*_\varepsilon - \varepsilon \rightharpoonup w^*$ weakly in $H^1_0(\Omega')$, the second term in the right hand side converges to zero. Let us consider the first term

$$\langle A^*_\varepsilon w^* - A^* w^*, w^* - w^*_\varepsilon\rangle = \int_\Omega \Big(\sum_{i,j=1}^{n} (a^\varepsilon_{ji} - a_{ji}) D_j w^* D_i (w^* - w^*_\varepsilon)\Big) dx\,.$$

As a_{ji} and (a^ε_{ji}) are bounded, $|(a^\varepsilon_{ji} - a_{ji}) D_j w^*| \le M |D_j w^*| \in L^2(\Omega')$, $(a^\varepsilon_{ji} - a_{ji}) D_j w^*$ converges pointwise a.e. to zero, by the Lebesgue Dominated Convergence Theorem, the convergence is also in $L^2(\Omega')$. As $D_i(w^* - w^*_\varepsilon)$ converges weakly in $L^2(\Omega')$ to zero we get that the first term converges to zero and so $\|D(w^* - w^*_\varepsilon)\|^2_{L^2(\Omega')} \to 0$, that is $w^*_\varepsilon - \varepsilon \to w^*$ strongly in $H^1_0(\Omega')$.

We shall continue now the proof of the existence of a solution of (0.3).

Let us consider the function $w^*_\varepsilon \vee \varepsilon$. As $A^* w^*_\varepsilon \le 1$ and $A^*\varepsilon \le 1$ we have that $1 - A^*(w^*_\varepsilon \vee \varepsilon) \ge 0$.(See, e.g., [8], Chapter II, Theorem

6.6.) Since $w_\varepsilon^* - \varepsilon \to w^*$ strongly in $H_0^1(\Omega')$ and $w^* \geq 0$, also $(w_\varepsilon^* \vee \varepsilon) - \varepsilon \to w^* \vee 0 = w^*$ strongly in $H_0^1(\Omega')$. As $(w_\varepsilon^* \vee \varepsilon) \in W^{1,\infty}(\Omega)$, by Lemma 2.1 there exists a function $u_\varepsilon \in H_0^1(\Omega)$ such that

$$(2.1)_\varepsilon \qquad \begin{aligned} &\langle A_\varepsilon u_\varepsilon, (w_\varepsilon^* \vee \varepsilon)\varphi\rangle - \langle A_\varepsilon^*(w_\varepsilon^* \vee \varepsilon), u_\varepsilon \varphi\rangle + \langle 1, u_\varepsilon \varphi\rangle = \\ &= \langle f, (w_\varepsilon^* \vee \varepsilon)\varphi\rangle \qquad \forall\, \varphi \in H_0^1(\Omega). \end{aligned}$$

By Lemma 2.2 we have $\|u_\varepsilon\|_{H_0^1(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)}$, so that, up to a subsequence, u_ε converges weakly to a function $u \in H_0^1(\Omega)$. We shall consider now test-functions $\varphi \in C_0^\infty(\Omega)$ and by passing to the limit in $(2.1)_\varepsilon$ we get that the limit function u is a solution of (0.3). As $u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$ and $\|u_\varepsilon\|_{H_0^1(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)}$, from the lower semicontinuity of the norm we obtain that $\|u\|_{H_0^1(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)}$.

Proposition 2.8. *Let $w^* \in K^*$ and $f \in H^{-1}(\Omega)$. If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution of (0.3) then u satisfies the equation for any test-function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover, if u_1, $u_2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ are solutions of (0.3) then $u_1 = u_2$.*

Proof. Let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution of (0.3) and let $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Since $w^* \in K^*$ is also in $L^\infty(\Omega)$ the products $u\varphi$ and $w^*\varphi$ belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$, hence all terms of the equation are well-defined. There exists a sequence φ_h of functions in $C_0^\infty(\Omega)$, bounded in $L^\infty(\Omega)$, that converges strongly in $H_0^1(\Omega)$ to φ. We consider in (0.3) φ_h as test-function, pass to the limit in the equation

$$\langle Au, w^*\varphi_h\rangle - \langle A^*w^*, u\varphi_h\rangle + \langle 1, u\varphi_h\rangle = \langle f, w^*\varphi_h\rangle$$

and obtain that u satisfies the equation with φ as test-function.

