THURSTON'S SOLITAIRE TILINGS OF THE PLANE (*)

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Sommario. - *Dato un numero di Pisot β e un insieme finito D di interi algebrici in $\mathbb{Q}(\beta)$, è possibile rappresentare i numeri complessi in base β con cifre in D. Se D è ordinato si può dire quali sono le rappresentazioni preferite, ed esiste un automa a stati finiti che riconosce tali rappresentazioni. Questo conduce a tassellazioni del piano tali che tramite l'espansione di fattore β ogni tegola della tassellazione viene mandata in una unione di tegole. Questo lavoro espande idee di Thurston.*

Summary. - *Given a Pisot number β and a finite set D of algebraic integers in $\mathbb{Q}(\beta)$, one can represent complex numbers in base β using digits D. If D has an order one can say which representations are preferred, and there exists a finite state automaton which recognizes such representations. This leads to tilings of the plane such that under the β-expansion each tile maps to a union of tiles. This paper expands ideas of Thurston.*

We will describe in this paper a construction due to Bill Thurston [11] of self-similar tilings of the plane (the name solitaire, not used in the sequel, is due to him). The basic idea of this construction is to define representations of complex numbers with respect to a given base using a given set of digits (just as the positive real numbers are represented in base 10 with digits $0, 1, ..., 9$). If the base β is a Pisot number and the digits are algebraic integers in $\mathbb{Q}(\beta)$ then there exists a finite state automaton which determines what are the "preferred representations" (in the previous example, both $0.9999\cdots$ and $1.0000\cdots$ are representations of 1: this is actually not a good example, as in our construction we will have to consider *both* of them as

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preferred representations of 1; but in other examples one really rules out some representations). Having these "preferred representations" one can group up complex numbers according to the "integer part" of their representation; the result is a self-similar tiling of the plane.

Apparently there has been no detailed account in the literature of this beautiful idea of Thurston, which combines the topology of the plane with number theory and the theory of automata. On the other hand (self-similar) tilings (of the plane) are important objects of interest in classical and modern mathematics (see [8] for a comprehensive introduction, then [11] again and [6], [7]; the recent works [2] and [1] also deal with relations of the theory of tilings with automata and special algebraic numbers, but from different viewpoints).

With respect to the original paper of Thurston (apart from giving full proofs of all the results) we will prove that the failure test for a state of the machine recognizing the preferred representations can be itself performed by a finite state machine: in the original paper this failure test was expressed in a somewhat implicit way (and at first it was not clear to the author how to implement it). This fact (together with a number of minor results whose scope is to keep the size of the automata involved in reasonable terms) has enabled us to write computer programs (using Mathematica) which actually allow to draw tilings of the plane; we will include at the end of the paper some pictures produced using these programs. We also prove the rather surprising fact that the tiling might have fewer tile types than the number of states of the machine.

In this paper we will assume that the reader is familiar with the basics of the theory of finite state automata (see e.g. [3] and [5], and also [4] where the notion of finite state automaton is beautifully applied to problems in group theory and geometry).

A previous version [10] of this paper (where the reader will find some proofs and related results omitted here) was written when the author was visiting the University of Warwick. The author expresses his sincere gratitude to this institution for its hospitality, and to the Scuola Normale Superiore di Pisa for financial support. He is especially grateful to Professor David Epstein for the very many friendly discussions from which this paper originates. The computing facilities supplied by SERC to Professor Epstein were essential to this research. The author wishes to record his thanks for this.

1. The main construction.

Let $\beta \in \mathbb{C}$ and $D \subset \mathbb{C}$ be such that $|\beta| > 1$ and $\#D < \infty$. We define a set $W(\beta, D) \subset \mathbb{C}$ as the set of the points z such that there exists $\{d_i\}_{i=0}^{\infty} \in D^{\mathbb{N}}$ with the property that the sequence $\{z_i\}$ recursively defined by

$$\begin{cases} z_0 = z \\ z_{i+1} = \beta(z_i - d_i) \end{cases}$$

is bounded.

The first few results are easily established and we omit their proof (see [10]).

Lemma 1.1. *If* $z \in W(\beta, D)$ *and* $\{d_i\} \in D^{\mathbb{N}}$ *is as in the definition then* $z = \sum_{i=0}^{\infty} d_i \cdot \beta^{-i}$. *Conversely if* $z = \sum_{i=0}^{\infty} d_i \cdot \beta^{-i}$ *for some* $\{d_i\} \in D^{\mathbb{N}}$ *then the sequence defined as above is bounded, so* $z \in W(\beta, D)$.

Corollary 1.2. $W(\beta, D) = \{ \sum_{i=0}^{\infty} d_i \cdot \beta^{-i} : \{d_i\} \in D^{\mathbb{N}}\}$.

Lemma 1.3. *If* $W \subset \mathbb{C}$ *is compact and* $W = D + \beta^{-1} \cdot W$ *then* W *is* $W(\beta, D)$, *and conversely* $W(\beta, D)$ *satisfies these two properties.*

Since β and D are fixed forever we set $W = W(\beta, D)$.

Let D be endowed with a total order. We define $\{d_i\}_{i=0}^{\infty} \in D^{\mathbb{N}}$ a *strictly preferred* sequence (or a strictly preferred representation of the number $\sum_{i=0}^{\infty} d_i \cdot \beta^{-i}$) if for all $\{d'_i\}_{i=0}^{\infty} \in D^{\mathbb{N}}$ such that $\sum_{i=0}^{\infty} d_i \cdot \beta^{-i} = \sum_{i=0}^{\infty} d'_i \cdot \beta^{-i}$ we have $\{d_i\}_{i=0}^{\infty} \geq \{d'_i\}_{i=0}^{\infty}$ with respect to the lexicographic order on $D^{\mathbb{N}}$ induced by the order on D. Of course every element of $W(\beta, D)$ admits a unique strictly preferred representation.

We define $\{d_i\}_{i=0}^{\infty} \in D^{\mathbb{N}}$ a *weakly preferred* sequence (or a weakly preferred representation of the number $\sum_{i=0}^{\infty} d_i \cdot \beta^{-i}$) if for all $k \in \mathbb{N}$ there exist $d'_{k+1}, d'_{k+2}, \ldots \in D$ such that the sequence $d_0, \ldots, d_k, d'_{k+1}, d'_{k+2}, \ldots$ is strictly preferred.

From now on we shall assume that β *is a Pisot number*, *i.e.* an algebraic integer with modulus bigger than 1 whose conjugates (apart from the number itself and its complex conjugate) have modulus strictly less than 1.

In the space $\mathbb{Q}(\beta)\otimes\mathbb{R}$ we fix the canonical basis $1\otimes 1$, $\beta\otimes 1$, $\ldots,\beta^{d-1}\otimes 1$, where d is the degree of β. We denote by B the multiplication by β in this space: we recall that if the minimal polynomial of β is $x^d+a_{d-1}x^{d-1}+\ldots+a_0$ then B is represented by the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{d-1} \end{pmatrix}.$$

Let us also recall that the minimal polynomial of B is the minimal polynomial of β, so the eigenvalues of B are the conjugates of β. To every eigenvalue γ of B we can associate a B-invariant subspace: the (1-dimensional) eigenspace if γ is real, and the (2-dimensional) span of the real and complex part of a complex eigenvector if γ is not real. We denote by $U\subset\mathbb{Q}(\beta)\otimes\mathbb{R}$ the *unstable* space of B (the B-invariant subspace associated to the eigenvalues β and $\overline{\beta}$) and by $S\subset\mathbb{Q}(\beta)\otimes\mathbb{R}$ the *stable* space (the span of the B-invariant subspaces associated to the other eigenvalues). We denote by π the projection of $\mathbb{Q}(\beta)\otimes\mathbb{R}$ onto S along U. Remark that $B\pi=\pi B$.

