

THE FAST DIFFUSION EQUATION WITH STRONG ABSORPTION: THE INSTANTANEOUS SHRINKING PHENOMENON (*)

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SOMMARIO. - *Questo lavoro verte sul fenomeno della contrazione istantanea del supporto delle soluzioni non negative di una classe di equazioni nonlineari paraboliche singolari. Vengono anche forniti risultati di esistenza, unicità e confronto per i problemi di Dirichlet e di Cauchy, nonché alcune stime sul comportamento iniziale della frontiera libera.*

SUMMARY. - *This paper deals with the phenomenon of the instantaneous shrinking of the support of the nonnegative solutions of a class of nonlinear singular parabolic equations. Existence, uniqueness and comparison results for the Dirichlet and the Cauchy problems are also given, together with some estimate of the initial behaviour of the free boundary.*

1. Introduction.

In this paper we are concerned with the following class of nonlinear parabolic equations:

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$$u_t = \Delta u^m - u^p,$$

and its generalization:

$$u_t = \Delta \varphi(u) - F(u).$$

The phenomenon of the instantaneous shrinking of the support, or briefly I.S., was originally discovered in an unpublished work by L. Tartar in the semilinear case $m = 1$, $0 < p < 1$, and then proved by L. C. Evans and B. F. Knerr (see [Ev-Kn]) in the general slow diffusion case with strong absorption (e.g. when $m \geq 1$ and $0 < p < 1$). A nonnegative continuous solution u of the Cauchy problem:

$$\begin{aligned} \text{(AP)} \quad & u_t = \Delta \varphi(u) - F(u) && \text{on } \mathbf{R}^N \times (0, T] \\ & u(x, 0) = u_0(x) && \text{on } \mathbf{R}^N \end{aligned}$$

has the I.S. property if for any $t > 0$ its support at time t , i.e. $S(t) := \{x \in \mathbf{R}^N \mid u(x, t) > 0\}$ is bounded even if it is unbounded for $t = 0$. In [Ev-Kn] it was proved that, if u_0 is a positive bounded continuous function such that $\lim_{\|x\| \rightarrow \infty} u_0(x) = 0$, the solution of (AP) with slow diffusion and strong absorption possess the I.S. property. For the power case this means $0 < p < 1 \leq m$; for general φ and F they assume:

- i) (linear or slow diffusion) $\varphi \in C^1([0, \infty))$, $\varphi(0) = 0$, strictly increasing and convex;
- ii) $F \in C([0, \infty))$, nondecreasing, $F(0) = 0$, $F(x) > 0$ for $x > 0$ and $\int_0^1 [F(s)]^{-1} ds < \infty$;
- iii) $\int_0^1 [s\eta(s)]^{-1/2} ds < \infty$, where $\eta := F \circ \varphi^{-1}$.

Results on I.S. can also be found in [Ke-Ni] for the porous media equation with variable coefficients.

In Section 4 of the present paper we prove that I.S. for initial data tending to 0 at infinity also occurs in the fast diffusion case, if and only if the absorption is stronger than the diffusion. For the power case this means simply that $0 < p < m < 1$. For general φ and F we assume the following ‘‘hypotheses B’’:

- B1) (fast diffusion) $\varphi \in C([0, \infty)) \cap C^3((0, \infty))$, $\varphi(0) = 0$, $\varphi'(s) > 0$ when $s \geq 0$, $\lim_{s \rightarrow 0^+} \varphi'(s) = +\infty$;

B2) (*absorption*) $F \in C([0, \infty))$, nondecreasing, $F(0) = 0$, $F(x) > 0$ for $x > 0$ and F is locally Lipschitz continuous on $(0, \infty)$;

B3) (*strong absorption*) as iii).

We have not found in the literature any result of well posedness of the (AP) problem in the case of our interest, i.e. fast diffusion with strong absorption. Nevertheless the works of J. Filo, [Fi], M. Bertsch, [Be], and M. Bertsch, R. Kersner, L. A. Peletier, [Be-Ke-Pe], provide a good outline to follow both for Dirichlet and Cauchy problems under assumptions weaker than hypotheses B (see hypotheses A, Section 2). As a consequence, we have chosen to collect the fundamental ideas of well posedness of Dirichlet and Cauchy problems respectively in Sections 2 and 3. The main tool of our proofs, besides the techniques of the previously mentioned papers, is to approximate the weak solution of (AP) by classical positive bounded solutions following well-known techniques (see e.g. [La-So-Ur], [Ol]).

