G-MANIFOLDS AND STRATIFICATIONS (*)

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SOMMARIO. - *Nel presente lavoro viene esposta in dettaglio la stratificazione canonica per "tipo di orbita" delle G-varietà e dei relativi spazi di orbite, compresa la prova della regolarità nel senso di Whitney. In quest'ambito viene pure inserita la stratificazione per "tipo d'orbita normale", mostrandone la relazione con la precedente.*

SUMMARY. - *In the present paper the canonical stratification of G-manifolds and related orbit spaces by means of the notion of "orbit type" is described in detail, included the proof of Whitney regularity. The notion of stratification by "normal orbit type" is considered in this frame as well, showing its relationship with the former one.*

Introduction.

It is a well known fact that the orbit space M/G of a smooth action of a compact Lie group G on a smooth manifold M is not in general a manifold. Anyway M/G turns out to be a stratified space, i.e. a topological space endowed with a partition into smooth locally closed manifolds enjoying good properties of incidence. The partition is the *stratification* and its parts are the *strata*. A wide class of "singular spaces" admits stratifications (a list of examples is

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given in Section 1); as matter of fact, stratifications turned out to be one of the most powerful tools to investigate on spaces bearing singularities since they were introduced by Whitney, Thom, Mather in the Sixties.

Among the various questions concerning stratifications, one of the most interesting is the existence of "canonical" stratifications for a certain class of spaces. In the case of orbit spaces, one has a canonical stratification obtained by means of the conjugation classes of isotropy groups, which is usually called "stratification by orbit types". More precisely, M can be stratified by taking the connected components of the subsets of points whose isotropy group are conjugated: these are invariant submanifolds which induce a stratification on M/G. Another remarkable property of the stratification by orbit types is that it is regular in the sense of Whitney: hence it admits a family of controlled tubes, which implies local topological triviality along the strata. Since we may take these tubes G-invariant, we get a family of controlled tubes for M/G too. The stratification by orbit types has been used in many occasions (e.g. [AS], [B1], [B2], [D1], [D2], [HS], [L], [LS], [MS]) and the subject was described in extent, also proving Whitney regularity, in some thesis ([Le], [S], [Sj]). Part of the stratification axioms are checked in [Br], in the frame of "locally smooth actions" and without considering the idea of stratified space.

The present paper has its origin in a seminar held by the author at the Department of Mathematics "U.Dini" of the University of Florence in January '94 with the aim of exposing in a detailed way the stratification by orbit types. Section 1 contains basic definitions of stratified spaces. Section 2 is concerned with the essential tools needed to build the stratification by orbit type , such as isotropy type, linear tubes and Slice Theorem. All these subjects can be found in classical references for G-manifolds, such as [P], [Br] and the recent [O], so proofs are mostly omitted with exception of some remarks on the relationship between "orbit type" and "normal orbit type" (see [J], [D1]) which I could not find in the current literature. An example shows that just taking the partition by orbit types one does not get "strata", in the sense of equidimensional manifolds: so connected components are needed and the strata by normal orbit types are just collections of components. In Section 3 the stratification theorem is proved essentially reorganizing and completing the material contained in [Br]. Finally, in Section 4 stratification by orbit types is shown to be Whitney regular and an outline on the existence of controlled tubes is given.

The seminar above was part of a cycle of seminars on smooth actions of Lie groups inspired and organized by prof. Franco Tricerri. Just while writing down these pages I was informed that prof. Tricerri and all his family were victims of a tragic air disaster. I dedicate to them this paper in order to remember, together with his family, an outstanding mathematician and a dearest friend.

1. Stratified spaces.

In this section some basic notions on stratifications are introduced. Let W be a locally compact Hausdorff topological space with countable basis. In all the paper, "smooth" will mean "C^∞".

DEFINITION 1.1. *A* stratification *on W is a partition Σ of W in locally closed sets such that:*

a) *Any $X \in \Sigma$ is a smooth, finite-dimensional manifold without boundary;*

b) *(Local finiteness) Σ is locally finite;*

c) *(Frontier) If X, Y belong to Σ and $X \cap \overline{Y} \neq \emptyset$, then $X \subseteq \overline{Y}$;*

d) *(Dimension) If X, Y belong to Σ and $X \subseteq \overline{Y}, X \neq Y$ then $dimX < dimY$.*

The elements of Σ are called the *strata*. Of course one may add other properties, for instance that the strata are *connected*; this will be our case.

The couple (W, Σ) will be called a *stratified space*. We shall always assume that $dimW = sup\{dimX; X \in \Sigma\} < \infty$. The subspace $W^k = \bigcup\{X \in \Sigma; dimX \leq k\}$ carries naturally a stratification and it is called the *k-skeleton* of (W, Σ) . The family $\{W^k\}$ forms a filtration of W which is said to be the *dimensional filtration*. In particular, if $\dim W = m$, we have that $W^m \setminus W^{m-1}$ is open in W; it can be seen as the "nonsingular" part of W.

Stratifications were introduced by Whitney to study the structure of algebraic varieties; afterwards Thom and Mather gave more general definitions and deepened the knowledge of this subject which is fundamental in the study of singularities (see [W] , [T], [M1], [M2], [V]). We give here some examples of important classes of spaces admitting stratifications.

i) Manifolds with boundary ($\Sigma = \{IntM, \partial M\}$);

ii) Cell-complexes ($\Sigma = \{cells\}$);

iii) Manifolds with corners;

iv) Algebraic varieties and analytic spaces (on R or C);

v) Semi-algebraic/analytic sets and subanalytic sets;

vi) If W is manifold endowed with a smooth symmetric 2-tensor g, then the sets $W_r = \{x \in W; rk(d_x g) = r\}$ give rise to a stratification.

The existence of stratifications (with fine “incidence” properties, see Section 4) is crucial in the study of classes *iv), v)* and *vi)*.