We denote by $v:\mathbb{Q}(\beta)\otimes\mathbb{R}\to\mathbb{C}$ the value homomorphism:

$$v:\sum_{i=0}^{d-1}(y_i\cdot\beta^i\otimes 1)\mapsto\sum_{i=0}^{d-1}y_i\beta^i.$$

LEMMA 1.4. *$Ker(v)=S$.*

Proof. Let us consider the complexification $\mathbb{Q}(\beta)\otimes\mathbb{C}$ and the natural extensions to it of B and v. Since v is a non-zero homomorphism it is sufficient to prove that if y is an eigenvector relative to an eigenvalue γ different from β then $v(y)=0$. It easily follows from the definition that $v(By)=\beta\cdot v(y)$; so $\beta\cdot v(y)=\gamma\cdot v(y)$ and the conclusion is obvious. ◊

We will define now a norm $\|\,.\,\|$ on S. Let us choose in every 1-dimensional eigenspace of β in S a non-zero vector x and in every 2-dimensional B-invariant subspace of S (associated to a non-real eigenvalue) a basis $\{y, z\}$ with respect to which B is expressed as a similarity (*i.e.* a scalar multiple of a rotation). We globally have a basis of S of the form $x_1, x_2, ..., y_1, z_1, y_2, z_2, ...$; we define the norm of a vector $\sum_i a_i x_i + \sum_j (b_j y_j + c_j z_j)$ as $\left(\sum_i a_i^2 + \sum_j (b_j^2 + c_j^2)\right)^{1/2}$.

Of course the norm thus defined is not unique, but in the sequel we will never refer to its construction: we will only use its property given by the next lemma. Moreover we will see in Section 3 that the objects we will construct, after a suitable simplification, will not depend on the norm.

We set:

$$\varepsilon = \max\{|\gamma| : \gamma \text{ conjugate of } \beta,\ \gamma \neq \beta, \overline{\beta}\}.$$

LEMMA 1.5. *For all $y \in S$ we have $\|By\| \leq \varepsilon \|y\|$.*

Proof. Inequality $\|By\| \leq \varepsilon \|y\|$ is true for the elements of the basis of S used in the definition of the norm, and the conclusion follows at once. ♢

From now on we will only deal with $\mathbb{Q}(\beta)$, not with the whole of $\mathbb{Q}(\beta) \otimes \mathbb{R}$. We will keep denoting by S, U, B, π, v the intersection with (or restriction to) $\mathbb{Q}(\beta) \cong \mathbb{Q}(\beta) \otimes 1$ of the corresponding objects in $\mathbb{Q}(\beta) \otimes \mathbb{R}$.

We will also assume from now on that *D is a set of algebraic integers in $\mathbb{Q}(\beta)$*; in particular the elements of D are vectors, not numbers: the corresponding numbers are obtained by applying the homomorphism v. Before turning to the construction we are really interested in we recall a well-known fact (see e.g. [9]):

LEMMA 1.6. *The algebraic integers in $\mathbb{Q}(\beta)$ form a lattice.*

We define now

$$\sigma = \max\{|v(d - d')| : d, d' \in D\}$$
$$\tau = \max\{\|\pi(d - d')\| : d, d' \in D\}$$

and $\mathcal{F}$ as the set of all algebraic integers y in $\mathbb{Q}(\beta)$ such that

$$|v(y)| \leq \sigma \cdot \frac{|\beta|}{|\beta| - 1} \qquad \|\pi(y)\| \leq \tau \cdot \frac{\varepsilon}{1 - \varepsilon}.$$

By 1.4 and 1.7 we have that $\mathcal{F}$ is a finite set.

PROPOSITION 1.7. *Given two sequences* $\{d_i\},\{d'_i\}\in D^{\mathbb{N}}$ *we have that they represent the same number if and only if all the elements of the sequence in* $\mathbb{Q}(\beta)$ *defined by*

$$\begin{cases} y_0 = 0 \\ y_{i+1} = B(y_i - (d_i - d'_i)) \end{cases}$$

are in $\mathcal{F}$.

Proof. Two sequences represent the same number if and only if their difference is a representation of 0 with $D-D$ replacing D; by Lemma 1.1, this fact is equivalent to boundedness of the sequence $\{v(y_i)\}$.

If $y_i\in\mathcal{F}$ for all i then of course $\{v(y_i)\}$ is bounded.

For the converse, we first have that the y_i's are certainly algebraic integers in $\mathbb{Q}(\beta)$. Moreover one can easily see that if $\{v(y_i)\}$ is bounded then for all i

$$v(y_i)=\sum_{j=0}^{\infty}(d_{i+j}-d'_{i+j})\beta^{-j},$$

and hence the first inequality to check, $|v(y_i)|\le\sigma|\beta|/(|\beta|-1)$, is easily established. We prove the second inequality, $\|\pi(y_i)\|\le\tau\varepsilon/(1-\varepsilon)$, by induction on i; the case $i=0$ is obvious, and for the inductive step, using Lemma 1.5, we have

$$\|\pi(y_{i+1})\|\le\varepsilon(\|\pi(y_i)\|+\tau)\le\varepsilon(\tau\varepsilon/(1-\varepsilon)+\tau)=\tau\varepsilon/(1-\varepsilon).$$

The proof is complete. ◇

Even if it is not strictly necessary now, we rephrase the previous result in terms of finite state automata. We recall that if M is a machine its language is denoted by $\mathcal{L}(\mathsf{M})$. In the definition of automata which follow we will use “fail state” as a synonym of “non-accept state”; but actually all our machines turn out to have prefix-closed language, so our use of the term “fail” is consistent with the usual one.

Let SN be the automaton with states $\mathcal{F} \cup \{*\}$, initial state 0, alphabet $D \times D$, fail state $*$ and transition

$$(y,(d,d')) \to \begin{cases} B(y-(d-d')) & \text{if } y \neq * \text{ and this point is in } \mathcal{F} \\ * & \text{otherwise.} \end{cases}$$

(Remark that of course $\mathcal{L}(\mathsf{SN})$ is prefix-closed.) The following result, which is immediately deduced from 1.7, means that SN checks whether two sequences represent the Same Number.

Corollary 1.8. *Given* $\{d_i\},\{d_i'\}\in D^{\mathbb{N}}$ *we have that*

$$\sum_{i=0}^{\infty} d_i\beta^{-i} = \sum_{i=0}^{\infty} d_i'\beta^{-i}$$

if and only if $(d_0,d_0')\cdots(d_k,d_k')\in\mathcal{L}(\mathsf{SN})$ *for all* $k\in\mathbb{N}$.

We define now the machine WPR which recognizes Weakly Preferred Representations. The states of WPR are the subsets of $\mathcal{F}$, the alphabet is D, the initial state is $\emptyset$, the d-arrow from the state F leads to

$$\Big(\{B(f-(d-d')) : f\in F, d'\in D\} \bigcup \{-B(d-d') : d'\in D,\, d'>d\}\Big) \bigcap \mathcal{F}$$

and a state F is a fail state if $W \subset v(F)+W$.

Theorem 1.9. *A sequence* $\{d_i\}_{i=0}^{\infty}\in D^{\mathbb{N}}$ *is weakly preferred if and only if for all* $k\in\mathbb{N}$ *its finite prefix* $d_0\cdots d_k$ *is accepted by* WPR.

The proof of this result requires the following preliminary fact. Let us recall that D^* denotes the language with alphabet D, that is the set of all strings (including the empty one) of elements of D.

Lemma 1.10. *In* WPR *the word* $d_0\cdots d_k\in D^*$ *leads from* $\emptyset$ *to*

$$\Big\{ -\sum_{i=0}^{k} B^{k+1-i}(d_i-d_i') \;:\; d_0'\cdots d_k'\in D^*,\, d_0'\cdots d_k' > d_0\cdots d_k, \\ -\sum_{i=0}^{j} B^{j+1-i}(d_i-d_i') \in \mathcal{F}\ \forall j=0,\dots,k\Big\}.$$

Proof. Denote by F_k this set. The proof is by induction on k. The case $k=0$ is obvious. So we must check that if F is the target of the d_{k+1}-arrow from F_k then $F=F_{k+1}$.

First inclusion: $F \subset F_{k+1}$.

Let $f=-\sum_{i=0}^{k} B^{k+1-i}(d_i-d_i') \in F_k$. Then for any $d_{k+1}' \in D$ we have $d_0' \cdots d_k' d_{k+1}' > d_0 \cdots d_k d_{k+1}$; moreover

$$B(f-(d_{k+1}-d_{k+1}')) = -\sum_{i=0}^{k+1} B^{k+2-i}(d_i-d_i')$$

so if $B(f-(d_{k+1}-d_{k+1}')) \in \mathcal{F}$ we have of course $B(f-(d_{k+1}-d_{k+1}')) \in F_{k+1}$.

