A NOTE ON TRACE EQUIVALENCE IN PSL(2,Z) (*)

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This paper is dedicated to the memory of Wilhelm Magnus, the author's teacher, mentor and friend.

Sommario. - *In questo lavoro si mostra che in* SL(2,Z) *è possibile costruire una successione di matrici per le quali tanto la successione delle tracce quanto la corrispondente successione dei numeri della classe di coniugio divergono ad infinito. Questa indagine è motivata da un problema proposto da Wilhelm Magnus. Si osserva inoltre che tale proprietà sussiste anche in* PSL(2,Z).

Summary. - *We show that in* SL(2,Z) *one can construct a sequence of matrices for which their sequence of trace values tends to infinity, and the corresponding sequence of conjugacy class numbers also tends to infinity. This is motivated by a problem suggested by Wilhelm Magnus. We observe that this result holds also in* PSL(2,Z).

1. Introduction.

In 1986 W. Magnus, in a letter to this author, considered the following problem for elements of SL(2,Z), the homogeneous modular group:

The author wishes to thank T. Jorgensen for several very informative discussions regarding the paper, and for calling the author's attention to the Lemma below. We also wish to thank G. Bachman for several discussions concerning the readability of the paper.

(*) Pervenuto in Redazione il 10 settembre 1994.

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Let t denote the trace of a given matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of SL(2,Z).

Let $s(t)$ denote the number of distinct conjugacy classes of elements of SL(2,Z) that have the same trace t. Determine the number $s(t)$ for this given t.

The approach to this problem, as suggested by him, was to prove that for a particular sequence $\{t_n\}$ of values of the trace the corresponding sequence of conjugacy class numbers $\{s(t_n)\}$ tends to $+\infty$. It is the purpose of this note to show that this is indeed the case.

We note that R. Horowitz [1] discovered that in a free group on two generators, $F_2 = \langle a, b\rangle$, there exists pairs of words $W_1(a,b)$, $W_2(a,b)$ such that W_1 and $W_2^{\pm 1}$ are not conjugate within F_2, and yet have the property that trace $W_1(A,B)=$ trace $W_2(A,B)$ for any mapping $a \mapsto A$, $b \mapsto B$, where $A, B \in GL(2,Z)$.

Theorem 8.1 of [1] shows that $s(t)$ is finite for any value of the trace t.

We will be consistent with the notation and terminology found e.g. in [3,4].

2. Construction of the Sequence.

We observe that SL(2,Z), the homogeneous modular group, is generated by

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

where $(P^{-1}QP^{-1})^2 = (P^{-1}Q)^3$, $(P^{-1}QP^{-1})^4 = I$.

The following observations are immediate consequences of this presentation. Their proofs can be found in [5].

Let $W = P^{n_i}Q^{m_i} \ldots P^{n_k}Q^{m_k}$, $n_i.\ m_i \in \mathbb{N}$; $i = 1, 2, \ldots k$, be a positive word in P and Q, of SL(2,Z); (i.e., a matrix all of whose entries are positive). The number k is called the *syllable length* of the word W, and the number $L = \sum_{i=1}^{k}(n_i + m_i)$ is the *length* of W.

1. Two positive words W and W' in P and Q of different syllable lengths can never be conjugate in GL(2,Z).

2. In GL(2,Z), P and Q are conjugate, since $P = XQX^{-1}$ where $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3. Two matrices M and M$'$ from SL(2,Z) can be conjugate in GL(2,Z) only if they have the same syllable lengths and if the sets of exponents $\{n_i\}$, $\{m_i\}$ on P, Q respectively, coincide apart from their rearrangement. Since P and Q are conjugate, we can exchange the exponents n_i on P with the m_i on Q.

We note that since PSL(2,Z) is the quotient group of SL(2,Z) modulo its center $\{\pm I\}$, we obtain an equivalence relation in PSL(2,Z) by declaring W_1, W_2 in SL(2,Z) to be *trace equivalent* iff trace (W_1) = trace (W_2).