In Section 4 we prove the I.S. in our case by comparison with a function similar to the one used by [Ev-Kn]; it will turn out that in the fast diffusion case the comparison has to be local in space-time. We conclude the paper providing some a priori estimates for the support $S(t)$ and for the solution u in the radially symmetric case as $t \rightarrow 0$ (Section 5).

2. The Dirichlet Problem.

In this Section we will provide well posedness results for the Dirichlet problem for the fast diffusion equation with strong absorption. As a matter of fact, we are concerned with a different type of nonlinearity of φ and so we give a different proof of Bertsch's Proposition 1.1 about uniqueness of solutions, making use of the techniques exposed in [Fi, Theorem 2.4]. Let's consider the problem:

$$\begin{array}{ll}
 (DP) & \begin{array}{l}
 u_t = \Delta\varphi(u) + f(u) \quad \text{on } \Omega \times (0, T] \\
 u(x, t) = U(x, t) \quad \text{on } \partial\Omega \times (0, T] \\
 u(x, 0) = u_0(x) \quad \text{on } \Omega,
 \end{array}
 \end{array}$$

where $T > 0$ and $\Omega \subset \mathbf{R}^N$ is a bounded connected domain with compact boundary $\partial\Omega$. Let's also assume that $\partial\Omega$ is regular, i.e. it is

piecewise of class C^1 and it satisfies the so-called exterior sphere condition (see e.g. [La-So-Ur]). We will assume throughout this Section that the functions φ, f, U and u_0 satisfies the “hypotheses A”:

A1) as B1);

A2) $f \in C([0, \infty))$, $f(0) = 0$, f is locally Lipschitz continuous on $(0, \infty)$ and uniformly Lipschitz continuous from above on $[0, \infty)$, in the sense that there exists a constant $K \geq 0$ such that $f(s) - f(r) \leq K(s - r)$, for any $0 \leq r \leq s$;

A3) $U \in C(\partial\Omega \times [0, T])$, $u_0 \in C(\overline{\Omega})$ and U, u_0 are nonnegative functions such that $u_0(x) = U(x, 0)$ for every $x \in \Omega$.

To be more clear, one can have in mind the model $u_t = \Delta u^m - \lambda u^p$ with $0 < m < 1$, $0 < p < 1$ and $\lambda \geq 0$.

DEFINITION 2.1. *A nonnegative function u defined on $\overline{\Omega} \times [0, T]$ is said to be a weak solution of problem (DP) on $[0, T]$ with data f, u_0 and U if*

i) $u \in C([0, T]; L^1(\Omega)) \cap L^\infty(\Omega \times [0, T])$;

ii) *for any test function $\zeta \in C^{1,0}(\overline{\Omega} \times [0, T]) \cap C^{2,1}(\Omega \times (0, T])$ such that $\zeta \geq 0$ on $\Omega \times (0, T]$ and $\zeta = 0$ on $\partial\Omega \times (0, T]$, u satisfies the integral identity:*

$$(2.1) \quad \int_{\Omega} u(t)\zeta(t) = \int_{\Omega} u_0\zeta(0) - \int_0^t \int_{\partial\Omega} \varphi(U)\partial_\nu\zeta + \\ + \int_0^t \int_{\Omega} [u\zeta_t + \varphi(u)\Delta\zeta + f(u)\zeta],$$

for any $0 \leq t \leq T$. Here, and in the following, $\nu := \nu(x)$ will denote the outward-directed normal vector pointed at $x \in \partial\Omega$, $\partial_\nu\zeta := \frac{\partial\zeta}{\partial\nu}$ and we use the common notation $\int_{\Omega} u(t)\zeta(t) := \int_{\Omega} u(x, t)\zeta(x, t)dx$.

Replacing the equality into (2.1) by \leq , u is said to be a weak subsolution of problem (DP) on $[0, T]$ with data f, u_0 and U .