Let $d_{k+1}' > d_{k+1}$. Then if we set $d_i' = d_i$ for $i=0,\dots,k$ we have $d_0' \cdots d_k' d_{k+1}' > d_0 \cdots d_k d_{k+1}$ and

$$-B(d_{k+1}-d_{k+1}') = -\sum_{i=0}^{k+1} B^{k+2-i}(d_i-d_i')$$

so if $-B(d_{k+1}-d_{k+1}') \in \mathcal{F}$ we have $-B(d_{k+1}-d_{k+1}') \in F_{k+1}$ (all the sums $-\sum_{i=0}^{j} B^{j+1-i}(d_i-d_i')$ with $j \leq k$ give 0, which belongs to $\mathcal{F}$).

Second inclusion: $F_{k+1} \subset F$.

Let $f=-\sum_{i=0}^{k+1} B^{k+2-i}(d_i-d_i') \in F_{k+1}$, where $d_0' \cdots d_k' d_{k+1}' > d_0 \cdots d_k d_{k+1}$. We have either $d_0' \cdots d_k' > d_0 \cdots d_k$ or $d_0' \cdots d_k' = d_0 \cdots d_k$ and $d_{k+1}' > d_{k+1}$. In the former case we have $f = B(f' - (d_{k+1}-d_{k+1}'))$ where $f' = -\sum_{i=0}^{k} B^{k+1-i}(d_i-d_i') \in F_k$; in the latter case we have $f=-B(d_{k+1}-d_{k+1}')$ where $d_{k+1}' > d_{k+1}$. In both cases $f \in F$ and the proof is complete. ◇

The following result immediately implies 1.9.

PROPOSITION 1.11. *A word* $d_0 \cdots d_k \in D^*$ *is not accepted by* WPR *if and only if it admits no strictly preferred extension.*

Proof. Let $d_0 \cdots d_k$ lead from $\emptyset$ to the state F (given by 1.10).

For the sake of simplicity in this proof we will say a point z of W *is* (or *is not*) *written in its strictly preferred representation* if a certain representation of z is thought to be fixed or evident from the context.

Assume that F is a fail state and, by contradiction, that there exist $z, z' \in W$ such that

$$z = \sum_{i=0}^{k} v(d_i)\beta^{-i} + \beta^{-k-1}z'$$

is a strictly preferred representation (of course z' itself must be written in its strictly preferred representation). Since $W \subset v(F) + W$ there exists $z'' \in W$ and a word $d'_0 \cdots d'_k$ bigger than $d_0 \cdots d_k$ such that

$$\begin{aligned} z' &= -\textstyle\sum_{i=0}^{k}(v(d_i) - v(d'_i))\beta^{k+1-i} + z'' \\ &\Rightarrow z = \textstyle\sum_{i=0}^{k} v(d'_i)\beta^{-i} + \beta^{-k-1}z'' \end{aligned}$$

and whatever representation of z'' we choose, this is a representation of z bigger than the previous one, This is a contradiction.

Assume that $d_0 \cdots d_k$ has no strictly preferred extensions. For $z \in W$ we have that

$$z' = \sum_{i=0}^{k} v(d_i)\beta^{-i} + \beta^{-k-1}z$$

is not written in its strictly preferred representation, whatever representation of z we choose. In particular if we choose the strictly preferred representation of z we have that a lexicographically bigger representation of z' must be bigger within the first $k+1$ terms. So there exist $d'_0 \cdots d'_k > d_0 \cdots d_k$ and $z'' \in W$ such that

$$\sum_{i=0}^{k} v(d_i)\beta^{-i} + \beta^{-k-1}z = \sum_{i=0}^{k} v(d'_i)\beta^{-i} + \beta^{-k-1}z''.$$

By 1.7 for $j = 0, ..., k$ we have

$$-\sum_{i=0}^{j} B^{j+1-i}(d_i - d'_i) \in \mathcal{F}$$

so $f = \sum_{i=0}^{k} B^{k+1-i}(d_i - d'_i)$ belongs to F; by direct calculation we have $z = v(f) + z''$. We have proved that $W \subset v(F) + W$, *i.e.* that F is a fail state. $\diamondsuit$

The following fact is easily deduced from the previous result.

COROLLARY 1.12. *The language of* WPR *is prefix-closed.*

We will see in Section 3 that a purely abstract manipulation of the automaton SN naturally leads to a machine which differs from WPR only for having a different (actually, much smaller) set of accept states. This is one of the reasons for having introduced the machine SN.

The only part of the construction of WPR which is not directly implementable, according to the above description, is the failure condition for the states. We will now automize this fail test; namely we will prove that for a state F of WPR there exists a finite state automaton such that F is a fail state if and only if the automaton accepts all the words. In Section 3 we shall describe various strategies which can be used to keep the size of the automata involved in the construction as reasonable as possible.

We recall that the states of WPR are the subsets of $\mathcal{F}$ and that a state F is fail if and only if $W \subset F + W$ (by simplicity from now on we will omit explicit mention of the value function v).

The basic instrument for checking the failure condition $W \subset F + W$ will be an automaton which can check equalities of the form

$$\sum_{i=0}^{\infty} d_i \beta^{-i} = f + \sum_{i=0}^{\infty} d'_i \beta^{-i}.$$

We recall that we have defined a machine SN which checks when two sequences represent the same number; since in the previous formula we have the perturbing element f the machine SN is not the right one, but a slightly different machine does the job: we will actually show that it is enough to change the start state.

We recall that an accessible state of WPR is a set of accessible accept states of SN. If f is an accessible accept state of SN (and hence an algebraic integer in $\mathbb{Q}(\beta)$) we define the machine SN(f) exactly as SN but using f instead of 0 as start state. The following result proves that $\mathsf{SN}(f)$ checks whether two sequences represent the Same Number apart from an initial perturbation f.

Proposition 1.13. *Given* $\{d_i\}, \{d'_i\} \in D^{\mathbb{N}}$ *we have that*

$$\sum_{i=0}^{\infty} d_i \beta^{-i} = f + \sum_{i=0}^{\infty} d'_i \beta^{-i}$$

if and only if $(d_0, d'_0) \cdots (d_k, d'_k) \in \mathcal{L}(\mathsf{SN}(f))$ *for all* $k \in \mathbb{N}$.

Proof. Since f is an accessible state of SN it can be written as

$$-\sum_{i=-p}^{-1} B^{-i}(d_i - d'_i),$$

so equality $\sum_{i=0}^{\infty} d_i \beta^{-i} = f + \sum_{i=0}^{\infty} d'_i \beta^{-i}$ is equivalent to equality

$$\sum_{i=0}^{\infty} d_{i-p} \beta^{-i} = \sum_{i=0}^{\infty} d'_{i-p} \beta^{-i}$$

and the conclusion easily follows from the properties of SN. $\diamondsuit$

The following proposition means that for any accessible state F of WPR there exists a machine $\mathsf{CF}(F)$ by means of which one can Check whether the state F is Fail or not.

Proposition 1.14. *Let* F *be an accessible state of* WPR. *Let* $\mathsf{CF}(F)$ *be the machine defined as follows: the states are the subsets of* $\mathcal{F}$, *the start state is* F, *the only fail state is* $\emptyset$, *the alphabet is* D *and the* d*-arrow from a state* G *leads to*

$$\{B(g - (d - d')) : \, g \in G, \, d' \in D\} \cap \mathcal{F}.$$

Then F *is a fail state of* WPR *if and only if* $\mathsf{CF}(F)$ *accepts all the words.*

Proof. We can rephrase the fail condition $W \subset F + W$ in the following terms: for all $\{d_i\} \in D^{\mathbb{N}}$ there exist $\{d'_i\} \in D^{\mathbb{N}}$ and $f \in F$ such that

$$\sum_{i=0}^{\infty} d_i \beta^{-i} = f + \sum_{i=0}^{\infty} d'_i \beta^{-i},$$

i.e., using 1.13, $(d_0, d'_0)\cdots(d_k, d'_k)\in\mathcal{L}(\mathsf{SN}(f))$ for all $k\in\mathbb{N}$.