In order to prove the uniqueness let us denote by u the difference $u_1 - u_2$. We have

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i \varphi\Big) w^*\, dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^* D_i \varphi\Big) u\, dx + \int_\Omega u\varphi\, dx = 0$$

$$\text{for every } \varphi \in C_0^\infty(\Omega)\,.$$

As w^* and u are bounded, this equality holds for any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then we can take u as test-function and obtain

(2.6)
$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i u\Big) w^* \, dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^* D_i u\Big) u \, dx + \int_\Omega u^2 \, dx = 0 \,.$$

We have that
$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^* D_i u\Big) u \, dx = \frac{1}{2} \langle A^* w^*, u^2 \rangle$$

and since $\langle 1 - A^* w^*, u^2 \rangle \geq 0$, we get
$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i u\Big) w^* \, dx + \frac{1}{2} \int_\Omega u^2 \, dx \leq 0 \,.$$

As the first term is nonnegative, by the ellipticity of (a_{ij}) and the positivity of w^*, we obtain that $u = 0$ a.e. in Ω and so the uniqueness is proved.

Theorem 2.9. *Let $w^* \in K^*$ and $f \in L^\infty(\Omega)$. Then there exists a unique solution of the problem*

$$(2.7) \qquad \begin{cases} u \in H_0^1(\Omega) \cap L^\infty(\Omega) \\ \langle Au, w^* \varphi \rangle - \langle A^* w^*, u\varphi \rangle + \langle 1, u\varphi \rangle = \langle f, w^* \varphi \rangle \\ \forall\, \varphi \in C_0^\infty(\Omega). \end{cases}$$

Moreover, u satisfies the equation for any test-function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and we have the following estimates $\|u\|_{H_0^1(\Omega)} \leq c \|f\|_{H^{-1}(\Omega)}$ and $|u| \leq \|f\|_{L^\infty(\Omega)} w \leq \|f\|_{L^\infty(\Omega)} w_0$ a.e. in Ω where w and w_0 are the solutions of (2.2) and (2.3), respectively, and c is a constant depending only on Ω and on the ellipticity constant α and not on w^*.

Proof. Let us consider the construction done in Theorem 2.7 for the proof of existence. If we denote by w_0^ε the solution of the Dirichlet problem $A_\varepsilon w_0^\varepsilon = 1$ in Ω, $w_0^\varepsilon \in H_0^1(\Omega)$, and by w_ε the solution of

$$\begin{cases} w_\varepsilon \in H_0^1(\Omega) \\ \langle A_\varepsilon w_\varepsilon, (w_\varepsilon^* \vee \varepsilon)\varphi \rangle - \langle A_\varepsilon^* (w_\varepsilon^* \vee \varepsilon), w_\varepsilon \varphi \rangle + \langle 1, w_\varepsilon \varphi \rangle = \\ \qquad = \langle 1, (w_\varepsilon^* \vee \varepsilon)\varphi \rangle \qquad \forall\, \varphi \in H_0^1(\Omega), \end{cases}$$

then by applying Lemma 2.5 and Lemma 2.4 to A_ε and $(w^*_\varepsilon \vee \varepsilon)$, we obtain that $|u_\varepsilon| \leq \|f\|_{L^\infty(\Omega)} w_\varepsilon \leq \|f\|_{L^\infty(\Omega)} w_0^\varepsilon$. Since the weak convergence of u_ε to u and of w_ε to w proved in Theorem 2.7 implies the pointwise convergence of a subsequence, by passing to the limit we obtain $|u| \leq \|f\|_{L^\infty(\Omega)} w \leq \|f\|_{L^\infty(\Omega)} w_0$. Since $w_0 \in L^\infty(\Omega)$, also $u \in L^\infty(\Omega)$. Then the uniqueness and the fact that the equation is satisfied for any test-function in $H^1_0(\Omega) \cap L^\infty(\Omega)$ follow from Proposition 2.8 and Theorem 2.7 gives the first estimate. $\diamondsuit$

