ITERATION OF HOLOMORPHIC FAMILIES (*)

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Sommario. - *Si dimostra che il comportamento asintotico sotto iterazione delle applicazioni di una famiglia olomorfa non dipende dal parametro.*

Summary. - *We prove that the asymptotic behaviour under iteration of the maps of a holomorphic family does not depend on the parameter.*

0. Introduction.

Let X be a taut manifold in the sense of Wu [W], that is such that the family $\mathrm{Hol}(\Delta, X)$ of holomorphic maps from the unit disk $\Delta \subset \mathbb{C}$ into X is a normal family (see the beginnning of section 1 for definitions). In [A1] and [A2] it was studied in some details the iteration theory of holomorphic self-maps of such a manifold, stressing in particular the asymptotic behaviour of the sequence $\{f^k\}$ of iterates of a holomorphic map $f: X \to X$.

It turned out that at the core of the theory there is a main dichotomy, between maps with compactly divergent sequence of iterates on one side, and maps whose sequence of iterates is not compactly divergent on the other side. It is possible to tell between them just looking at the orbit of one point:

Theorem 0.1 ([A2]). *Let $f: X \to X$ be a holomorphic self-map of a taut manifold X. Then the following assertions are equivalent:*

(i) *the sequence $\{f^k\}$ of iterates of f is not compactly divergent;*

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(ii) *for all* $z \in X$ *the sequence* $\{f^k(z)\}$ *is relatively compact in* X*;*

(iii) *there exists a* $z_0 \in X$ *such that the sequence* $\{f^k(z_0)\}$ *is relatively compact in* X.

When the sequence of iterates is not compactly divergent, we are able to describe the set of maps $h \in \mathrm{Hol}(X, X)$ which are limit of a subsequence of iterates. The main result in this direction is

THEOREM 0.2 ([A1, 2]). *Let* $f: X \to X$ *be a holomorphic self-map of a taut manifold* X *such that the sequence of iterates* $\{f^k\}$ *is not compactly divergent. Then there exists a holomorphic map* $\varphi: X \to X$ *with the following properties:*

(i) $\varphi^2 = \varphi$, *that is* φ *is a holomorphic retraction;*

(ii) *if* $h \in \mathrm{Hol}(X, X)$ *is limit of a subsequence* $\{f^{k_\nu}\}$ *of iterates of* f, *then* $h = \gamma \circ \varphi$, *where* γ *is an automorphism of the manifold* $M = \varphi(X)$*;*

(iii) φ *itself is limit of a subsequence of iterates of* f.

The map φ is the *limit retraction* of f; its image M (which is always a manifold; see [C] and [R]) is the *limit manifold* of f; the dimension of M is the *limit multiplicity* of f.

When the sequence of iterates is compactly divergent, to study its asymptotic behaviour is sensible to have a boundary available; for instance, we may consider taut domains in $\mathbb{C}^n$. In [A2] then we proved a generalization of the classical Wolff-Denjoy theorem to strongly pseudoconvex domains:

THEOREM 0.3 ([A2]). *Let* $f: D \to D$ *be a holomorphic self-map of a strongly pseudoconvex bounded domain* $D \subset\subset \mathbb{C}^n$ *such that the sequence of iterates* $\{f^k\}$ *is compactly divergent. Then there exists a point* $x_0 \in \partial D$ *such that* $f^k \longrightarrow x_0$ *uniformly on compact subsets.*

This short summary of iteration theory leads us to the argument of this paper. A *holomorphic family* of holomorphic self-maps of a complex manifold X is a holomorphic map $F: \Delta \times X \to X$; in other words, for every $\zeta \in \Delta$ the map $F(\zeta, \cdot): X \to X$ is a holomorphic self-map of X depending holomorphically on a complex parameter ζ. It is natural to wonder whether different maps of a holomorphic family have similar behaviour under iteration; the aim of this short note is exactly to answer this question.

Not much was known about this problem, at least as far as we know. Franzoni and Vesentini [FV] showed that a holomorphic family F of self-maps of a hyperbolic manifold containing an automorphism is necessarily constant, i.e., $F(\zeta, z)$ does not depend on ζ (we shall recover a weaker form of this result in Corollary 1.4). Hriljac [H], on the other hand, studied the situation for domains in $\mathbb{C}^2$, where he proved, for instance, that if one map of the family has an attractive fixed point (i.e., the sequence of iterates converges, uniformly on compact subsets, to a point in the domain), then all the maps of the family have one.