By definition, the state G to which a word $d_0\cdots d_k$ leads in $\mathsf{CF}(F)$ is obtained as follows: as f varies in F and $d'_0\cdots d'_k$ varies in D^{k+1} we consider the state to which the word $(d_0, d'_0)\cdots(d_k, d'_k)$ leads in $\mathsf{SN}(f)$; then G is the set of all such states which are accept.

This implies immediately that if F is a fail state then $\mathsf{CF}(F)$ accepts all the words.

Let us prove the converse. Let $\{d_i\}\in D^{\mathbb{N}}$: for all k we can find $f^k\in F$ and $d^k_0\cdots d^k_k$ such that $(d_0, d^k_0)\cdots(d_k, d^k_k)$ is accepted by $\mathsf{SN}(f^k)$. We can extract a subsequence $n\mapsto k_n$ and assume that f^{k_n} is constant (equal to f). Then by a diagonal extraction we can also assume that $d^{k_n}_i$ is constant for $n\geq i$ (equal to d'_i). Then all the prefixes of the word $(d_0, d'_0)(d_1, d'_1)\cdots$ are accepted by $\mathsf{SN}(f)$, and the conclusion follows at once. $\diamondsuit$

Remark 1.15. The need of extracting subsequences in the proof of the previous result essentially comes from the fact that given $\{d_i\}$ the predicates $\forall k$ and $\exists\{d'_i\}$ interchange their positions when we pass through the machine.

Since the aim of this construction is to actually produce computer programs which recognize weakly preferred sequences, it is worth stating the following:

Remark 1.16. The states of the machine SN are defined as the algebraic integers in $\mathbb{Q}(\beta)$ inside a certain "box" around the origin. To find all of them we would need a basis for the lattice of algebraic integers in $\mathbb{Q}(\beta)$. However, for the machines WPR and $\mathsf{CF}(F)$ we are only interested in the accessible states of SN. By the very expression of the transition function of SN, once the digits are chosen as algebraic integers in $\mathbb{Q}(\beta)$, starting from 0 we automatically get algebraic integers in $\mathbb{Q}(\beta)$, and then we only have to check if they are in the box or not.

We have now that the construction of the machine WPR is essentially directly implementable. However one can see that even in simple examples the number of accessible states of the machine SN can be rather big. This implies that the *a priori* bounds for the number of states of the machines WPR and CF(F) can be extremely big. In Section 3 we will describe some strategies which can be used to try to keep the size of these automata in reasonable terms.

2. The associated tilings.

We will see now how to associate tilings of the plane to the above construction. First of all we recall that a self-similar tiling of $\mathbb{C}$ with expansion γ (a complex number with modulus bigger than 1) is given by a family $\mathcal{T}$ of subsets of $\mathbb{C}$ with the following properties and additional structures:

1. $\mathcal{T}$ is a locally finite covering of $\mathbb{C}$.
2. Every element of $\mathcal{T}$ is compact and it is the closure of its interior.
3. The interiors of two different elements of $\mathcal{T}$ are disjoint.
4. Every element of $\mathcal{T}$ is given a label, in such a way that two elements of $\mathcal{T}$ having the same label are translates of each other, and the labels are globally finitely many.
5. The image of every element of $\mathcal{T}$ under the multiplication by γ is a (necessarily unique) union of elements of $\mathcal{T}$ (*subdivision property*).
6. Two elements of $\mathcal{T}$ having the same label subdivide in the same way, *i.e.* if $T, T' \in \mathcal{T}$ have the same label, $T' = T + v$ and $\gamma T = \cup_i \{T_i\}$, $\gamma T' = \cup_j T'_j$ then $\{T'_j\} = \{T_i + \gamma v\}$.

Given a word $d_{-p} \cdots d_{-1} \in D^*$ we define $T_{d_{-p} \cdots d_{-1}}$ as the set of all complex numbers of the form $\sum_{i=-p}^{\infty} d_i \beta^{-i}$ where $d_{-p} \cdots d_{-1} d_0 d_1 d_2 \cdots$ is a weakly preferred sequence (remark that this set is always defined, though it might be empty). We define the label of this set as the state of WPR to which the word leads from $\emptyset$. Remark that by the properties of WPR the label of this set is a fail state if and only if the set is empty (in the sequel we will also reprove this fact in a more direct way). We will use these $T_{d_{-p} \cdots d_{-1}}$'s as tiles for tilings of $\mathbb{C}$. We first deal with the essential topological properties, which will be deduced from a very explicit description which has independent interest.

LEMMA 2.1. *If W is a neighbourhood of 0 then it is the closure of its interior.*

Proof. Let W contain a disc of radius r. If $x = \sum_{i=0}^{\infty} d_i\beta^{-i} \in W$ then for all $k \geq 0$ the point $x_k = \sum_{i=0}^{k} d_i\beta^{-i}$ has distance at most $|\beta|^{-(k+1)} \cdot \mathrm{diam}(W)$ from x (remark that we may have $x_k \notin W$). Moreover within distance $|\beta|^{-(k+1)} \cdot \mathrm{diam}(W)$ from x_k there is a disc of radius $|\beta|^{-(k+1)}r$ completely contained in W, and hence a point of the interior of W. This implies the conclusion. ♢

Remark 2.2. The tilings we will describe will always require that W is a neighbourhood of 0. In particular β cannot be real; for, if β is real, then also the elements of D must have real value, so W is contained in $\mathbb{R}$. In this case one could consider the condition that W be a neighbourhood of 0 in $\mathbb{R}$, and prove results completey analogous to those we prove here for tilings of $\mathbb{R}$ instead of $\mathbb{C}$. We will not explicitly mention this generalization.

Remark 2.3. It is not hard to see that if β is not real then it is always possible to choose D in such a way that W is a neighbourhood of 0 (see [10]).

LEMMA 2.4. *$T_{d_{-p}\cdots d_{-1}}$ is a compact subset of the plane.*

Proof. Consider a sequence

$$x_k = \sum_{i=-p}^{-1} d_i\beta^{-i} + \sum_{i=0}^{\infty} d_i^k\beta^{-i}$$

in $T_{d_{-p}\cdots d_{-1}}$; a diagonal extraction allows us to assume that d_i^k is constant (equal to d_i) for $k \geq i$. Then of course x_k converges to $\sum_{i=-p}^{\infty} d_i\beta^{-i}$ which is a point of $T_{d_{-p}\cdots d_{-1}}$. ♢

For a word $d_{-p}\cdots d_{-1} \in D^*$ we define now $T'_{d_{-p}\cdots d_{-1}}$ as the set of points of the form $\sum_{i=-p}^{\infty} d_i\beta^{-i}$ where $d_{-p}\cdots d_{-1}d_0d_1\cdots$ is a strictly preferred sequence.

LEMMA 2.5. $T'_{d_{-p}\cdots d_{-1}}$ is dense in $T_{d_{-p}\cdots d_{-1}}$.

Proof. If $x = \sum_{i=-p}^{\infty} d_i\beta^{-i}$ and $d_{-p}\cdots d_0 d_1 \cdots$ is weakly preferred then for $k \geq 0$ if $d_{-p}\cdots d_0 \cdots d_k \tilde{d}_{k+1}\tilde{d}_{k+2}\cdots$ is strictly preferred then the point $\sum_{i=-p}^{k} d_i\beta^{-i} + \sum_{i=k+1}^{\infty} \tilde{d}_i\beta^{-i}$ has distance from x bounded by a constant which tends to 0 as k tends to ∞. ♢

For a word $d_{-p}\cdots d_{-1}$ we define now $T''_{d_{-p}\ldots d_{-1}}$ as the following subset of $\mathbb{C}$:

$$\left(\sum_{i=-p}^{-1} d_i\beta^{-i} + W\right) \setminus \bigcup_{\substack{d'_{-p},\ldots,d'_{-1}\in D,\\ d'_{-p}\cdots d'_{-1} > d_{-p}\cdots d_{-1}}} \left(\sum_{i=-p}^{-1} d'_i\beta^{-i} + W\right).$$

LEMMA 2.6. $T''_{d_{-p}\cdots d_{-1}} = T'_{d_{-p}\cdots d_{-1}}$.