PROPOSITION 2.10. *Let $f \in L^\infty(\Omega)$, let U be an open subset of Ω and let u and w^* be the solutions of the problems*

$$\begin{cases} u \in H^1_0(U) \\ Au = f \ \textit{in}\ U \end{cases} \qquad\qquad \begin{cases} w^* \in H^1_0(U) \\ A^*w^* = 1 \ \textit{in}\ U\,. \end{cases}$$

Then u is the solution of

$$(2.7) \qquad \begin{cases} u \in H^1_0(\Omega) \cap L^\infty(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \\ \forall\, \varphi \in C^\infty_0(\Omega). \end{cases}$$

Proof. Let $\varphi \in C^\infty_0(\Omega)$. Since $u,\ w^* \in H^1_0(U) \cap L^\infty(U)$, the products $u\varphi$ and $w^*\varphi$ belong to $H^1_0(U)$ so that can be considered as test-functions in the equations satisfied by w^* and u, respectively. We obtain that

$$\langle Au, w^*\varphi\rangle = \langle f, w^*\varphi\rangle, \qquad \langle A^*w^*, u\varphi\rangle = \langle 1, u\varphi\rangle, \qquad \forall \varphi \in C^\infty_0(\Omega)$$

and by subtraction, we get that u is the solution of (2.7). $\diamondsuit$

Let us study now the dependence on w^* of the solutions of (2.7).

THEOREM 2.11. *Let $f \in L^\infty(\Omega)$, let $w^*_h \in K^*$ and let u_h be the solution of the problem*

$$(2.8) \qquad \begin{cases} u_h \in H^1_0(\Omega) \cap L^\infty(\Omega) \\ \langle Au_h, w^*_h\varphi\rangle - \langle A^*w^*_h, u_h\varphi\rangle + \langle 1, u_h\varphi\rangle = \langle f, w^*_h\varphi\rangle \\ \forall\, \varphi \in C^\infty_0(\Omega). \end{cases}$$

Assume that w_h^* converges weakly in $H_0^1(\Omega)$ to a function $w^* \in K^*$. Then u_h converges weakly in $H_0^1(\Omega)$ to the solution u of (2.7).

Proof. The estimate $\|u_h\|_{H_0^1(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)}$, proved in Theorem 2.9, gives the existence of a subsequence, still denoted by u_h, that converges weakly in $H_0^1(\Omega)$ to some function u. As $|u_h| \leq \|f\|_{L^\infty(\Omega)} w_0$, where w_0 is the solution of (2.3), the sequence (u_h) is bounded in $L^\infty(\Omega)$, thus $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then

$$\langle Au_h, w_h^* \varphi \rangle - \langle A^* w_h^*, u_h \varphi \rangle + \langle 1, u_h \varphi \rangle = \langle f, w_h^* \varphi \rangle$$

can be written as

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i \varphi\Big) w_h^* \, dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w_h^* D_i \varphi\Big) u_h \, dx +$$

$$+ \int_\Omega u_h \varphi \, dx = \langle f, w_h^* \varphi \rangle ,$$

we may pass to the limit and obtain that u is a solution of (2.7). The uniqueness of the solution implies that the whole sequence (u_h) converges weakly in $H_0^1(\Omega)$ to the solution u of (2.7). ♢

LEMMA 2.12. *Let Ω_h be an arbitrary sequence of open subsets of Ω. Then there exist a subsequence, still denoted by Ω_h, and a function $w^* \in K^*$ such that for every $f \in L^\infty(\Omega)$ the solution u_h of (0.1) extended by* 0 *on $\Omega \setminus \Omega_h$ converges weakly in $H_0^1(\Omega)$ to the solution u of (2.7).*

Proof. Let w_h^* be the solution of $(0.2)_*$. As (w_h^*) is bounded in $H_0^1(\Omega)$ there exists a subsequence, still denoted by (w_h^*), that converges weakly in $H_0^1(\Omega)$ to a function w^*. Since $w_h^* \in K^*$ (Proposition 1.1) and K^* is weakly closed we obtain $w^* \in K^*$. Let u_h be the solution of (0.1). Then, as (u_h) is bounded in $H_0^1(\Omega)$, it has a subsequence that converges weakly in $H_0^1(\Omega)$ to some function u. By Proposition 2.10 u_h is the solution of (2.8) and by applying now Theorem 2.11 we deduce that the limit u is the solution of (2.7). ♢

3. A density result.

Our purpose is now to complete the characterization given by Theorem 2.12, that is we want to show that if $w^* \in K^*$, $f \in L^\infty(\Omega)$ and u is the solution of (2.7) then there exists a sequence of domains Ω_h such that the corresponding solutions u_h of (0.1) converge weakly in $H_0^1(\Omega)$ to u. From Theorem 2.11 of Section 2 it follows that we have only to prove that any $w^* \in K^*$ can be approximated by solutions of $(0.2)_*$.