The desired sequence will then be obtained from the above observations and as a consequence of the following result communicated by T. Jorgensen.

LEMMA. *Let* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a matrix in SL(2,Z). The greatest common divisor of* $\{a - d,\ b,\ c\}$ *is a conjugacy invariant for* M *in SL(2,Z).*

Proof. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,Z)$, and let $\delta = g.c.d.\{a - d\ , b, c\}$.

It suffices to show that δ is invariant when we conjugate M by the generators P, Q of SL(2,Z). Now,

$$PQMQ^{-1}P^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

where $x = 2a + c - 2b - d$, $y = -2a - c + 4b + 2d$, $z = a + c - b - d$, $w = -a - c + 2b + 2d$.

Let $\mu = g.c.d.\{x - w, y, z\}$. Then,

$$\begin{aligned} x - w &= 3(a - d) + 2c - 4b, \text{hence } \delta|(x - w)\ , \\ y &= -2(a - d) - c + 4b, \text{ hence } \delta|y\ , \\ z &= (a - d) + c - b, \text{ hence } \delta|z\ . \end{aligned}$$

Therefore, $\delta|\mu \quad (1)$.

Now, $b = (x - w) + y - z$, so $\mu|b$,

$$a - d = 3b - y - z \text{ so } \mu|(a - d),$$

$$c = (a - d) + (x - w) + y, \text{ so } \mu|c.$$

Therefore, $\mu|\delta \quad (2)$.

It follows from (1) and (2) that $\mu = \delta$.

This completes the proof.

To obtain our sequence of trace values, we define a sequence of matrices from SL(2,Z) as follows:

For each $n \in \mathbb{N}$,

$$W(2^{n-i}, 2^i) = P^{2^{n-i}} Q^{2^i} = \begin{bmatrix} 1 + 2^n & 2^{n-i} \\ 2^i & 1 \end{bmatrix}, \ i = 0, 1, 2, \ldots, n \ .$$

For each $n \in \mathbb{N}$, $i = 0, 1, 2, \ldots, n$, trace $[W(2^{n-i}, 2^i)] = t_n = 2 + 2^n$. Let $s(t_n)$ denote the number of distinct conjugacy classes of matrices in SL(2,Z) that have trace t_n. By the Lemma, we see that for each $n \in \mathbb{N}$, and $i = 0, 1, 2, \ldots, n$ we will obtain a family that includes nonconjugate matrices having trace t_n. The number of such nonconjugate matrices corresponds to the number of distinct values of the g.c.d. $\{2^n, 2^{n-i}, 2^i\}$ as i varies from 0 to n.

Let $R(n)$ denote this number, and let $d(n)$ denote the number of positive divisors of 2^n. The number $d(n)$ is too large a value for $R(n)$, since we can exchange exponents on P and Q. A counting argument gives

$$R(n) = \begin{cases} \frac{1}{2} d(n), \text{ if } d(n) \text{ is even }, \\ \\ \left[\frac{1}{2} d(n)\right] + 1, \text{ if } d(n) \text{ is odd }. \end{cases}$$

We observe that $R(n)$ gives a count only for words in P and Q of syllable length one. Therefore, $s(t_n) \geq R(n)$ for $n = 1, 2, 3, \ldots$. As

$n \to +\infty$, $t_n = 2 + 2^n \to +\infty$, and consequently $d(n) \to +\infty$, hence $R(n) \to +\infty$. Hence, $\lim_{n\to\infty} \sup s(t_n) = +\infty$.

Thus we see that the number of distinct conjugacy classes of elements of SL(2,Z) having trace t_n becomes large as t_n becomes large, as we were to show.

We observe that this result holds also in PSL(2,Z).

ACKNOWLEDGMENT. We wish to thank the referee for several useful suggestions that have improved the content of the paper.

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