In many cases, the existence of a stratification for a certain class of spaces is proved by showing that there is an open subset of the space to be stratified which is a manifold and such that the complement is in the same class of the space but with lower dimension. Then the result follows by induction. The proof for G-manifolds and for their orbit spaces is not of this kind, though the existence of a open, dense, G-invariant subset of W which is a manifold is proven (Principal Orbit Theorem, Section 2)

We see from *vi)* that a smooth manifold can have stratifications more interesting than the one formed by itself alone! This will be the case of G-manifolds , as we shall see. In general , if our stratified space W is a subspace of a smooth manifold M, one asks that the strata are *regular* (that is regularly embedded) submanifolds of M.

As usual, we have a notion of “stratified” also for maps.

Definition 1.2. *A map $f : (W, \Sigma) \to (W', \Sigma')$ between two stratified spaces is said to be* stratified *if for any $X \in \Sigma$ there is $X_f \in \Sigma'$ such that $f(X) \subseteq X_f$ and $f_{|X}$ is smooth. In particular, if f is an homeomorphism and f^{-1} is stratified, then we say that f is an* isomorphism of stratifications.

2. Actions of compact groups and orbit spaces.

We recall here, mostly without proofs, some basic facts which will be needed later. Main references will be [Br], [P]. From now on M will be a locally compact, Hausdorff, m-dimensional smooth manifold without boundary and with countable basis.

If G is a Lie group (with unit e) and if we assign a smooth action of G on M, $\mu : G \times M \to M$, the triple (M, G, μ) will be said a *G-manifold*. In general μ will be understood and we shall write (M, G) and gx for (M, G, μ) and $\mu(g, x)$ respectively.

If M is a (euclidean) vector space and G acts linearly (orthogonally) on G, (M, G) will be said a *(orthogonal) G-module.*

We shall identify an element g of G with the diffeomorphism (automorphism) of M that g induces via μ.

Let (N, G) another G-manifold. A smooth map $\varphi : M \to N$ is *G-equivariant* if $g\varphi(x) = \varphi(gx)$ for any $x \in M$ and $g \in G$. If φ is a diffeomorphism, φ^{-1} is clearly G-equivariant and φ is called a *G -equivalence.*

If H is a closed subgroup of G, then G/H is considered a G-manifold with the canonical action $g(aH) = gaH$, for $g, a \in G$.

As usual, for $x \in M$:

- G_x is the *isotropy group* of x. G_x is a closed sugroup of G.
- If $A \subseteq M$ and $H \subseteq G$, then $HA = \{ha; h \in H, a \in A\}$; A is *H-invariant* if $HA = A$. In particular, Gx is the *orbit* of x.
- M^G is the *fixed point set* ; if (M, G) is a G-module, M^G is a vector subspace.

The orbits define a partition of M in equivalence classes and the associated quotient space is called the *orbit space* and is denoted with M/G. The quotient map $\pi : M \to M/G$ is an *open* map which is called the *orbit map.*

If G is taken compact, we have many important properties which we condense in the following proposition (see [Br], Chap. I, VI):

PROPOSITION 2.1. *Let (M, G) be a G-manifold with G compact. Then:*

a) *M/G is a Hausdorff, locally compact space with countable basis and π is a proper map (in particular it is a closed map).*

b) *For any $x \in M$, its orbit Gx is a compact, G-invariant regular submanifold of M which is G-equivalent to G/G_x as G-manifold.*

From now on we shall assume that G is compact. To indicate that H is a compact subgroup of G we shall write $H < G$. Moreover we shall conventionally set $\pi(A) = A^*$ for $A \subseteq M$.

We introduce now the notions of isotropy type and orbit type.

Definition 2.1. *If $H, K < G$, we say that H and K have the same* isotropy type *if they are conjugated, that is if there is $g \in G$ such that $K = gHg^{-1}$. We denote the isotropy equivalence class of H by (H) . Then the set $\mathcal{I}(G) = \{(H); H < G\}$ is a poset with the partial order*

$$(K) \leq (H) \Leftrightarrow \exists g \in G \text{ such that } gHg^{-1} \subseteq K$$

that is H is conjugated to a subgroup of K. The maximal element is (e) while the minimal one is (G). We have ([Br], p.41):

Proposition 2.2. *Let $H, K < G$. Then*

a) *$(K) \leq (H)$ iff there is a G-equivariant submersion from G/H to G/K. In particular G/H and G/K are G-equivalent iff $(K) = (H)$.*

b) *If $Eq(G/H)$ is the group of self G-equivalences of G/H, then $Eq(G/H)$ is canonically isomorphic to $N(H)/H$, where $N(H)$ is the normalizer of H in G.*

The G-equivalence class of G/H is called its *orbit type*. By the proposition above, G/H and G/K have the same orbit type iff $(H) = (K)$. Let us consider now an H-module (V, H) and a K-module (W, K). If $K = gHg^{-1}$, W becomes naturally an H-module with the action $hw = \mu(h, w) = ghg^{-1}w$ for $h \in H$ and $w \in W$.

DEFINITION 2.2. *(V, H) and (W, K) are* equivalent *(and we write $(V, H) \sim (W, K)$) if $(H) = (K)$ and (V, H) and (W, H) are H-equivalent by a linear isomorphism.*

Let us go back to our G-manifold (M, G). Then , if $x \in M$ and $y \in Gx$, $(G_x) = (G_y)$ and $G/G_x, G/G_y$ have the same orbit type. If we set

$$\mathcal{I}(G, M) = \{(H) \in \mathcal{I}(G); \exists x \in M \text{ such that } (H) = (G_x)\}$$

we may define for $(H) \in \mathcal{I}(G, M)$

$$M_{(H)} = \{x \in M; (H) = (G_x)\} .$$

The family $\Gamma_{M,G} = \{M_{(H)}; (H) \in \mathcal{I}(G, M)\}$ is a partition of M in invariant subsets which induces a partition $\Gamma^*_{M,G} = \{M^*_{(H)}\}$ of M^*.

If $x \in M_{(H)}$, we denote with N_x the quotient vector space $T_x(M)/T_x(Gx)$. Then N_x is naturally a G_x-module with the action $gv = d_x g(v)$. We call (N_x, G_x) the *linear slice* at x. By previous arguments we can consider N_x as a H-module. It has to be pointed out that, for $x, y \in M_{(H)}$, $(N_x, G_x), (N_y, G_y)$ are not in general equivalent, as shown in this example.