Let us return to the sets K^* of all functions which satisfy (0.4) and H^* defined in Section 1 (before Proposition 1.1). We shall prove that the weak closure in $H_0^1(\Omega)$ of H^* is equal to K^*. As we have already remarked the closure of H^* is contained in K^* (Proposition 1.1). So we have only to prove that any function $w^* \in K^*$ can be approximated by functions in H^*. To this end let us define two auxiliary sets:

K_1^* - the set of all functions in K^* that are continuous and strictly positive on Ω and

K_2^* - the set of all functions in $H_0^1(\Omega)$ that satisfy $A^*w^*+bw^* = 1$ in the sense of distributions on Ω, for some continuous and positive function b.

REMARK 3.1. $K_2^* \subseteq C^0(\Omega) \cap K^*$.

Proof. Let $w^* \in K_2^*$. By De Giorgi's Theorem (see, e.g., [7] theorem 8.22) $w^* \in C^0(\Omega)$. Since $A^*w^* + bw^* = 1$ in the sense of distributions and $b \geq 0$ we get $w^* \geq 0$ and $A^*w^* \leq 1$. $\diamondsuit$

REMARK 3.2. The closure of K_2^* in the weak topology of $H_0^1(\Omega)$ contains K_1^*.

Proof. Indeed, let $w^* \in K_1^*$. Then $A^*w^* + \nu = 1$, where $\nu \in H^{-1}(\Omega)$ is a positive Radon measure. Since $w^* > 0$, we can define $\mu = \dfrac{\nu}{w^*}$. We can approximate μ strongly in $H^{-1}(\Omega)$ by continuous and positive functions b_ε.

Let $w^*_\varepsilon \in K^*_2$ be the solution of the following Dirichlet problem:

$$\begin{cases} w^*_\varepsilon \in H^1_0(\Omega), \\ A^* w^*_\varepsilon + b_\varepsilon w^*_\varepsilon = 1. \end{cases}$$

Then $w^*_\varepsilon \geq 0$ and, as $b_\varepsilon \geq 0$, $A^* w^*_\varepsilon \leq 1$ in the sense of distributions on Ω.

By De Giorgi's Theorem $w^*_\varepsilon \in C^0(\Omega)$ hence $1 - A^* w^*_\varepsilon = b_\varepsilon w^*_\varepsilon \in C^0(\Omega)$. As (w^*_ε) is bounded in $H^1_0(\Omega)$, there exists a function $\tilde{w} \in H^1_0(\Omega)$ such that w^*_ε converges to $\tilde{w}$ weakly in $H^1_0(\Omega)$. Since $w^*_\varepsilon \leq w^*_0$, where $w^*_0 \in H^1_0(\Omega)$ is the solution of $A^* w^*_0 = 1$ in Ω, and $w^*_0 \in L^\infty(\Omega)$ we have that (w^*_ε) is bounded in $L^\infty(\Omega)$, hence $\tilde{w} \in L^\infty(\Omega)$. We have

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^*_\varepsilon D_i \varphi\Big)\, dx + \int_\Omega b_\varepsilon w^*_\varepsilon \varphi\, dx = \int_\Omega \varphi\, dx \qquad \forall\, \varphi \in C^\infty_0(\Omega).$$

Passing to the limit as $\varepsilon \to 0$, we get

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j \tilde{w} D_i \varphi\Big)\, dx + \int_\Omega \tilde{w} \varphi \, \frac{d\nu}{w^*} = \int_\Omega \varphi\, dx \qquad \forall\, \varphi \in C^\infty_0(\Omega).$$

The above equation is satisfied for all $\varphi \in H^1_0(\Omega)$. This can be proved by density arguments using the fact that $\tilde{w} \in L^\infty(\Omega)$ and that $H^1_0(\Omega) \subseteq L^1_\mu(\Omega)$ for any positive Radon measure belonging to $H^{-1}(\Omega)$. As w^* is a solution in $H^1_0(\Omega)$, we get that $w^* = \tilde{w}$ a.e. in Ω. So, w^*_ε converges to w^* weakly in $H^1_0(\Omega)$. $\diamondsuit$

REMARK 3.3. The closure of K^*_1 in the weak topology of $H^1_0(\Omega)$ is equal to K^*.