Following [Be], we define supersolutions of the problem (DP) in this way: let's consider the problem (PS)

$$(PS) \quad \begin{aligned} u_t &= \Delta \varphi(u) + f(u) + h(x, t) && \text{on } \Omega \times (0, T] \\ u(x, t) &= U(x, t) && \text{on } \partial\Omega \times (0, T] \\ u(x, 0) &= u_0(x) && \text{on } \Omega, \end{aligned}$$

where $h \in L^\infty(\Omega \times [0, T])$, $h \geq 0$ on $\Omega \times (0, T]$.

DEFINITION 2.2. *A weak (sub-)solution of the problem (PS) on $[0, T]$ with data f, u_0, U and h is a function which satisfies Definition 2.1 after having added to the right-hand side of the (in-)equality (2.1) the term $\int_0^t \int_\Omega h \zeta$.*

DEFINITION 2.3. *A generalized supersolution of the problem (DP) on $[0, T]$ with data f, u_0 , and U is a function u such that there exist functions f^*, u_0^* and U^* satisfying hypotheses A, and there exists a function $h \in L^\infty(\Omega \times [0, T])$ such that $f \leq f^*$ on $[0, \infty)$, $u_0 \leq u_0^*$ on Ω , $U \leq U^*$ on $\partial\Omega \times (0, T]$, $h \geq 0$ a.e. on $\Omega \times [0, T]$ and u is a weak solution of problem (PS) on $[0, T]$ with data f^*, u_0^*, U^* and h .*

EXISTENCE

THEOREM 2.1. *Assuming hypotheses A, the problem (DP) admits a weak solution on $[0, T]$ with data f, u_0 and U .*

The proof, which we summarize for sake of brevity, is a straightforward adaptation of that one proposed in [Be]. The ideas are: to regularize the absorbing term with a sequence $\{f_n\}_{n \in \mathbf{N}}$, $f_n \in C^1([0, \infty))$ such that on $[0, \infty)$ it holds:

$$\begin{aligned} (i) \quad & f \leq f_n \leq f + 1/n \\ (ii) \quad & f_{n-1} \leq f_n \\ (iii) \quad & f'_n \leq K, \end{aligned}$$

where K is the constant defined in A2); to define $A_n := \max\{[f(s)]^-, 0 \leq s \leq 1/n\}$, where $[r]^- = \max(0, -r)$; to consider the unique classical solutions of the problems:

$$(DP)_n \quad \begin{aligned} u_t &= \Delta \varphi(u) + f_n(u) + A_n && \text{on } \Omega \times (0, T] \\ u(x, t) &= U(x, t) + 1/n && \text{on } \partial\Omega \times (0, T] \\ u(x, 0) &= u_0(x) + 1/n && \text{on } \Omega; \end{aligned}$$

to show that $\{u_n\}$ is a positive decreasing sequence bounded from above by a constant depending only from U, u_0, K and T ; to define $u := \lim_{n \rightarrow \infty} u_n$ and to prove that u is a weak solution of (DP).

UNIQUENESS

We begin considering the simpler problem:

$$(BP) \quad \begin{array}{ll} u_t = \Delta \varphi(u) + g(x, t) & \text{on } \Omega \times (0, T] \\ u(x, t) = U(x, t) & \text{on } \partial\Omega \times (0, T] \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{array}$$

where $g \in L^\infty(\Omega \times (0, T])$ is a given function.

By a *weak solution* of (BP) we mean a function u satisfying Definition 2.1 with the term $f(u)$ replaced by g . A *weak subsolution* (*supersolution*) of (BP) on $[0, T]$ with same data is a nonnegative function u which verifies the same definition, replacing the equality sign in (2.1) by \leq (\geq).