Here we extend and complete these results showing that, roughly speaking, the behaviour under iteration of the maps of a holomorphic family does not depend on the parameter. We shall first show that the sequence of iterates of one map of the family is compactly divergent if and only if this happens for all the maps of the family (Proposition 1.1); so the main dichotomy represents itself for holomorphic families too.

If the sequence of iterates of any map of the holomorphic family is not compactly divergent, then we prove (Proposition 1.3) that the limit multiplicity does not depend on the parameter. In particular, we generalize to any taut manifold Hriljac's result (Corollary 1.4). But possibly the most striking result is the one describing the compactly divergent situation: if F is a holomorphic family of maps with compactly divergent sequence of iterates in a strongly pseudoconvex domain D, then there exists a single point $x_0 \in \partial D$ such that the sequence of iterates of any map of the family converges, uniformly on compact subsets, to x_0 (Theorem 1.5).

After the completion of this work I became aware of similar results obtained independently by Jean-Pierre Vigué; see [V].

1. Holomorphic families.

Let X, Y be complex manifolds, and let $\mathrm{Hol}(X, Y)$ denote the space of holomorphic maps from X into Y, endowed with the compact-open topology (i.e., with the topology of uniform convergence on compact subsets). We shall denote by $\mathrm{Aut}(X)$ the group of biholomorphic automorphisms of X.

A sequence $\{f_k\} \subset \mathrm{Hol}(X,Y)$ is *compactly divergent* if for any pair of compact subsets $H \subset X$ and $K \subset Y$ there is $k_0 \in \mathbb{N}$ such that

$$\forall k \geq k_0 \quad f_k(H) \cap K = \varnothing.$$

In a certain sense, a compactly divergent sequence is converging to infinity, uniformly on compact subsets.

A family $\mathcal{F} \subset \mathrm{Hol}(X,Y)$ is *normal* if every sequence in $\mathcal{F}$ admits a subsequence which is either convergent uniformly on compact subsets or compactly divergent. A complex manifold X is *taut* (see [W]) if the family $\mathrm{Hol}(\Delta, X)$ is normal, where Δ is the unit disk in $\mathbb{C}$. By Montel's theorem, Δ is taut. It is known that if X is taut then $\mathrm{Hol}(Y,X)$ is a normal family for any complex manifold Y; see [A1] for this and other properties of taut manifolds.

Let X be a complex manifold. A *holomorphic family* of holomorhic self-maps of X is a holomorphic map $F\colon \Delta \times X \to X$; for every $\zeta \in \Delta$ the map $f_\zeta \equiv F(\zeta, \cdot)\colon X \to X$ is a holomorphic self-map of X, depending holomorphically on the parameter ζ.

As we have seen in [A1, A2], in studying iteration theory of holomorphic maps on taut manifolds the main dichotomy is between maps with compactly divergent sequence of iterates and maps with non-compactly divergent sequence of iterates. So our first result shows that, in a holomorphic family, either all maps have compactly divergent sequence of iterates or none has:

PROPOSITION 1.1. *Let $F\colon \Delta \times X \to X$ be a holomorphic family of holomorphic self-maps of a taut manifold X. Then the following assertions are equivalent:*

(i) *there is $\zeta_0 \in \Delta$ such that the sequence of iterates $\{(f_{\zeta_0})^k\} \subset \mathrm{Hol}(X,X)$ is compactly divergent;*

(ii) *the sequence of iterates $\{(f_\zeta)^k\} \subset \mathrm{Hol}(X,X)$ is compactly divergent for all $\zeta \in \Delta$.*