Proof. Let us first remark that by definition, $K^*_1 \subseteq K^*$. Let $w \in K^*$. Then $\nu = 1 - A^* w^*$ is a positive Radon measure that belongs to $H^{-1}(\Omega)$. We can approximate it strongly in $H^{-1}(\Omega)$ by a sequence of positive smooth functions ν_ε. Let us consider the solution v_ε of the Dirichlet problem

$$\begin{cases} v_\varepsilon \in H^1_0(\Omega) \\ A^* v_\varepsilon + \nu_\varepsilon v_\varepsilon = 1. \end{cases}$$

By the maximum principle $v_\varepsilon \geq 0$ and by De Giorgi's Theorem $v_\varepsilon \in C^0(\Omega)$. By the same arguments as before we obtain the weak convergence in $H_0^1(\Omega)$ of v_ε to w^*. In order to obtain a sequence of functions in K_1^* let us consider the solution $w_0^* \in H_0^1(\Omega)$ of $A^* w_o^* = 1$. By the strong maximum principle [9] we have that $w_0^* > 0$ and, by De Giorgi's Theorem, $w_0^* \in C^0(\Omega)$. We define then $w_\varepsilon^* = (1-\varepsilon)v_\varepsilon + \varepsilon w_0^*$. It is easy to see that $w_\varepsilon^* \in K_1^*$ and $w_\varepsilon^* \rightharpoonup w^*$ weakly in $H_0^1(\Omega)$. $\diamondsuit$

The conclusion of the Remarks 3.1-3.3 is that K_2^* is dense in K^* with respect to the weak topology of $H_0^1(\Omega)$, hence in order to prove that H^* is dense in K^* it is enough to show that every element of K_2^* can be approximated by elements of H^*.

THEOREM 3.4. *The closure of H^* with respect to the weak topology of $H_0^1(\Omega)$ contains K^*.*

Proof. As we have mentioned above it suffices to show that the closure of H^* with respect to the weak topology of $H_0^1(\Omega)$ contains K_2^*. Let $w^* \in K_2^*$. This means that $w^* \in H_0^1(\Omega)$ and there exists a continuous, positive function b on Ω such that $A^*w^* + bw^* = 1$ on Ω in the sense of distributions.

In order to get a sequence of functions in H^* that converges weakly in $H_0^1(\Omega)$ to w^* we shall use the method of Cioranescu and Murat [4] following the lines of [6]. There exist a sequence of open subsets Ω_h of Ω, a sequence z_h of functions in $H^1(\Omega)$ that converges weakly in $H^1(\Omega)$ to 1 and two sequences λ_h and ν_h of measures in $H^{-1}(\Omega)$ such that $A^*z_h = \nu_h - \lambda_h$ in Ω, λ_h converges to b weakly in $H^{-1}(\Omega)$, ν_h converges to b strongly in $H^{-1}(\Omega)$ and $\langle \lambda_h, \varphi \rangle = 0$ for every function $\varphi \in H_0^1(\Omega_h)$. For the construction see [6]. We may assume that $0 \leq z_h \leq 1$ a.e. in Ω.

Let us define $u_h = z_h w^*$. By construction $u_h \in H_0^1(\Omega_h)$. From the weak convergence of z_h to 1 we deduce that u_h converges to w^* weakly in $H_0^1(\Omega)$.