Following the ideas of [Ar-Cr-Pe, Proposition 9] we begin with a useful inequality:

LEMMA 2.2. *Let $g, g^* \in L^\infty(\Omega \times (0, T])$ and suppose that φ satisfies hypothesis A1) and U, u_0, V, v_0 hypothesis A3). Assume also that u is a weak subsolution of (BP) on $[0, T]$ with data U, u_0, g and v is a weak supersolution of (BP) on $[0, T]$ with data V, v_0, g^* . If $U \leq V$ then for any $\lambda \geq 0$ and $0 \leq t \leq T$,*

$$e^{\lambda t} \int_{\Omega} [u(t) - v(t)]^+ \leq \int_{\Omega} [u_0 - v_0]^+ + \int_0^t \int_{\Omega} e^{\lambda s} [g - g^* + \lambda(u - v)]^+.$$

Proof. The proof is the same as in [Fi, Theorem 2.5]. We summarize it here because a similar proof is needed later in Section 3 (see Proposition 3.2). According to the definitions, for any test function ζ one has:

$$\begin{aligned} & \int_{\Omega} [u(t) - v(t)]\zeta(t) - \int_0^t \int_{\Omega} [\varphi(u) - \varphi(v)](a\zeta_s + \Delta\zeta) \leq \\ & \leq \int_{\Omega} (u_0 - v_0)\zeta(0) + \int_0^t \int_{\Omega} (g - g^*)\zeta - \int_0^t \int_{\partial\Omega} [\varphi(U) - \varphi(V)]\partial_\nu\zeta, \end{aligned}$$

where

$$(2.2) \quad a = \begin{cases} [u - v]/[\varphi(u) - \varphi(v)] & u \neq v \\ 0 & u = v. \end{cases}$$

Since $\varphi(U) \leq \varphi(V)$ we get:

$$(2.3) \quad \int_{\Omega} [u(t) - v(t)]\zeta(t) - \int_0^t \int_{\Omega} [\varphi(u) - \varphi(v)](a\zeta_s + \Delta\zeta) \leq \\ \leq \int_{\Omega} (u_0 - v_0)\zeta(0) + \int_0^t \int_{\Omega} (g - g^*)\zeta.$$

Now one regularize a , which is a bounded function (see A1)), with the sequence $a_n := R_{\varepsilon} a + 1/n$, where R_{ε} is a mollifier with $\text{Supp}(R_{\varepsilon}) \subseteq \mathbb{B}(0, \varepsilon)$ and $\varepsilon > 0$ is chosen small enough in order to have

$$\|a - R_{\varepsilon} a\|_{L^2(\Omega \times [0, T])} < \frac{1}{n}.$$

It is easy to see that the functions a_n are smooth and the following properties hold:

$$(2.4) \quad \frac{1}{n} \leq a_n \leq \|a\|_{L^{\infty}(\Omega \times [0, T])} + \frac{1}{n} \\ (a_n - a)/\sqrt{a_n} \rightarrow 0 \quad \text{in } L^2(\Omega \times [0, T]).$$

Now fix an arbitrary $\chi \in C_0^{\infty}(\Omega)$, $0 \leq \chi \leq 1$, and consider the solutions ζ_n of the parabolic backward problems:

$$(2.5) \quad \begin{aligned} a_n(x, s)(\zeta_n)_s + \Delta\zeta_n &= \lambda a_n(x, s) && \text{on } \Omega \times [0, T] \\ \zeta_n(x, s) &= 0 && \text{on } \partial\Omega \times [0, T] \\ \zeta_n(x, T) &= \chi && \text{on } \Omega \times \{T\}. \end{aligned}$$

As in [Fi, Theorem 2.4] we can show, by means of the maximum principle and of standard integral estimates obtained from the equation (2.5), that:

$$(2.6) \quad 0 \leq \zeta_n(x, s) \leq e^{-\lambda(t-s)} \quad \text{on } \Omega \times [0, T] \\ \int_0^t \int_{\Omega} a_n [(\zeta_n)_s]^2 \leq C.$$

Substituting the functions ζ_n into (2.3) one obtains:

$$(2.7) \quad \int_{\Omega} [u(t) - v(t)]\chi - \int_0^t \int_{\Omega} [\varphi(u) - \varphi(v)]\lambda a_n \zeta_n \leq \\ \leq \int_{\Omega} (u_0 - v_0)\zeta(0) + \int_0^t \int_{\Omega} (g - g^*)\zeta + \int_0^t \int_{\Omega} (a - a_n)[\varphi(u) - \varphi(v)](\zeta_n)_s.$$