Proof. Let us consider $G \in \mathrm{Hol}(\Delta \times X, \Delta \times X)$ given by

$$G(\zeta, z) = (\zeta, F(\zeta, z)). \tag{1.1}$$

Since the product of taut manifolds is taut (see, e.g., [A1]), we can apply Theorem 0.1; so the sequence $\{G^k\}$ is not compactly divergent iff the orbit of any point $(\zeta, z) \in \Delta \times X$ is relatively compact in $\Delta \times X$ iff the orbit of a single point $(\zeta_0, z_0) \in \Delta \times X$ is relatively compact in $\Delta \times X$. But

$$\{G^k(\zeta_0, z_0)\} = \{\zeta_0\} \times \{(f_{\zeta_0})^k(z_0)\}$$

is relatively compact in $\Delta \times X$ iff $\{(f_{\zeta_0})^k(z_0)\}$ is relatively compact in X, and we are done. $\diamondsuit$

We shall say that a holomorphic family $F: \Delta \times X \to X$ is *compactly divergent* if the sequence of iterates of one (and hence all) map of the family is compactly divergent.

Take now a holomorphic family F not compactly divergent. Then, as discussed in the introduction, we can associate to any f_ζ a holomorphic retraction $\varphi_\zeta: X \to M_\zeta \subset X$, which more or less completely describes the interesting part of the dynamics of f_ζ. It is easy to find examples showing that the limit manifold M_ζ of f_ζ can depend on ζ. For instance, take $F: \Delta \times \Delta^2 \to \Delta^2$ given by

$$F(\zeta, (z, w)) = (\zeta, w^2);$$

the limit manifold of f_ζ is the point $(\zeta, 0)$.

On the other hand, the limit multiplicity — the dimension of the limit manifold — is independent from the parameter. To prove this result we need a preliminary lemma:

LEMMA 1.2. *Let $f: X \to X$ be a holomorphic self-map of a taut manifold X such that the sequence of iterates $\{f^k\}$ is not compactly divergent. Then the limit retraction φ of f is the only holomorphic retraction of X which is limit of a subsequence of iterates of f.*

Proof. Let $\rho: X \to X$ be a holomorphic retraction of X limit of a subsequence of iterates of f. Then, by Theorem 0.2, $\rho = \gamma \circ \varphi$ for a suitable automorphism γ of the limit manifold M. But we should have

$$\gamma \circ \varphi = \rho = \rho^2 = (\gamma \circ \varphi) \circ (\gamma \circ \varphi) = \gamma^2 \circ \varphi,$$

because φ is the identity on M, and hence $\gamma^2 = \gamma$. Therefore $\gamma = \mathrm{id}_M$, and $\rho = \varphi$. $\diamondsuit$

Then

PROPOSITION 1.3. *Let $F: \Delta \times X \to X$ be a holomorphic family of holomorphic self-maps of a taut manifold. Assume that F is not compactly divergent. Then the limit multiplicity of f_ζ does not depend on ζ.*

Proof. Let us consider again the map $G: \Delta \times X \to \Delta \times X$ defined in (1.1). By assumption, $\{G^k\}$ is not compactly divergent; let $\Phi: \Delta \times X \to \Delta \times X$ be its limit retraction, and choose a subsequence $\{G^{k_\nu}\}$ converging to it. Now,

$$G^{k_\nu}(\zeta, z) = (\zeta, (f_\zeta)^{k_\nu}(z));$$

therefore

$$\Phi(\zeta, z) = (\zeta, \varphi_\zeta(z)),$$

for a suitable $\varphi_\zeta \in \mathrm{Hol}(X, X)$, where

$$\varphi_\zeta = \lim_{\nu \to \infty} (f_\zeta)^{k_\nu}.$$

Since Φ is a retraction, each φ_ζ is; therefore, by Lemma 1.2, φ_ζ is the limit retraction of f_ζ. In particular, the limit manifold of G fibers over Δ in the limit manifolds of the f_ζ, which therefore must have constant dimension. ♢

In particular, if a map of the family has an attractive fixed point then all of them have one; and if a map is an automorphism of X then all of them are.

COROLLARY 1.4. *Let $F: \Delta \times X \to X$ be a holomorphic family of holomorphic self-maps of a taut manifold. Assume that F is not compactly divergent. Then:*

(i) *there is a $\zeta_0 \in \Delta$ such that f_{ζ_0} has an attractive fixed point iff f_ζ has an attractive fixed point for all $\zeta \in \Delta$;*

(ii) *there is $\zeta_0 \in \Delta$ such that f_{ζ_0} is an automorphism of X iff $f_\zeta \in \mathrm{Aut}(X)$ for all $\zeta \in \Delta$.*

Proof. (i) f_ζ has an attractive fixed point iff its limit multiplicity is zero.