Proof. Let $d_{-p}\cdots d_0 d_1\cdots$ be strictly preferred; of course $\sum_{i=-p}^{\infty} d_i\beta^{-i}$ belongs to $\sum_{i=-p}^{-1} d_i\beta^{-i} + W$; moreover it cannot belong to any $\sum_{i=-p}^{-1} d_i\beta^{-i} + W$ with $d'_{-p}\cdots d'_{-1} > d_{-p}\cdots d_{-1}$ for otherwise $d_{-p}\cdots d_0 d_1\cdots$ would not be strictly preferred.

Conversely let $x = \sum_{i=-p}^{\infty} d_i\beta^{-i} \in T''_{d_{-p}\cdots d_{-1}}$, and assume that $d_0 d_1\cdots$ alone is strictly preferred. Assume by contradiction that $d_{-p}\cdots d_0 d_1\cdots$ is not strictly preferred: then a bigger representation must be bigger within the first p terms; it easily follows that x belongs to some $\sum_{i=-p}^{-1} d'_i\beta^{-i} + W$ with $d'_{-p}\cdots d'_{-1} > d_{-p}\cdots d_{-1}$, which is a contradiction. ♢

The following result, which is easily deduced from 2.5 and 2.6, gives a description of the tiles which does not explicitly involve the notion of preferred representation.

COROLLARY 2.7. *$T_{d_{-p}\cdots d_{-1}}$ is the closure of $T''_{d_{-p}\cdots d_{-1}}$.*

PROPOSITION 2.8. *If W is a neighbourhood of 0 for any word $d_{-p}\cdots d_{-1}$ the set $T_{d_{-p}\cdots d_{-1}}$ is the closure of its interior.*

Proof. Let us omit the subscripts $d_{-p}\cdots d_{-1}$. Since T is the closure of T'' of course it is sufficient to prove that the interior of T'' is dense in T''. In fact, let $x=\sum_{i=-p}^{-1} d_i\beta^{-i}+w\in T''$; we can find $\varepsilon>0$ such that

$$\operatorname{dist}\left(x, \sum_{i=-p}^{-1} d'_i\beta^{-i}+W\right)\geq\varepsilon$$

for all words $d'_{-p}\cdots d'_{-1}$ bigger than $d_{-p}\cdots d_{-1}$. Since $w\in W$ and W is the closure of its interior for all $n>0$ we can find w_n in the interior of W such that $|w-w_n|\leq\min\{1/n,\varepsilon/2\}$. Let $\delta_n>0$ be such that $\delta_n<\varepsilon/2$ and the disc of radius δ_n centred at w_n is contained in W. Then it is easily checked that the disc of radius δ_n centred at $\sum_{i=-p}^{-1} d_i\beta^{-i}+w_n$ is contained in T'', and hence $\sum_{i=-p}^{-1} d_i\beta^{-i}+w_n$ is in the interior of T''. The conclusion follows at once. ◊

Proposition 2.9. *If $d_{-p}\cdots d_{-1}<d'_{-p}\cdots d'_{-1}$ then $T_{d_{-p}\cdots d_{-1}}$ does not intersect the interior of $T_{d'_{-p}\cdots d'_{-1}}$.*

Proof. The interior of $T_{d'_{-p}\cdots d'_{-1}}$ is contained in the interior of $\sum_{i=-p}^{-1} d'_i\beta^{-i}+W$, which, by the characterization given in 2.6 and 2.7, of course does not meet $T_{d_{-p}\cdots d_{-1}}$. ◊

Corollary 2.10. *If $d_{-p}\cdots d_{-1}\neq d'_{-p}\cdots d'_{-1}$ then $T_{d_{-p}\cdots d_{-1}}$ and $T_{d'_{-p}\cdots d'_{-1}}$ have disjoint interior.*

It is remarkable that the proofs of 2.8 and 2.10 follow from 2.7 by purely topological methods.

The description of the tile $T_{d_{-p}\cdots d_{-1}}$ we have obtained in 2.7 gives us a better insight to the construction of the machine WPR.

Proposition 2.11. *Let F be the state to which a word $d_{-p}\cdots d_{-1}$ leads in* WPR*; then we have that $T_{d_{-p}\cdots d_{-1}}$ is the closure of*

$$\sum_{i=-p}^{-1} d_i\beta^{-i}+(W\setminus(F+W)).$$

Proof. First of all we easily have from 2.7 that $T_{d_{-p}\cdots d_{-1}}$ is the closure of

$$\sum_{i=-p}^{-1} d_i\beta^{-i} + \left(W \setminus \bigcup_{\substack{d'_{-p},\ldots,d'_{-1}\in D,\\ d'_{-p}\cdots d'_{-1} > d_{-p}\cdots d_{-1}}} \left(-\sum_{i=-p}^{-1} (d_i - d'_i)\beta^{-i} + W \right) \right)$$

and it is not hard to see that

$$\left\{ -\sum_{i=-p}^{-1} (d_i - d'_i)\beta^{-i} : \ d'_{-p}\cdots d'_{-1} > d_{-p}\cdots d_{-1} \right\} \supset F \supset$$

$$\supset \Bigg\{ -\sum_{i=-p}^{-1} (d_i - d'_i)\beta^{-i} : \ d'_{-p}\cdots d'_{-1} > d_{-p}\cdots d_{-1},$$

$$W \cap \left(-\sum_{i=-p}^{-1} (d_i - d'_i)\beta^{-i} + W \right) \neq \emptyset \Bigg\}$$

which implies the conclusion at once. ♢

We will construct in Section 3 a machine WPR_1 which is a sort of optimized version of WPR. It is worth remarking soon this fact, whose proof will be a straight-forward consequence of the properties of WPR_1:

Remark 2.12. If F is the state of the machine WPR_1 to which a word $d_{-p}\cdots d_{-1}$ leads from $\emptyset$, then we have exactly

$$F = \Bigg\{ -\sum_{i=-p}^{-1} (d_i - d'_i)\beta^{-i} : d'_{-p}\cdots d'_{-1} > d_{-p}\cdots d_{-1},$$

$$W \cap \left(-\sum_{i=-p}^{-1} (d_i - d'_i)\beta^{-i} + W \right) \neq \emptyset \Bigg\}.$$

We start now the description of the tilings. We define $\mathcal{T}_p$ as the set of all non-empty sets of the form $T_{d_{-p}\cdots d_{-1}}$. Remark that $\mathcal{T}_0 = \{W\}$. We also define $\mathcal{T}$ as the union of all the $\mathcal{T}_p$'s.

By 2.8 and 2.10 we easily have the following:

COROLLARY 2.13. *If W is a neighbourhood of* 0 *then $\mathcal{T}_p$ is a tiling of the region it covers.*

PROPOSITION 2.14.

1. *The different labels of the elements of* $\mathcal{T}$ *are finitely many.*

2. *Two elements of* $\mathcal{T}$ *having the same label differ by a translation.*

3. *If* $T \in \mathcal{T}_p$ *then* βT *is a union of elements of* $\mathcal{T}_{p+1}$ *(*subdivision *property).*

4. *Two elements of* $\mathcal{T}$ *having the same label subdivide in the same way; namely, if* $T \in \mathcal{T}_p$ *and* $S \in \mathcal{T}_k$ *have the same label and* $v \in \mathbb{C}$ *is such that* $S = T + v$*, the subdivisions*

$$\beta T = \bigcup_i T_i \quad \{T_i\} \subset \mathcal{T}_{p+1} \qquad\qquad \beta S = \bigcup_j S_j \quad \{S_j\} \subset \mathcal{T}_{k+1}$$

are such that $\{S_j\} = \{T_i + \beta v\}$*, and the natural bijection between these sets respects the labeling.*

Proof.

1. This is obvious: the labels are states of the machine WPR.

2. Let us define (for further purpose as well) for an accessible accept state F of WPR the set $M_F \subset \mathbb{C}$ as the set of all sums $\sum_{i=0}^{\infty} d_i \beta^{-i}$ where $\{d_i\}_{i=0}^{\infty}$ is such that all its prefixes lead in WPR from F to an accept state. Then if $T_{d_{-p}\cdots d_{-1}}$ has label F we simply have

$$T_{d_{-p}\cdots d_{-1}} = \sum_{i=-p}^{-1} d_i \beta^{-i} + M_F$$

which implies the conclusion at once.