EXAMPLE A. If $n > 1$, let S^n be the standard n-sphere in R^{n+1} and let $f : S^n \to S^n$ be the diffeomorphism of S^n with itself given by the restriction of the map $(x_0, ..., x_{n-1}, x_n) \to (x_0, ..., x_{n-1}, -x_n)$. Since $f^2 = Id$, f defines an action of Z_2 on S^n, and since $f(-x) = -f(x)$ for any $x \in S^n$, we have an induced action of Z_2 on RP^n. Let $RP^n = M$ and $Z_2 = G$; then $M_{(G)} = M^G$ has two components of different dimensions, that is $M^G = \{p(e_n)\} \sqcup p(S^n \cap \{x_n = 0\})$, where $p : S^n \to RP^n$ is the quotient map. Hence, if $\eta \in p(S^n \cap \{x_n = 0\})$ and $\xi = p(e_n)$, the G-modules (N_ξ, G) and (N_η, G) cannot be equivalent. In fact, G acts on N_ξ as $\{I, -I\}$ on R^n, while the action of G on N_η is the same of the one of $\{I, J\}$, where J is the matrix

$$\begin{pmatrix} I_{n-1} & 0_{n-1,1} \\ 0_{1,n-1} & -1 \end{pmatrix}$$

and the actions on R^n are the standard ones given by multiplication. Anyway, if $y = gx, g \in G$, then the linear isomorphism between N_x and N_y induced by $d_x g$ gives a linear H-equivalence.

If $(G) \in \mathcal{I}(G,M)$ then $M_{(G)} = M^G$; in particular, if (M,G) is a G-module, $G_0 = G$, hence $(G) \in \mathcal{I}(G,M)$. Then we get directly from definitions that

PROPOSITION 2.3. *If (M,G) is a G-module and $x, y \in M^G$, (N_x, G_x) and (N_y, G_y) are equivalent.*

Let $H < G$ and let (V,H) be an H-module . Then the orbit space of the action on $G \times V$ given by $h(g,v) = (gh^{-1}, hv)$ is denoted with $G \times_H V$ and is called the *twisted product* of G with (V,H). The equivalence class of (g,v) in $G \times_H V$ is indicated with $[g,v]$. In general, if $A \subseteq G$ and $B \subseteq V$, we set $H(A \times B) = A \times_H B$.

We have (see e.g. [KN]):

PROPOSITION 2.4. *$G \times_H V$ is a G-manifold with the action given by $a[g,v] = [ag,v]$. Moreover the map*

$$p : G \times_H V \to G/H \ , \ p([g,v]) = gH$$

is G-invariant and defines a fiber bundle with fiber V (the G-fiber bundle associated to the G-principal bundle $G \to G/H$ with fiber V).

The useful properties stated in the following lemma are easily checked:

LEMMA 2.1. *Let $[g,v] \in G \times_H V$ and $(K) \in \mathcal{I}(G, G \times_H V)$. Then:*

a) $(H) \leq (K)$;

b) $G_{[g,v]} = gH_v g^{-1}$;

c) $(G \times_H V)_{(K)} = G \times_H V_{(K)}$;

d) $(G \times_H V)_{(H)} = G/H \times V^H$.

Let us see that equivalence of modules implies G-equivalence between the relative twisted products:

LEMMA 2.2. *If (V,H) and (W,K) are equivalent then $G \times_H V$ and $G \times_K W$ are G-equivalent.*

Proof. Let $K = aHa^{-1}$ for an $a \in G$ and let $L : V \to W$ be a linear H-equivalence. Then $\alpha : G \times_K W \to G \times_H W$ given by $\alpha([g, w]) = [ga, w]$ and $\beta : G \times_H W \to G \times_H V$ given by $\beta([g, w]) = [g, L^{-1}(w)]$ are G-equivalences; so $\beta \circ \alpha$ is the G-equivalence we were looking for. $\diamondsuit$

The fundamental tool to study orbit spaces is the well known (see e.g. [P], [Br], [O])

Slice Theorem. *Let (M, G) be a G-manifold, with G compact. Then for any $x \in M$ there is a G-equivalence*

$$\Phi : G \times_{G_x} N_x \to T,$$

where T is a G-invariant open neighbourhood of Gx in M.

By Lemma 2.2, we may define a *linear tube* at the orbit Gx as a G-equivalence $\Phi : G \times_H V \to T$, where T is a G-invariant neighbourhood of Gx in M and (V, H) is equivalent to (N_x, G_x).

We have the following corollaries:

Corollary 2.1. *T^* is homeomorphic to V/H.*

Corollary 2.2. *The elements of $\Gamma_{M,G}$ and $\Gamma^*_{M,G}$ are locally connected.*

Proof. If $\Phi : G \times_H V \to T$ is a linear tube at Gx, by Lemma 2.1d)

$$\Phi((G \times_H V)_{(H)}) = \Phi(G/H \times V^H) = M_{(H)} \cap T\ .$$

$\diamondsuit$

Next propositions are consequences of the Slice Theorem.

Proposition 2.5. ([Br] p.182 and Corollary 2.2) *The elements of $\Gamma_{M,G}$ and $\Gamma^*_{M,G}$ are locally closed in M , M^* as well as their connected components.*

Lemma 2.3. *If $(H) \in \mathcal{I}(G, M)$ and X is a connected component of $M^*_{(H)}$, then $\pi^{-1}(X)$ is union of connected components $\{X_i\}_{i \in I}$ of*

$M_{(H)}$ such that $X_i^* = X$ for any i.

Proof. It is immediate that any connected component of $M_{(H)}$ which meets $\pi^{-1}(X)$ is in fact contained in $\pi^{-1}(X)$; hence $\pi^{-1}(X) = \bigsqcup X_i$, being X_i connected components of $M_{(H)}$.