(ii) f_ζ is an automorphism of X iff its limit multiplicity is equal to the dimension of X. ♢

Now assume that the holomorphic family F is compactly divergent; as customarily, to study this case we need a boundary, and so we limit ourselves to the case of a bounded domain in $\mathbb{C}^n$. The main available tool in this case is provided by the horospheres; let us recall the definitions.

Let $D \subset\subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$. Then the *small horosphere* $E^D_{z_0}(x, R)$ and the *large horosphere* $F^D_{z_0}(x, R)$ of *center* $x \in \partial D$, *radius* $R > 0$ and *pole* $z_0 \in D$ of the domain D are defined by

$$E^D_{z_0}(x, R) = \{z \in D \mid \limsup_{w \to x}[k_D(z, w) - k_D(z_0, w)] < \frac{1}{2}\log R\},$$

$$F^D_{z_0}(x, R) = \{z \in D \mid \liminf_{w \to x}[k_D(z, w) - k_D(z_0, w)] < \frac{1}{2}\log R\},$$

where k_D denotes the Kobayashi distance of the domain D (see [A1] for more informations about horospheres and the Kobayashi distance).

The proofs described up to now suggest that we need some informations about horospheres in product domains. Namely:

LEMMA 1.5. *Let $D \subset\subset \mathbb{C}^n$ be a complete hyperbolic domain, and fix $z_0 \in D$. Then*

$$E^{\Delta\times D}_{(\zeta_0,z_0)}((\zeta_1, x), R) = \Delta \times E^D_{z_0}(x, R),$$

$$F^{\Delta\times D}_{(\zeta_0,z_0)}((\zeta_1, x), R) = \Delta \times F^D_{z_0}(x, R),$$

for all $x \in \partial D$, $R > 0$ and ζ_0, $\zeta_1 \in \Delta$.

Proof. Indeed if $(\eta_k, w_k) \in \Delta \times D$ goes to $(\zeta_1, x) \in \Delta \times \partial D \subset \partial(\Delta \times D)$ then $k_\Delta(\zeta, \eta_k)$ stays bounded for any $\zeta \in \Delta$, whereas $k_D(z, w_k)$ goes to $+\infty$ for any $z \in D$. This means that eventually

$$k_{\Delta\times D}((\zeta, z), (\eta_k, w_k)) = \max\{k_\Delta(\zeta, \eta_k), k_D(z, w_k)\} = k_D(z, w_k).$$

So

$$\begin{aligned} k_{\Delta\times D}((\zeta,z),(\eta_k,w_k)) &- k_{\Delta\times D}((\zeta_0,z_0),\ (\eta_k,w_k)) \\ &= k_D(z,w_k)-k_D(z_0,w_k) \end{aligned}$$

eventually, and the assertion follows. ♢

A domain D is *F-convex* if

$$\overline{F^D_{z_0}(x,R)}\cap\partial D=\{x\} \tag{1.2}$$

for all $x\in\partial D$, $R>0$ and $z_0\in D$. The main class of F-convex domains is the class of strongly pseudoconvex domains (see [A1]). On the other hand, the product of two F-convex domains is in general not F-convex, as shown in the previous lemma.

The iteration theory of compactly divergent maps is particularly nice in F-convex domains, and so the main — and final — theorem of this short note comes as no surprise:

THEOREM 1.6. *Let $D\subset\subset\mathbb{C}^n$ be a complete hyperbolic F-convex (e.g., strongly pseudoconvex) domain, and let $F\colon\Delta\times D\to D$ be a compactly divergent holomorphic family of holomorphic self-maps of D. Then there is $x_0\in\partial D$ such that the sequence of iterates $\{(f_\zeta)^k\}$ converges uniformly on compact subsets to x_0 for every $\zeta\in\Delta$.*