3. If $T_{d_{-p}\cdots d_{-1}}$ has label F and $a_1, \ldots, a_n$ are the letters accepted from F in WPR then we easily have

$$\beta T_{d_{-p}\cdots d_{-1}} = \bigcup_{j=1}^{n} T_{d_{-p}\cdots d_{-1} a_j}$$

and of course this is a union of elements of $\mathcal{T}_{p+1}$.

4. Let $T = T_{d_{-p}\cdots d_{-1}}$ and $S = T_{d'_{-k}\cdots d'_{-1}}$ both have label F, and let $a_1, \ldots, a_n$ be the letters accepted from F in WPR; denote the

states to which these letters lead by $F_1, ..., F_n$ respectively. We have as in 2

$$T = \sum_{i=-p}^{-1} d_i \beta^{-i} + M_F \qquad\qquad S = \sum_{i=-k}^{-1} d'_i \beta^{-i} + M_F$$

and then we have $S = T + v$ with $v = \sum_{i=-k}^{-1} d'_i \beta^{-i} - \sum_{i=-p}^{-1} d_i \beta^{-i}$; the subdivision rules are

$$\beta T = \bigcup_{j=1}^{n} \Big(\beta \sum_{i=-p}^{-1} d_i \beta^{-i} + \beta a_j + M_{F_j}\Big)$$

$$\beta S = \bigcup_{j=1}^{n} \Big(\beta \sum_{i=-k}^{-1} d'_i \beta^{-i} + \beta a_j + M_{F_j}\Big)$$

and the proof is complete. $\diamondsuit$

According to this result, the obvious idea to obtain self-similar tilings of the plane from this construction is just to put all the various tilings $\mathcal{T}_p$ together; in general we cannot do this directly, as two different $\mathcal{T}_p$'s may disagree on some region. The following result gives a very natural condition under which the different $\mathcal{T}_p$'s do agree; this case is the one we are really interested in: however we shall show below how to obtain self-similar tilings in the general situation.

PROPOSITION 2.15. *If $0 \in D$ is the biggest element of D then for all $p \geq 0$ $\mathcal{T}_p \subset \mathcal{T}_{p+1}$; hence if W is a neighbourhood of 0 the union $\mathcal{T}$ of all the $\mathcal{T}_p$'s is a self-similar tiling of the plane with expansion β.*

Proof. The 0-transition from $\emptyset$ in WPR leads to $\emptyset$ again; this implies that a tile $T_{d_{-p} \cdots d_{-1}} \in \mathcal{T}_p$ can be naturally identified (respecting the labeling) with the tile $T_{0 d_{-p} \cdots d_{-1}} \in \mathcal{T}_{p+1}$. All the properties of the definition of a self-similar tiling easily follow from 2.8, 2.10 and 2.14 (only local-finiteness requires an easy argument which we leave to the reader). $\diamondsuit$

We can show now how to obtain self-similar tilings of the plane also in the general case.

Proposition 2.16. *Let W be a neighbourhood of* 0. *Let $p,k\in\mathbb{N}$, $k\neq 0$, $T\in\mathcal{T}_p$ be such that the subdivision of β^kT into elements of $\mathcal{T}_{p+k}$ contains a set $T+v$ with the same label as T which is completely contained in the interior of β^kT; define u as $(1-\beta^k)^{-1}v$. For $n\geq 0$ express $\beta^{nk}T$ as a union of elements of $\mathcal{T}_{p+kn}$, and translate all these elements by the vector $\beta^{nk}u$; denote by $\mathcal{U}_n$ the resulting family of subsets of $\mathbb{C}$. Then $\mathcal{U}_n\subset\mathcal{U}_{n+1}$ and the union of the $\mathcal{U}_n$'s naturally defines a self-similar tiling of $\mathbb{C}$ with expansion β^k.*

Proof. This fact is essentially straight-forward. The translation is just defined in such a way that $\mathcal{U}_n\subset\mathcal{U}_{n+1}$, and all the properties of a self-similar tiling are easily verified. The condition that $T+v$ is contained in the interior of β^kT easily implies that 0 is in the interior of $T+u$, which implies that the tiling covers $\mathbb{C}$. $\diamondsuit$

Proposition 2.17. *If W is a neighbourhood of* 0 *there always exist p, k and T satisfying the assumptions of* 2.16.

Proof. The proof is carried out by contradiction.

Let us denote by δ the maximal diameter of the tiles (there are finitely many up to translation). Let us choose $\varepsilon>0$ such that all the tiles contain an open ball of radius ε (again, we are using the fact that the tiles are finitely many up to translation, together with the fact that they have non-empty interior). We can choose $k\geq 1$ such that $2\delta|\beta|^{-k}<\varepsilon$. For any tile T we define its "core" as

$$c(T)=\{x\in T\,:\,\operatorname{dist}(x,\partial T)\geq 2\delta|\beta|^{-k}\}$$

which, by the choice of k, is non-empty. Moreover we easily have that for $h\geq k$ the following holds:

1. If a tile in the subdivision of β^hT intersects $\beta^hc(T)$ then it is contained in the interior of β^hT.
2. There exist tiles T' in the subdivision of β^hT such that $T'\subset\beta^hc(T)$ (to prove this, let $x\in T$ be the center of a ε-ball contained in T, and choose T' such that $\beta^hx\in T'$).

For the conclusion of the proof, it is convenient to contract the "p-generation" tiles by the factor β^{-p} to define tilings always of the set W; namely, we define $\tilde{\mathcal{T}}_p$ as $\{\beta^{-p}T\,:\,T\in\mathcal{T}_p\}$; the fact that for all

p's this is a tiling of W is obvious. Remark that now the subdivision rule just means that any element of $\tilde{\mathcal{T}}_p$ is a union of elements of $\tilde{\mathcal{T}}_{p+1}$. If $S = \beta^{-p}T \in \tilde{\mathcal{T}}_p$ we define $c(S)$ as $\beta^{-p}c(T)$, and the label of S as the label of T (remark that now tiles with the same label are similar but not obtained from each other by a translation). The contradiction hypothesis now implies that if $S \in \tilde{\mathcal{T}}_p$ and $h \geq k$ then no element of $\tilde{\mathcal{T}}_{p+h}$ intersecting $c(S)$ has the same label of S. (This explains why we have contracted the tilings; in fact we have now that if $S' \in \tilde{\mathcal{T}}_{p+1}$ is contained, as a set, in $S \in \tilde{\mathcal{T}}_p$, then $\beta^{p+1}S' \in \mathcal{T}_{p+1}$ is in the subdivision of $\beta^p S \in \mathcal{T}_p$.)

Let us define $S_0 = W \in \tilde{\mathcal{T}}_0$. For $h \geq k$ no element of $\tilde{\mathcal{T}}_h$ intersecting $c(S_0)$ has the same label of S_0. Let us choose $S_1 \in \tilde{\mathcal{T}}_k$ such that $S_1 \subset c(S_0)$, which implies that S_1 has not the same label as S_0. Now any tile in $\tilde{\mathcal{T}}_h$ with $h \geq 2k$ which intersects $c(S_1)$ must have label different from S_0 and S_1. We can choose $S_2 \in \tilde{\mathcal{T}}_{2k}$ such that $S_2 \subset c(S_1)$, and similarly continue. The tiles $\{S_i\}$ thus defined all have different labels, and this is a contradiction. $\diamondsuit$

We conclude this section with a remark concerning the definition of self-similar tiling. To get rid of some pathologies which may occur (*e.g.* tilings of $\mathbb{C}$ by squares of two sizes with irrational ratio) Thurston suggests to assume that the tiling is *quasi-homogeneous* in the following sense: for $z \in \mathbb{C}$ and $r > 0$ define the (z,r)-local arrangement as the pattern (types and relative posititions) of the tiles which intersect the disc of radius r at z. The tiling is called quasi-homogeneous if for all $r > 0$ there exists $R > 0$ satisfying the following property: given any $z, y \in \mathbb{C}$ there exists $w \in \mathbb{C}$ such that $|y - w| \leq R$ and the (z,r) and (w,r)-local arrangements are identical. Heuristically this means that all the r-local arrangements occur more or less uniformly all over $\mathbb{C}$.