Since $(M_{(H)} \cap A)^* = M^*_{(H)} \cap A^*$ for any $A \subseteq M$, we get from Proposition 2.1 that $\pi|_{M_{(H)}}$ is open and closed (for the induced topology). By Corollary 2.2, the X_i are open and closed in $M_{(H)}$, then any X_i^* is open and closed in $M^*_{(H)}$. This gives $X_i^* = X$. $\diamondsuit$

Proposition 2.6. *Let $(H) \in \mathcal{I}(G,M)$ and let $x, y \in M_{(H)}$. If $\pi(x)$ and $\pi(y)$ belongs to the same connected component of $M^*_{(H)}$, then (N_x, G_x) and (N_y, G_y) are equivalent.*

Proof. Let X be the connected component containing $\pi(x)$ and $\pi(y)$. Then, by Lemma 2.3, $\pi^{-1}(X) = \bigsqcup X_i$, being X_i connected components of $M_{(H)}$. We define an equivalence relationship on $\pi^{-1}(X)$: "$p \sim q$" iff p and q belong to the same connected component of $M_{(H)}$ and (N_p, G_p) , (N_q, G_q) are equivalent. Then each equivalence class is open by Slice Theorem, Proposition 2.3 and Lemma 2.2 and it is contained in an X_i; hence it has to coincide with X_i. If $x \in X_i$, $y \in X_j$, since $\pi(X_j) = X$, there is $z \in X_j$ such that $\pi(x) = \pi(z)$. Then $(N_x, G_x) \sim (N_z, G_z) \sim (N_y, G_y)$. $\diamondsuit$

It can be checked with some computations that $(N_x,\, G_x)$ and (N_y, G_y) are equivalent if and only if they are in the same "normal orbit type" (see [J], [D1]). Then Proposition 2.6 above shows that "normal orbit types" strata are just collections of connected components of the $M_{(H)}$.

One of the most important consequences of the Slice Theorem is the

Principal Orbit Theorem. (see [Br]) *If (M,G) is a G-manifold with G compact, then there is $(U) \in \mathcal{I}(G,M)$ such that:*

a) *$(U) \geq (H)$ for any $(H) \in \mathcal{I}(G,M)$;*

b) *$M_{(U)}$ and $M^*_{(U)}$ are open and dense in M and M^* respectively;*

c) *If M^* is connected, then $M^*_{(U)}$ is connected too.*

(U) is said to be the *principal isotropy type* and the corresponding orbits are the *principal orbits* . An orbit which is not principal is said to be *singular*.

From Slice Theorem and Lemma 2.1 we have

PROPOSITION 2.7. *Let $\Phi : G \times_H V \to T$ be a linear tube at Gx. Then*

a) *If $x \in T$ and $x = \Phi([g, v])$, then $(G_x) = (H_v)$;*

b) *Gx is principal iff H acts trivially on V.*

3. Stratifications of M and M/G.

We want to show in this section that the connected components of the elements of $\Gamma_{M,G}$ and $\Gamma^*_{M,G}$ form stratifications of M and M^* respectively. We define

$$\Sigma_{M,G} =$$

$$= \{X; X \text{ is a connected component of } M_{(H)} \text{ for } (H) \in \mathcal{I}(G, M)\}$$

and

$$\Sigma^*_{M,G} =$$

$$= \{X; X \text{ is a connected component of } M^*_{(H)} \text{ for } (H) \in \mathcal{I}(G, M)\}.$$

Then $\Sigma_{M,G}$ and $\Sigma^*_{M,G}$ are respectively partitions of M and M^* in locally closed sets (Proposition 2.5) and we have to check the axioms for a stratification.

The following theorem gives the differentiable structure on our strata.

THEOREM 3.1. *Any $X \in \Sigma_{M,G}$ is a regular submanifold of M. Moreover , if $X \in \Sigma^*_{M,G}$ and $X \subseteq M^*_{(H)}$, then $\pi : \pi^{-1}(X) \to X$ is a smooth fiber bundle with fiber G/H and structural group $N(H)/H$.*

Proof. Let $x \in M_{(H)}$ and set $H = G_x$. If $\Phi : G \times_H V \to T$ is a linear tube at Gx, as in Corollary 2.2, $\Phi(G/H \times V^H) = M_{(H)} \cap T$.

Being $G/H \times V^H$ smooth and Φ a diffeomorphism, $M_{(H)}$ is locally a regular submanifold of M. Since dimension is locally constant, each connected component of $M_{(H)}$ is a regular submanifold. By Corollary 2.1, there is an homeomorphism $\Phi^* : V^H \to M^*_{(H)} \cap T^*$ such that the diagram

$$\begin{tikzcd} G/H \times V^H \arrow[r, "\Phi"] \arrow[d] & M_{(H)} \cap T \arrow[d, "\pi"] \\ V^H \arrow[r, "\Phi^*"'] & M^*_{(H)} \cap T^* \end{tikzcd}$$

commutes.

Then each $M^*_{(H)}$ is locally a topological manifold , which gives that its connected components are topological manifolds and $\pi|_{M_{(H)}}$ is locally trivial.

Let now X be the stratum of $\Sigma^*_{M,G}$ containing $\pi(x)$ and let $y \in M, \pi(y) \in X$. Then Proposition 2.6 and Lemma 2.2 give that $G \times_H V$ and $G \times_{G_y} N_y$ are G-equivalent, so we can take a linear tube at Gy of the form $\Phi' : G \times_H V \to T'$. If $T \cap T' \neq \emptyset$, the set $A = M_{(H)} \cap T \cap T'$ is a nonempty invariant open subset of $M_{(H)}$. Let $\Phi^{-1}(A) = G/H \times W$ and $\Phi'^{-1}(A) = G/H \times W'$ where W and W' are open subsets of V^H. Then we get the commutative array

$$\begin{tikzcd} G/H \times W' \arrow[r, "\Phi^{-1} \circ \Phi'"] \arrow[d] & G/H \times W \arrow[d] \\ W' \arrow[r, "\Phi^{*-1} \circ \Phi'^*"'] & W \end{tikzcd}$$

where $\Phi^{-1} \circ \Phi'$ is a G-equivalence of the form

$$\Phi^{-1} \circ \Phi'(gH, v) = (\theta(gH), \Phi^{*-1} \circ \Phi'^*(v))$$

with $\theta \in Eq(G/H)$. Now $\Phi^{*-1} \circ \Phi'^*$ is clearly a diffeomorphism and our theorem follows from Proposition 2.2 b). $\diamondsuit$

We can now give a stronger version of Lemma 2.3.