Let w_h^* be the solution of the Dirichlet problem

$$\begin{cases} w_h^* \in H_0^1(\Omega_h), \\ A^* w_h^* = 1 \text{ on } \Omega_h. \end{cases}$$

We extend w_h^* by zero on $\Omega \backslash \Omega_h$. Then (w_h^*) has a subsequence that converges to some function v weakly in $H_0^1(\Omega)$. We want to prove that $w^* = v$. (As a consequence the whole sequence (w_h^*) converges to w^*.) The properties of z_h and w^* imply that there exists $c_1 > 0$ such that $\|u_h\|_{L^\infty(\Omega)} \leq c_1$. There exists also $c_2 > 0$ such that $\|w_h^*\|_{L^\infty(\Omega)} \leq c_2$. We have

$$\langle A^* u_h, u_h - w_h^* \rangle = \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j u_h D_i (u_h - w_h^*)) dx =$$

$$= \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j z_h w^* D_i (u_h - w_h^*)) dx + \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j w^* z_h D_i (u_h - w_h^*)) dx =$$

$$= \int_\Omega \sum_{i,j=1}^n a_{ji} D_j z_h D_i (w^*(u_h - w_h^*)) dx - \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j z_h D_i w^*)(u_h - w_h^*) dx +$$

$$+ \int_\Omega \sum_{i,j=1}^n a_{ji} D_j w^* D_i (z_h (u_h - w_h^*)) dx - \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j w^* D_i z_h)(u_h - w_h^*) dx =$$

$$= \int_\Omega w^*(u_h - w_h^*) d\nu_h + \int_\Omega (1 - bw^*) z_h (u_h - w_h^*) dx -$$

$$- \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j z_h D_i w^*)(u_h - w_h^*) dx - \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j w^* D_i z_h)(u_h - w_h^*) dx =$$

$$= I_1 + I_2 - I_3 - I_4,$$

where we have used the fact that $w^*(u_h - w_h^*) \in H_0^1(\Omega_h)$ so that $\langle \lambda_h, w^*(u_h - w_h^*) \rangle = 0$.

As $w^*, (u_h), (w_h^*)$ are bounded in $L^\infty(\Omega)$, the product $w^*(u_h - w_h^*)$ converges to $w^*(w^* - v)$ weakly in $H_0^1(\Omega)$. Then the strong convergence of ν_h to b in $H^{-1}(\Omega)$ implies the convergence of I_1 to $\int_\Omega w^*(w^* - v) b dx$. Since $u_h - w_h^* \to w^* - v$ in $L^2(\Omega)$, the second term I_2 converges to $\int_\Omega (1 - bw^*)(w^* - v) dx$. From the weak convergence of $D_j z_h$ to 0 in $L^2(\Omega)$, the boundedness in $L^\infty(\Omega)$ of $(u_h - w_h^*)$ and its strong convergence to $w^* - v$ in $L^2(\Omega)$ we deduce that $I_3 \to 0$ and the same arguments hold for I_4. So that

$$\alpha \|u_h - w_h^*\|^2_{H_0^1(\Omega)} \leq \langle A^*(u_h - w_h^*), u_h - w_h^* \rangle =$$
$$= \langle A^* u_h, u_h - w_h^* \rangle - \langle A^* w_h^*, u_h - w_h^* \rangle =$$
$$\langle A^* u_h, u_h - w_h^* \rangle - \langle 1, u_h - w_h^* \rangle = Z_h$$

Since Z_h converges to $\int_\Omega w^*(w^* - v) b dx + \int_\Omega (1 - bw^*)(w^* - v) dx - \langle 1, w^* - v \rangle = 0$ we get $w^* = v$. So, for any $w^* \in K_2^*$ there exists a sequence of functions w_h^* in H^* such that w_h^* converges to w^* weakly in $H_0^1(\Omega)$, hence H^* is dense in K^*. $\diamondsuit$

THEOREM 3.5. *Let* $w^* \in K^*$ *and* $f \in L^\infty(\Omega)$. *If* u *is the solution of (2.7) then there exists a sequence* Ω_h *of open subsets of* Ω *such that the corresponding solutions* u_h *of (0.1) extended by* 0 *on* $\Omega \backslash \Omega_h$ *converge to* u *weakly in* $H_0^1(\Omega)$.

Proof. Theorem 3.4 gives the existence of a sequence Ω_h of open subsets of Ω such that the solution w_h^* of $(0.2)_*$ converges weakly in $H_0^1(\Omega)$ to w^*. Then the corresponding solutions u_h of (0.1) converge weakly in $H_0^1(\Omega)$ to u. This can be seen for example by using Proposition 2.10 and Theorem 2.11. $\diamondsuit$

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