Proof. Let us consider, as usual, the map $G\in\mathrm{Hol}(\Delta\times D,\Delta\times D)$ defined in (1.1), and fix $(\zeta_0,z_0)\in\Delta\times D$. By assumption, the sequence $\{G^k\}$ is compactly divergent; being $\Delta\times D$ complete hyperbolic, we have

$$\lim_{k\to\infty}k_{\Delta\times D}((\zeta_0,z_0),G^k(\zeta_0,z_0))=+\infty.$$

Fix $p\in\mathbb{N}$. We claim that there is a subsequence $\{G^{k_\nu}\}$ such that

$$\begin{aligned} \forall\nu\in\mathbb{N}\quad & k_{\Delta\times D}((\zeta_0,z_0),G^{k_\nu}(\zeta_0,z_0)) \\ & < k_{\Delta\times D}((\zeta_0,z_0),\ \ G^{k_\nu+p}(\zeta_0,z_0)). \end{aligned} \tag{1.3}$$

Indeed, it suffices to take as k_ν the largest integer k satisfying

$$k_{\Delta\times D}((\zeta_0,z_0),G^k(\zeta_0,z_0))\le\nu.$$

We know, by Theorem 0.3, that $\{(f_{\zeta_0})^k\}$ converges, uniformly on compact subsets, to a point $x_0 \in \partial D$. This implies that

$$w_\nu = G^{k_\nu}(\zeta_0, z_0) = (\zeta_0, (f_{\zeta_0})^{k_\nu}(z_0)) \longrightarrow (\zeta_0, x_0),$$

and that

$$G^p(w_\nu) = (\zeta_0, (f_{\zeta_0})^{k_\nu + p}(z_0)) \longrightarrow (\zeta_0, x_0).$$

Furthermore, (1.3) yields

$$\limsup_{\nu\to\infty}[k_{\Delta\times D}((\zeta_0, z_0), w_\nu) - k_{\Delta\times D}((\zeta_0, z_0), G^p(w_\nu))] \leq 0.$$

Now take $(\zeta, z) \in E^{\Delta\times D}_{(\zeta_0,z_0)}((\zeta_0, x_0), R)$. Then

$$\begin{aligned}
&\liminf_{w\to(\zeta_0,x_0)}[k_{\Delta\times D}(G^p(\zeta, z), w) - k_{\Delta\times D}((\zeta_0, z_0), w)] \\
&\quad\leq \liminf_{\nu\to\infty}[k_{\Delta\times D}(G^p(\zeta, z), G^p(w_\nu)) - k_{\Delta\times D}((\zeta_0, z_0), G^p(w_\nu))] \\
&\quad\leq \liminf_{w\to\infty}[k_{\Delta\times D}((\zeta, z), w_\nu) - k_{\Delta\times D}((\zeta_0, z_0), G^p(w_\nu))] \\
&\quad\leq \limsup_{\nu\to\infty}[k_{\Delta\times D}((\zeta, z), w_\nu) - k_{\Delta\times D}((\zeta_0, z_0), w_\nu)] \\
&\qquad + \limsup_{\nu\to\infty}[k_{\Delta\times D}((\zeta_0, z_0), w_\nu) - k_{\Delta\times D}((\zeta_0, z_0), G^p(w_\nu))] \\
&\quad\leq \limsup_{w\to(\zeta_0,x_0)}[k_{\Delta\times D}((\zeta, z), w) - k_{\Delta\times D}((\zeta_0, z_0), w)] \\
&\quad< \frac{1}{2}\log R.
\end{aligned}$$

In other words, we have proved that

$$G^p\left(E^{\Delta\times D}_{(\zeta_0,z_0)}((\zeta_0, x_0), R)\right) \subset F^{\Delta\times D}_{(\zeta_0,z_0)}((\zeta_0, x_0), R),$$

for every $p \in \mathbb{N}$ and $R > 0$. Thus Lemma 1.5 yields

$$G^p\left(\Delta \times E^D_{z_0}(x_0, R)\right) \subset \Delta \times F^D_{z_0}(x_0, R)$$

for every $p \in \mathbb{N}$ and $R > 0$. But this implies that

$$(f_\zeta)^p\left(E^D_{z_0}(x_0, R)\right) \subset F^D_{z_0}(x_0, R),$$

for all $p \in \mathbb{N}$, $R > 0$ and $\zeta \in \Delta$. Since we know that each sequence $\{(f_\zeta)^p\}$ is compactly divergent, and that D is F-convex, we find that

$$(f_\zeta)^p \longrightarrow x_0$$

for all $\zeta \in \Delta$, as claimed. $\Diamond$

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