The tilings obtained by the construction we have described do not satisfy in general this quasi-homogeneity property, and this is the reason for not having included it in the definition of self-similar tiling. On the other hand we have that by definition our tilings are obtained by successive expansion and subdivision from a single tile, so in particular all local arrangements have an ancestor which is a single tile (a property which Rick Kenyon calls *purity* in [6] and [7], and which allows anyway to get rid of some unpleasant special cases).

So, for an easy example of the non-quasi-homogeneity of a tiling obtained as in 2.5, consider the base $\beta = i\sqrt{2}$ and the digits $D = \{-1, 1, 0\}$ (in increasing order). It is easily proved that $W(\beta, D) = [-2, 2] \times [-\sqrt{2}, \sqrt{2}]$. Using 2.7 one can see that the resulting tiling of $\mathbb{C}$ is as represented in Fig. 1. In particular the base tile W occurs only once, and hence of course the tiling is not quasi-homogeneous.

Figure 1. A non-quasi-homogeneous self-similar tiling of the plane

3. Shortcuts and related ideas.

In this section we present miscellaneous facts related to the above construction, and methods to make the implementation of the machine WPR more effective.

A. Interchanging the predicates for the fail test.

It is easily seen that the machine $\mathsf{CF}(F)$ defined in 1.14 is obtained by first applying the "or" predicate to the machines $\mathsf{SN}(f)$ as $f \in F$ (*i.e.* taking the machine whose language is the union of the languages of these $\mathsf{SN}(f)$'s) and then applying the "there exists" predicate to the second letter. (Remark however that the general abstract method for performing these two steps leads to a machine more complicated than $\mathsf{CF}(F)$: we have exploited the fact that the starting machines $\mathsf{SN}(f)$ are very much related to each other.) On the other hand, this abstract description of $\mathsf{CF}(F)$ starting from the $\mathsf{SN}(f)$'s implies that if we apply the predicates "or" and "there exists" in the reverse order we still get a machine which checks the failure condition.

We explicit this construction and then explain why it might be useful.

If f is an accessible state of SN we define the machine $\mathsf{ESN}(f)$ by applying the "there exists" predicate to the second letter in $\mathsf{SN}(f)$. Namely by definition

$$\mathcal{L}(\mathsf{ESN}(f)) = \{d_0 \cdots d_k : \quad k \in \mathbb{N}, \exists d'_0, ..., d'_k \text{ s.t. } (d_0, d'_0) \cdots (d_k, d'_k) \in \mathcal{L}(\mathsf{SN}(f))\}.$$

Now, if F is an accessible state of WPR, let $\mathsf{NCF}(F)$ be the machine whose language is the union of the languages of the $\mathsf{ESN}(f)$'s as f varies in F. As we have remarked, this $\mathsf{NCF}(F)$ is a New machine which Check whether F is a Fail state or not:

Proposition 3.1. *$F \subset \mathcal{F}$ is a fail state if and only if $\mathcal{L}(\mathsf{NCF}(f)) = D^*$.*

The calculus of predicates can be quite easily explicitly carried out in the construction of $\mathsf{NCF}(F)$, and it leads to the following result. If $F = \{f_1, ..., f_p\}$ is an accessible state of WPR then $\mathsf{NCF}(F)$ is the machine with states $(\wp(\mathcal{F} \cup \{*\}))^p$, initial state $(\{f_1\}, ..., \{f_p\})$, single fail state $(\{*\}, ..., \{*\})$, alphabet D and transition as follows: the arrow d leads from $(A_1, ..., A_p)$ to $(A'_1, ..., A'_p)$, where

$$A'_i = \{T(a, d, d') : a \in A_i, d' \in D\}$$

$$T(a, d, d') = \begin{cases} B(a - (d - d')) & \text{if } a \neq * \text{ and this point is in } \mathcal{F} \\ * & \text{otherwise.} \end{cases}$$

It is quite evident that the machine thus described is generally bigger than $\mathsf{CF}(F)$. Let us remark however that since the passage from the $\mathsf{ESN}(f)$'s to $\mathsf{NCF}(F)$ is purely abstract, we must not think that this is "the" machine $\mathsf{NCF}(F)$: we can replace every $\mathsf{ESN}(f)$ by an equivalent machine, and we still get a machine which checks if F is fail.

This is the reason why this strategy for checking the fail test may turn out to be useful: before taking the union machine one can minimize the $\mathsf{ESN}(f)$'s. Moreover it seems to be "experimentally" true that the machines $\mathsf{ESN}(f)$ are small enough to be minimized in reasonable time and that their minimization leads to an important reduction in the number of states.

B. Cutting the dead branches.

In SN there may be "dead branches", *i.e.* non-fail states from which one is sure to fail when he reads a long enough word. Of course we can cut these dead branches without affecting the truth of 1.8. Let us denote by SN_1 the machine thus obtained (more precisely, one should say that the dead branches and their arrows are all merged with the fail state $*$; of course one can easily describe algorithms to do this). Let $\mathcal{F}_1$ be the set of all non-fail states of SN_1.

PROPOSITION 3.2. *Let* WPR_1 *be the machine constructed exactly in the same way as* WPR *with* $\mathcal{F}_1$ *replacing* $\mathcal{F}$. *Then* WPR_1 *recognizes weakly preferred representations (in the sense of 1.9).*

Proof. The scheme of the argument is exactly as in the proof of Theorem 1.9. First of all one proves for WPR_1 an analogue of Lemma 1.10 which describes the state at which the machine is after reading a word; we only have to replace $\mathcal{F}$ by $\mathcal{F}_1$ and the proof is exactly the same.

The proof that a word leading to a fail state admits no strictly preferred extension is unchanged.

The converse is proved by a similar argument using the fact that if $\{d_i\}$ and $\{d'_i\}$ represent the same number then

$$-\sum_{j=0}^{k} B^{(k+1-j)}(d_j - d'_j)$$

is the state in which SN is after having read a prefix of an infinite word all the prefixes of which are accepted; so it is a state of SN_1, *i.e.* an element of $\mathcal{F}_1$. ♢

Using this same idea of cutting the dead branches we have now a simple method which could allow simplifications in the machine $\mathsf{NCF}(F)$ we have introduced in the previous paragraph. In fact if for all the machines $\mathsf{SN}(f)$ we cut the dead branches (and denote the resulting machines by $\mathsf{SN}(f)_1$), then of course 1.13 is still true, so by applying the "there exists" predicate to these machines and then taking the union as $f \in F$ we obtain a new machine which checks the failure condition for F. And this machine is potentially smaller than $\mathsf{NCF}(F)$.

We can prove now that cutting the dead branches from the $\mathsf{SN}(f)$'s leads to minimized machines, so any further simplification of $\mathsf{NCF}(F)$ can be performed only after having applied the "there exists" predicate.

LEMMA 3.3. *The machine* $\mathsf{SN}(f)_1$ *is the minimal machine which accepts its language.*

Proof. Of course all the states of $\mathsf{SN}(f)_1$ are accessible. According to the well-known characterization of minimal machines, it is sufficient to show that if starting from two states f_1 and f_2 the same words are accepted then $f_1 = f_2$. Since f_1 and f_2 are accessible non-fail states of SN_1, the words accepted from them in $\mathsf{SN}(f)_1$ are the same as the words accepted from them in SN_1. By definition of SN_1 we can find an infinite sequence $\{(d_i, d_i')\}_{i=0}^{\infty}$ all the prefixes of which are accepted from f_1 (and hence from f_2); this implies that

$$v(f_1) = v(f_2) = -\sum_{i=0}^{\infty}(d_i - d_i')\beta^{-i}$$

and hence $f_1 = f_2$. ♢

C. Changing the norm.

Let us recall that the construction of the machine SN was based on the determination of a certain set $\mathcal{F}$ of algebraic integers in $\mathbb{Q}(\beta)$, which in turn required the definition of a norm $\|\,.\,\|$ on the stable space $S \subset \mathbb{Q}(\beta)$ satisfying a certain property (namely that B is a contraction in S with respect to this norm). Of course the choice of the norm is not unique, so the machine SN is not unique. But it easily follows from its description that the machine SN_1 is indeed uniquely associated to β and D. A more accurate choice of the norm can allow us to replace the set $\mathcal{F}$ by a smaller one, and then to replace SN by a smaller machine which still satisfies 1.8, but after cutting the dead branches we will always get SN_1. Of course it is always nicer to cut the dead branches from a small machine than from a big one, so it can be of some use anyway to choose "better" norms. It is not difficult to devise methods to do this —see [10] for details.