COROLLARY 3.1. *If X is a connected component of $M^*_{(H)}$ and k is the number of connected components of G/H , then $\pi^{-1}(X)$ is the union of k connected components of $M_{(H)}$. Moreover if Y is one of these components, then $\pi_{|Y} : Y \to X$ is a smooth fiber bundle with fiber a connected component of G/H.*

We turn now to local finiteness.

THEOREM 3.2. *$\Sigma_{M,G}$ and $\Sigma^*_{M,G}$ are locally finite in M and M^* respectively.*

Proof. We use induction on $m = dimM$. If $m = 0$ the theorem is obvious. Let us assume that $m > 0$ and that our statement holds for any (M', G') with $dimM' < m$.

Let $x \in M$; then $x \in M_{(K)}$ for a $(K) \in \mathcal{I}(G, M)$. As usual we put $G_x = K$ and we consider a linear tube $\Phi : G \times_K V \to T$ at Gx such that $\Phi([e, 0]) = x$. Then $T \cap M_{(H)} = \emptyset$ for $(H) < (K)$ by Lemma 2.1 a). Let us fix a K-invariant scalar product on V and set $W = (V^K)^{\perp}$. Then (W, K) is an orthogonal K-module such that $W_{(K)} = \{0\}$ and $(V^K \times W, K)$ with the action $\mu(a, (v, w)) = a(v, w) = (v, aw)$, $a \in K$, is K-equivalent to (V, K) by the map $\psi((v, w)) = v + w$. If S is the unit sphere in W with center in the origin, (S, K) is a K-manifold with $dimS < m$. From our inductive hypothesis and compactness of S, we get that $\Sigma_{S,K}$ (and, *a fortiori*, $\Gamma_{S,K}$) is finite. Now, due to the linearity of the action on V, for any $v \in V$ and $\lambda \neq 0$, $K_{\lambda v} = K_v$. Then $\mathcal{I}(K, V) = \mathcal{I}(K, S) \cup (K)$ and $V_{(H)} = \psi(c_\circ(S_{(H)}) \times V^K)$ for $(H) > (K)$, where $c(\bullet)$ denotes the cone with vertex 0 over a set and $c_\circ(\bullet) = c(\bullet) \setminus \{0\}$. Moreover, if Y is a connected component of $V_{(H)}$, then $Y = \psi(c_\circ(Z) \times V^K)$, where Z is a connected component of $S_{(H)}$. The above arguments prove that:

1) T meets only a finite number of elements of $\Gamma_{M,G}$, that is $\Gamma_{M,G}$ is locally finite in M;
2) $\Sigma_{V,K}$ is finite.

Let now $A = \Phi(G_\circ \times_K V)$, where $G_\circ$ is the connected component of e in G; A is an open connected neighbourhood of x in T and, by 1), we reach our goal if we prove that, for each $(H) \in \mathcal{I}(G, M)$ with

$(H) > (K)$, A meets only finitely many connected components of $M_{(H)}$. If such an (H) is fixed, let X be a connected component of $M_{(H)}$ such that $X \cap A \neq \emptyset$ and let $y \in X \cap A$. Then $y = \Phi([g, v])$ with $g \in G_\circ$ and $v \in V_{(H)}$; let Y be the connected component of v in $V_{(H)}$. Then $\Phi(G_\circ \times_K Y)$ is connected , contained in $M_{(H)} \cap A$ and meets X. Hence $\Phi(G_\circ \times_K Y) \subseteq X \cap A$ and X is the unique connected component of $M_{(H)}$ with this property. So the number of connected components of $M_{(H)}$ which meet A is less or equal to the number of connected components of $V_{(H)}$, which is finite by 2). Now Lemma 2.3 shows directly that $\Sigma^*_{(M,G)}$ is locally finite in M^*. $\diamondsuit$

As a byproduct of this proof we get

Proposition 3.1. *If (M, G) is an orthogonal G-module, $\Sigma_{M,G}$ and $\Sigma^*_{M,G}$ are finite.*

We show now the frontier property.

Theorem 3.3. *If X, Y belong to $\Sigma_{M,G}$ or to $\Sigma^*_{M,G}$ and $X \cap \overline{Y} \neq \emptyset$, then $X \subseteq \overline{Y}$.*

Proof. We begin with $\Sigma_{M,G}$. Let $X \subseteq M_{(K)}$, $Y \subseteq M_{(H)}$ and let $Z = X \cap \overline{Y}$. If $x \in Z$ and if $\Phi : G \times_K V \to T$ is a linear tube at Gx, then $T \cap M_{(H)} \neq \emptyset$; hence $(K) \leq (H)$. If $(K) = (H)$, then $X = Y$ trivially. So let $(K) < (H)$. The set Z is closed in X, so, if we prove that it is open as well, we get $X = Z$, that is $X \subseteq \overline{Y}$. For a fixed $x \in Z$, assume that the linear tube at Gx is such that $\Phi([e, 0]) = x$. Then $D = \Phi(G_\circ/K \times V^K)$ is a connected open neighbourhood of x in $X \cap T$. We show that $D \subseteq Z$.

Let $z \in D$; since $x \in \overline{Y}$, there is a sequence $\{y_n\} \subset Y$ such that $y_n \to x$ and $y_n \in Y \cap T$ for n big enough. Then we may assume that, for any n, $y_n = \Phi([g_n, v_n])$ with $g_n \to e, v_n \to 0$ and $(K_{v_n}) = (H)$. If $z = \Phi([g, v])$ with $g \in G_\circ$ and $v \in V^K$, it is immediate to check that $K_{v_n} = K_{v_n + v}$. Therefore $z_n = \Phi([gg_n, v + v_n])$ defines a sequence in $M_{(H)}$ converging to z; so $z \in \overline{M}_{(H)}$. To go on with our proof, we observe that , for any $t \in R$ and n, $K_{v_n + tv} = K_{v_n}$. Then, if we take an arc $\gamma : [0,\ 1] \to G_\circ$ connecting e with g, for any n, the arc $\Phi([\gamma(t)g_n, v_n + tv])$ has support in $M_{(H)}$ and connects y_n with z_n, hence $\{z_n\} \subset Y$, which gives $z \in \overline{Y}$.

As for $\Sigma^*_{M,G}$, the result is easily deduced from the above proof and Corollary 3.1. $\diamondsuit$

The last step is to show the dimension property.

THEOREM 3.4. *If* X, Y *belong to* $\Sigma_{M,G}$ *or to* $\Sigma^*_{M,G}$ *and* $X \subseteq \overline{Y}, X \neq Y$ *then* $dimX < dimY$. *Moreover, if* $X, Y \in \Sigma_{M,G}$ *with* $X \subseteq M_{(K)}$, $Y \subseteq M_{(H)}$ *and* $dim(G/H) > dim(G/K)$, *then* $dimX < dimY - 1$.

Proof. Let $X, Y \in \Sigma_{M,G}$ with $X \subseteq \overline{Y}, X \neq Y$ and $X \subseteq M_{(K)}$, $Y \subseteq M_{(H)}$; then $(K) < (H)$. If $x \in X$ and if $\Phi : G \times_H V \to T$ is a linear tube at Gx such that $\Phi([e,\ 0]) = x$, then $X \cap T = \Phi(G_\circ/K \times V^K)$ (Corollary 3.1). In order to compute dimensions, we may assume that $Y \cap T$ is connected and more precisely that $Y \cap T = \Phi(G_\circ \times_K Z)$, where Z is a connected component of $V_{(H)}$. Then

$$dimX = dim(G/K) + dimV^K = dim(G/K) + dimX^*$$

$$dimY = dim(G/H) + dim(Z/K) = dim(G/H) + dimY^* \ .$$

Since any element of $\Sigma^*_{M,G}$ is of the form X^* for some $X \in \Sigma_{M,G}$, our thesis will be fully proved if we show that (Corollary 3.1):

$$dimV^K < dim(Z/K) = dimZ - dim(K/H) \ .$$

Since $(K) < (U)$, $V^K \neq V$, by Proposition 2.7 b). Let us fix a K-invariant scalar product on V and let us set $W = (V^K)^\perp$; then, as in Theorem 3.2 proof, (V, K) is K-equivalent to the the K-module $(V^K \times W, K)$ by the map ψ , and we can write $Z = \psi(V^K \times L)$, where L is a connected component of $W_{(H)}$. So we are reduced to prove that $dimL > dim(K/H)$. Now, by Corollary 3.1, $\pi : L \to L/K$ is a fiber bundle with fiber K/H, hence $dimL \geq dim(K/H)$. If $dimL = dim(K/H)$, we had $dim(L/K) = 0$, that is L/K should be a point. Since $\pi(0) \in \overline{L/K} \setminus L/K$, this is absurd. $\diamondsuit$

We can now enounce the final statement which follows from Theorems 3.1, 3.2, 3.3 and 3.4.

THEOREM 3.5. *$(M, \Sigma_{M,G})$ and $(M^*, \Sigma^*_{M,G})$ are stratified spaces and $\pi : M \to M^*$ is a continuous stratified map. If $dim(G/U) = d$, then $dim M^* = m - d$.*

We see that, is our case, the "nonsingular" parts are given by principal orbits, that is $M_{(U)}$ and $M^*_{(U)}$. We may observe that, if we set

$$M_{(V,H)} = \{x \in M; (N_x, G_x) \sim (V, H)\}$$

for any H-module (V, H), the family of the $M_{(V,H)}$ which are not empty forms a stratification of M whose strata are union of elements of $\Sigma_{M,G}$. Analogously, we can stratify M^* by means of $M^*_{(V,H)}$. These are the statifications by "normal orbit type" used in [Le], [D1] and [D2]. We remark that, as in [D1], it is possible to show directly that the $M_{(V,H)}$ are smooth manifolds with $dim M_{(V,H)} = dim G/H + dim V^H$ and to deduce in this way that they are union of elements of $\Sigma_{M,G}$.

As a final consideration, we give an example which shows that, in general, $\Gamma_{M,G}$ and $\Gamma^*_{M,G}$ do not fulfil Theorem 3.3 and that the dimensional filtration of $\Sigma_{M,G}$ (or $\Sigma^*_{M,G}$) is not compatible with $\Gamma_{M,G}$ (or $\Gamma^*_{M,G}$).

EXAMPLE B. In the same situation of Example A, let us set $f = f_1$ and let $f_2 : S^n \to S^n$ be the diffeomorphism of S^n with itself given by the restriction of the map $(x_0, ...x_{n-1}, x_n) \to (x_0, ..., -x_{n-1}, x_n)$. Then we have the abelian group $G = \{Id, f_1, f_2, f_3 = f_1 \circ f_2\}$ (the Klein group) which acts on S^n. The nontrivial subgroups of G are $H_i = \{Id, f_i\}$ for $i = 1, 2, 3$; they are all isomorphic to Z_2 and conjugated only to themselves in force of the abelianity. As in Example A, we have an induced action on $M = RP^n$. It easy to check that $\mathcal{I}(G, M) = \{(Id), (H_1), (H_2), (G)\}$ and that

$M_{(G)} = \{p(e_n)\} \sqcup \{p(e_{n-1})\} \sqcup p(S^n_{(G)})$,

$M_{(H_1)} = p(S^n_{(H_1)}) \setminus \{p(e_{n-1})\}$,

$M_{(H_2)} = p(S^n_{(H_2)}) \setminus \{p(e_n)\}$,

where

$S_{(G)} = S^n \cap \{x_{n-1} = x_n = 0\}$

$S_{(H_1)} = S^n \cap \{x_{n-1} \neq 0, x_n = 0\}$,

$S_{(H_2)} = S^n \cap \{x_n \neq 0, x_{n-1} = 0\}$.

Then $M_{(G)} \cap \overline{M}_{(H_i)} \neq \emptyset$ but $M_{(G)} \nsubseteq \overline{M}_{(H_i)}$ for $i = 1,\ 2$.

Moreover $dimM_{(H_1)} = dimM_{(H_2)}$ though H_1 and H_2 are not conjugated.