D. Geometric shortcuts for the fail test.

Some very simple geometric conditions under which one can immediately answer to the fail test are deduced from the next result, whose proof is immediate.

PROPOSITION 3.4. *Let K be a non-empty compact subset of $\mathbb{C}$.*

1. *If $u \in \mathbb{C}$ and $K \subset u + K$ then $u = 0$.*
2. *If $u, v \in \mathbb{C}$, $u \notin \mathbb{R}_- \cdot v$ and $K \subset (u+K) \cup (v+K)$ then $u \cdot v = 0$.*
3. *If $U \subset \mathbb{C}$ and there exists $u_0 \in \mathbb{C}$ such that $\langle u | u_0 \rangle > 0$ for all $u \in U$ (where $\langle\, . \,|\, . \,\rangle$ is the standard scalar product in $\mathbb{R}^2 \cong \mathbb{C}$), then $K \not\subset U + K$.*

Some sufficient conditions, and some necessary ones, for a state to be fail are deduced from the fact that W can be approximated by taking finite sums up to a certain level, and the accuracy of this approximation can be controlled. We omit explicit statements and refer the reader to [10].

E. A related construction

We mention here an idea which unfortunately (and quite surprisingly) does not work, but nonetheless gives a better insight to the weakly preferred representations acceptor. We first quickly recall that we have constructed an automaton SN_1 with alphabet $D \times D$ which recognizes if two sequences represent the same number in the following precise sense:

- given $\{d_i\}, \{d'_i\} \in D^{\mathbb{N}}$ we have $\sum_{i=0}^{\infty} d_i \beta^{-i} = \sum_{i=0}^{\infty} d'_i \beta^{-i}$ if and only if for all $k \in \mathbb{N}$ the word $(d_0, d'_0) \cdots (d_k, d'_k)$ is accepted by SN_1;
- if a word $(d_0, d'_0) \cdots (d_k, d'_k)$ is accepted by SN_1 then there exist extensions $\{d_i\}, \{d'_i\} \in D^{\mathbb{N}}$ such that $\sum_{i=0}^{\infty} d_i \beta^{-i} = \sum_{i=0}^{\infty} d'_i \beta^{-i}$.

Moreover the machine SN_1 is obtained from SN by a general abstract method (cutting the dead branches). Now, starting from SN_1, one could hope to construct an automaton recognizing weakly preferred representations by means of purely abstract operations on automata. We describe the idea and why it does not work.

Let M be the machine obtained starting from SN_1 in the following way:

- let LEX be the machine with alphabet $D \times D$ which recognizes strict lexicographic inequality, and apply the predicate "and" to SN_1 and LEX;
- apply the predicate "there exists" to the second letter in the previous machine;
- apply the predicate "not" to the previous machine.

The machine M accepts a word if there exists no lexicographically bigger word such that the pair is accepted by SN_1; since SN_1 checks if two strings represent the same number, one may conjecture that M recognizes weakly preferred representations, *i.e.* that M accepts a word if and only if it has strictly preferred extensions; actually only one of these implications is true (the proof is easy).

LEMMA 3.5. *If a word admits no strictly preferred extensions then it is not accepted by* M.

EXAMPLE 3.6. *Let* $\beta = 2$ *and* $D = \{-1, 1, 0\}$ *(in this order). Of course* $1111\cdots$ *is a strictly preferred sequence; but* $1 < 0$ *and the extensions* $1(-1)(-1)(-1)\cdots$ *and* $0000\cdots$ *represent the same* *number, so* 1 *is not accepted by* M. *A variation on this argument proves that no word containing a* 1 *is accepted by* M. *Similarly one can see that no word containing a* -1 *is accepted.*

The following result shows that (as in the previous example) the words accepted by M are always dramatically less than the prefixes of the strictly preferred sequences. For the proof the reader is referred to [10].

Proposition 3.7. *The machine* M *coincides with the machine* WPR_1, *with the only difference that the initial state of* WPR_1 *is the unique accept state for* M.

F. Minimal machine vs. minimal tiling.

In Section 4 the reader will find some pictures which illustrate tilings arising from the construction described above. For some of these examples, using the various strategies described in this Section, we have completely computed the machine WPR_1 and we have minimized it. The number of states of the minimized machine tends to be quite big (for instance in the first example there are 101 states). Therefore one would expect the combinatorial structure of the self-similar tiling to be rather complicated. Actually, the author's first conjecture was that, knowing the base and the digits, the language $\mathcal{L}(\mathsf{WPR}_1)$ could be recovered directly from the self-similar structure of the tiling: in particular the minimal number of tile types necessary to describe the combinatorial structure of the tiling would have been equal to the number of accept states of the minimized version of WPR_1. Surprisingly enough this is not the case, as the following result shows:

Proposition 3.8. *Let* $\beta = i\sqrt{2}$, $D = \{-1, 1, 0\}$ *be the example considered at the end of Section 2. Then 4 tile types are sufficient to describe the self-similar structure of the resulting tiling, whereas the minimized version of* WPR_1 *involves 9 accept states (plus the fail state).*

Proof. The first assertion is obvious: there are 4 tile shapes in

Fig. 1, and one easily sees that tiles of the same shape subdivide in the same way.

For the second assertion we explicitly show in Fig. 2 the minimized version of WPR_1 (all missing arrows lead to the fail state, which is not shown). We have symbolically represented the states by the tiles they give rise to, which also explains why there are 9 types rather than only 4. Every tile has been equipped by its "capital", whose meaning is the following: pick a state/tile T; first think of T as a tile and choose its position in $\mathbb{C}$ so that its capital is 0, and call T_0 the subset of $\mathbb{C}$ you get; now think of T as a state and consider the set of complex numbers having a representation which is accepted starting from T; what you get is T_0 again.

Figure 2. A minimal weakly preferred representation acceptor

Since in Fig. 2 the 9 patterns (tile,capital) are all different from each other one sees that the machine cannot be minimized (it is also very easy to check this directly). $\diamondsuit$

4. Examples.

We show here some pictures of tilings obtained with the method described above.

For every picture we mention the Pisot number β used, the minimal polynomial $p(x)$ of β and the set D of digits in vector form (we write the elements of D in increasing order).

We always show the tiles of first and second generation: the black tile is W and the whole picture illustrates its subdivision rule.

The phrase "easy fail test" means that (at least for the states which have been examined to produced the figure) only the states containing 0 are fail.

Figure 3

$\beta \cong 1.766 + 1.20282i$ $\qquad p(x) = x^3 - 2x^2 + 2x + 1$
$D = \{(1,0,0),(0,-1,0),(-1,0,0),(0,0,0)\}$
Easy fail test.

Figure 4

$\beta \cong 1.766 + 1.20282i \qquad p(x) = x^3 - 2x^2 + 2x + 1$
$D = \{(1,0,0), (0,-1,0), (1,0,1), (-1,0,0), (0,0,0)\}$

Figure 5

$\beta = i\sqrt{2} \qquad p(x) = x^2 + 2$
$D = \{(1,0), (0,-1), (-1,0), (0,0)\}$

Figure 6

$\beta \cong 1.766 + 1.20282i \qquad p(x) = x^3 - 2x^2 + 2x + 1$
$D = \{(1,0,0), (0,-1,0), (1,0,-1), (0,0,0)\}$
Easy fail test.

Figure 7

$\beta = i\sqrt{3}$ $\qquad p(x) = x^2 + 3$
$D = \{(1,0), (1/2, 1/2), (3/2, -1/2), (0,0)\}$
Easy fail test.

Figure 8

$\beta \cong -0.696323 + 1.43595i \qquad p(x) = x^3 + x^2 + 2x - 1$
$D = \{(-1,1,0),(0,0,-1),(-1,0,0),(0,0,0)\}$

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