4. Whitney regularity of $\Sigma_{M,G}$.

In this final section we prove that $\Sigma_{M,G}$ is a regular stratification in the sense of Whitney: this has many consequences, for instance local topological triviality along the strata and the existence of system of controlled tubes. We give in advance some definitions. Main references are [W], [GWPL], [M1] and [M2] for generalities on Whitney regularity and [Le], [S] and [Sj] for the regularity of $\Sigma_{M,G}$ and the existence of systems of G-invariant tubular neighbourhoods.

Definition 4.1. *Let $X, Y \subseteq R^m$ be locally closed regular submanifolds. We say that the pair (X,Y) is* Whitney regular *(or (b)-regular) at $x \in X \cap \overline{Y}$ if, for all sequences $\{x_n\} \subset X$, $\{y_n\} \subset Y$ such that*

1) $lim\ x_n = lim\ y_n = x$,

2) *$T_{y_n}(Y)$ converges to T in $\mathbf{G}(dimY, m)$,*

3) *$\frac{x_n - y_n}{|x_n - y_n|}$ converges to λ in S^{m-1},*

we have $\lambda \subset T$.

Definition 4.2. *Let M be a smooth m-dimensional manifold and let $X, Y \subseteq M$ be locally closed regular submanifolds. We say that the pair (X,Y) is* Whitney regular *at $x \in X \cap \overline{Y}$ if there is a local chart (U, ϕ) of M in x such that the pair $(\phi(X \cap U), \phi(Y \cap U))$ is Whitney regular at $\phi(x)$ in R^m.*

The pair (X,Y) is said to be *Whitney regular* if it is Whitney regular at any point of $X \cap \overline{Y}$. Moreover a stratification Σ of a locally closed subset of M is said to be *Whitney regular* if any pair (X,Y) with $X, Y \in \Sigma$ is Whitney regular.

All the examples at page 2 admit Whitney regular stratifications (*i), ii), iii)* after a suitable embedding).

Of course, $X \cap \overline{Y} = \emptyset$ implies Whitney regularity trivially. On the other hand, in a stratification, $X \cap \overline{Y} \neq \emptyset$ is equivalent to $X \subseteq \overline{Y}$.

The following proposition makes Definition 4.2 intrinsic:

PROPOSITION 4.1. *If $f : M \to N$ is a diffeomorphism between two manifolds and (X, Y) is a Whitney regular pair of submanifolds at $x \in M$, then the pair of submanifolds $(f(X), f(Y))$ in N is Whitney regular at $f(x)$.*

In what follows we shall need the next fact which is easy to check.

PROPOSITION 4.2. *If M is a manifold and (X, Y) is a Whitney regular pair of submanifolds of M then, for any manifold N, $(X\times N, Y \times N)$ is a Whitney regular pair of submanifolds of $M \times N$.*

We show now that the stratification we defined in Section 3 for a G-manifold is Whitney regular in the above sense.

THEOREM 4.1. *If (M, G) is a G-manifold, the stratification $\Sigma_{M,G}$ is Whitney regular.*

Proof. By Proposition 4.1 and the Slice Theorem, it is enough to prove that for any K-module (V, K), $K < G$, the pair $(G/K \times V^K, G \times_K V_{(H)})$ is Whitney regular for any $(H) \in \mathcal{I}(G, G \times_K V)$, $(H) > (K)$. Let $dimG/K = d$ and let $\chi : U \to G/K$ be a local smooth parametrization of G/K, where U is a open in R^d. Then a local parametrization $\overline{\chi} : U \times V \to G \times_K V$ of $G \times_K V$ is defined as follows :

$$\overline{\chi}((\xi, v)) = [a_\xi, v]$$

where $a_\xi K = \chi(\xi)$.

Then it easy to check that, for any $(H) \in \mathcal{I}(G, G \times_K V)$,

$$(G \times_K V_{(H)}) \cap \overline{\chi}(U \times V) = \overline{\chi}(U \times V_{(H)}) \ .$$

Thus, by Proposition 4.1 and Proposition 4.2, we are reduced to prove that the pair $(V^K, V_{(H)})$ is Whitney regular in V. Now, as in Theorem 3.2 proof, if we fix a K-invariant scalar product on V and if we set $W = (V^K)^\perp$, (V, K) is K-equivalent to the the K-module

$(V^K \times W, K)$ by the map ψ. Then it will be enough to prove that the pair $(\{0\}, W_{(H)})$ is Whitney regular in W. But we know that $W_{(H)} = c_\circ(S_{(H)})$, where S is the unit sphere in W, and Whitney regularity condition is immediately checked for the pair $(\{0\}, c_\circ(X))$, with X regular submanifold of W such that $0 \notin \overline{X}$ (we identify W with an euclidean space R^n). $\diamondsuit$

It can be proved that if the pair (X, Y) is Whitney regular and formed by connected manifolds, then $dim X < dim Y$; hence first part of Theorem 3.4 is direct consequence of Theorem 4.1.

DEFINITION 4.3. *Let X be a G-invariant submanifold of M; a triple (U, π, ρ), where:*

a) *U is a G-invariant neighbourhood of X in M,*

b) *$\pi : U \to X$ is a G-equivariant map and $\rho : U \to R^+$ is a G-invariant function such that $\pi|_X = Id$, $\rho_X^{-1}(0) = X$ and $(\pi, \rho) : U \to X \times R^+$ is a submersion, is said to be a G-invariant tubular neighbourhood of X.*

Whitney regularity condition implies the existence of a family of tubular neighbourhoods of the strata which are "compatible" each other. In our case this tubular neighbourhoods can be taken G-invariant. More precisely we have :

THEOREM 4.2. *To any $X \in \Sigma_{M,G}$ we may associate a G-invariant tubular neighbourhood (U_X, π_X, ρ_X) such that:*

a) *If $X \subset \overline{Y}$, the map $(\pi_X, \rho_X)|_Y : U_X \cap Y \to X \times R^+$ is a submersion,*

b) *$U_X \cap U_Y \neq \emptyset$ iff $X = Y$, $X \subset \overline{Y}$ or $Y \subset \overline{X}$,*

c) *If $X \subset \overline{Y}$, then $\pi_X \circ \pi_Y = \pi_X$ and $\rho_X \circ \pi_Y = \rho_X$ where both sides of these equations are defined.*

Theorem 4.2 proof is essentially the same as in the general case of a Whitney regular stratification (see [GWPL], [M2]), up to verify G-invariance. We just recall two key lemmas.

LEMMA 4.1. *Any $X \in \Sigma_{M,G}$ admits a G-invariant tubular neighbourhood in M.*

Proof. Just fix a G-invariant Riemannian metric (see e.g. [Br]) and take the image through the exponential map of a suitable neighbourhood of the zero section of the normal bundle of X in TM.

$\diamondsuit$

LEMMA 4.2. *Let M and N be G-manifolds, $f : M \to N$ a G-equivariant map and X a G-invariant submanifold of M such that $f|_X$ is a submersion. Let $A \subseteq B$ be relatively open G-invariant subsets of X such that $\overline{A} \cap X \subseteq B$. If (U_B, π_B, ρ_B) is a G-invariant tubular neighbourhood of B, then there is a G-invariant tubular neighbourhood (U, π, ρ) of X such that*

$$(U, \pi, \rho)|_B = (U_B, \pi_B, \rho_B) \ .$$

Lemma 4.2 is proved as its not invariant correspondent in [GWPL] (Theorem (1.6) at page 39) up to take a Riemannian metric and a partition of the unity which are G-invariant.

The family $\{(U_X, \pi_X, \rho_X); X \in \Sigma_{M,G}\}$ is called a *system of controlled tubular neighbourhoods* for $\Sigma_{M,G}$. In our case, the fact that the tubular neighbourhoods are G-invariant allows to induce an analogous structure on M^* associating to each X^* in $\Sigma^*_{M,G}$ the triple $(U^*_X, \pi_{X^*}, \rho_{X^*})$, where $\pi_{X^*}(\xi) = \pi(\pi_X(x))$ and $\rho_{X^*}(\xi) = \rho_X(x)$ for $\xi = \pi(x)$. It is immediate that π_{X^*} and ρ_{X^*} are stratified maps (recall Definition 1.2 and Theorem 3.1) and that properties a), b) and c) of Theorem 4.2 hold true for this family. We get in this way a *system of controlled stratified tubular neighbourhoods* for $\Sigma^*_{M,G}$, which gives to M^* the structure of a Thom-Mather (or abstract) stratified space (see [M1] and [V]).

We end this paper recalling an important feature of the stratification $\Sigma^*_{M,G}$. If (M, G) is a G-module, M/G can be embedded in a space R^n as a semialgebraic subset A by means of a map γ induced on M/G by a minimal set of generators of the G-invariant polynomials on M ([Sc]). Then γ is an isomorphism of stratifications with a Whitney regular stratification of A ([B1]).

References

[AS] Abud M. and Sartori G., *The geometry of orbit space and natural minima of Higgs potentials*, Physics Letters **104B** n. 2 (1981), pp.147-152.

[B1] Bierstorne E., *Lifting isotopies from orbit spaces*, Topology **14** (1975), pp.245-252.

[B2] Bierstorne E., *The structure of orbit spaces and the singularities of equivariant mappings*, Monografias de Matematica, IMPA, Rio de Janeiro (1980).

[Br] Bredon G., *Introduction to compact transformation groups*, Academic Press (1972).

[D1] Davis M., *Smooth G-manifolds as collections of fiber bundles*, Pac. Jour. of Maths. **77**, n.2 (1978), pp.315-363.

[D2] Davis M., *Universal G-manifolds*, Am. Jour. of Maths. **103**, n.1 (1981), pp.103-141.

[GWPL] Gibson C., Wirthmüller K., du Plessis A. and Looijenga E., *Topological stability of smooth mappings*, Lec. Notes in Maths. **552**, Springer (1976)

[HS] Hector G. and Saralegi M., *Intersection homology of S^1-actions*, Trans. of the A.M.S. **338**, n.1 (1993), pp.263-288.

[J] Jänich K., *Differenzierbare G-Manningfaltigkeiten*, Lec. Notes in Maths. **59**, Springer (1968).

[KN] Kobayashi S. and Nomizu K., *Foundations of Differential Geometry I*, Wiley Interscience (1963).

[L] Lander L., *Stratification by orbits with an application to $J^2(2,2)$*, Jour. of the London Math. Soc. (2) **8** (1974), pp.443-450.

[Le] Lellmann W., *Orbitraüme von G-Manningfaltigkeiten und Stratifizierte Mengen*, Diplomarbeit, University of Bonn (1975).

[LS] Lerman E. and Sjaamar R., *Stratified symplectic spaces and reduction*, Ann. of Maths. (2) **134** n.2 (1991), pp.375-422 .

[M1] Mather J., *Notes on topological stability*, Lecture Notes, Harvard University (1970).

[M2] Mather J., *Stratification and mappings*, Proc. of the Dynamical Systems Conference, Academic Press (1971).

[MS] Matumoto T. and Shiota M., *Unique triangulation of the orbit space of a differentiable transformation group and its applications*, Adv. Studies in Pure Maths. **9** (1986) pp.41-55.

[O] Onishchik A., *Lie groups and Lie algebras I*, Springer Verlag (1993).

[P] Palais R., *The classification of G-spaces*, Memoirs of the A.M.S. **36** (1960).

[S] Saralegi M., *Sobre un teorema de dualidad entre la homologia de* *intersección y la cohomologia* L^2, Thesis, University of Pais Basco, Euskal Herriko (1986).

[Sc] SCHWARZ G., *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), pp.63-68.

[Sj] SJAAMAR R., *Singular orbit spaces in Riemannian and Symplectic Geometry* Thesis, University of Utrecht (1990).

[T] THOM R., *Ensembles et morphismes stratifiés*, Bull. of the A.M.S. **75** (1969), pp.240-284.

[V] VERONA A., *Stratified mappings - Structure and triangulability*, Lec. Notes in Maths. **1102**, Springer (1984).

[W] WHITNEY H., *Tangents to an analytic variety*, Ann. of Maths. **81** (1965), pp.496-549.