By means of Hölder's inequality and of (2.4), (2.6), one can show that the last addendum converges to zero. Therefore, passing to the limit in (2.7),

$$\int_{\Omega}[u(t) - v(t)]\chi \leq \int_{\Omega}[u_0 - v_0]^+ e^{-\lambda t} + \int_0^t \int_{\Omega}[g - g^* + \lambda(u - v)]^+ e^{\lambda(s-t)}.$$

Now choosing any sequence of $\chi_n \in C_0^\infty(\Omega)$ pointwise converging to the function $\text{sign}([u(t) - v(t)]^+)$ and letting n tend to infinity, the Lemma remains proved.

Now we can prove the following results:

PROPOSITION 2.3. *Assuming hypotheses A, let u, v be weak solutions of (DP) with initial data u_0, v_0 respectively but the same data U and f . If it happens that $u(x, t) \geq v(x, t)$ a.e. on Ω and for any $0 \leq t \leq T$, then*

$$\|u(t) - v(t)\|_{L^1(\Omega)} \leq e^{Kt} \|u_0 - v_0\|_{L^1(\Omega)}.$$

Proof. As in [Be] we may apply Lemma 2.2 to the functions u and v with $\lambda = 0$, $g := f(u)$ and $g^* := f(v)$. Then, using A2) one finds that for $0 \leq t \leq T$:

$$\int_{\Omega}[u(t) - v(t)]^+ \leq \int_{\Omega}[u_0 - v_0]^+ + K \int_0^t \int_{\Omega}[u - v]^+.$$

So the result follows from Gronwall's Lemma and the fact that for $0 \leq t \leq T$, $\int_{\Omega}[u(t) - v(t)]^+ = \|u(t) - v(t)\|_{L^1(\Omega)}$, since $u \geq v$.

PROPOSITION 2.4. *Assuming hypotheses A, let u be the weak solution constructed in Theorem 2.1 and let v be any weak subsolution of (DP) on $[0, T]$ with the same data u_0, U and f . Then $u(\cdot, t) \geq v(\cdot, t)$ a.e. on Ω , for any $0 \leq t \leq T$.*

The proof is the same as the one of [Be, Lemma 3.1] and we omit it here; the main ideas, anyway, are to use Lemma 2.2 and to prove that $v \leq u_n$ for any n using assumption A2).

THEOREM 2.5. *Assuming hypotheses A, for any $T > 0$ the problem (DP) admits a unique weak solution on $[0, T]$ with data u_0, U and f .*

Proof. If u is the weak solution constructed in Theorem 2.1 and v is another weak solution with the same data then, by Proposition 2.4, $v \leq u$. So Proposition 2.3 states that $u = v$ a.e. on $\Omega \times [0, T]$.

COMPARISON

THEOREM 2.6. *Assuming hypotheses A, let u be the weak solution of (DP) on $[0, T]$ of data u_0, U and f . i) If v is a weak subsolution of (DP) on $[0, T]$ with the same data, then $v \leq u$ a.e. on Ω , for any $0 \leq t \leq T$. ii) If w is a generalized supersolution of (DP) on $[0, T]$ with the same data, then $w \geq u$ a.e. on Ω , for any $0 \leq t \leq T$.*

The first part of this Theorem comes from the uniqueness result together with Proposition 2.4. The proof of part ii) is the same of the one proposed in [Be, Thm 0.2] and we don't reproduce it here, recalling that one needs the following result:

LEMMA 2.7. *Let $h \in L^\infty(\Omega \times [0, T])$, $h \geq 0$ a.e. on $\Omega \times (0, T]$ and assume hypotheses A. Then the theorems 2.1, 2.4 and Theorem 2.5-i) still hold replacing "weak (sub)solution of problem (DP) on $[0, T]$ with data u_0, U and f " by "weak (sub)solution of problem (PS) on $[0, T]$ with data u_0, U, f and h ".*

Note that in [Be, Lemma 4.1, Lemma 4.2] two sufficient conditions are proved in order to ensure that a function v is a weak subsolution or a generalized supersolution of (DP). We will need only the first one, which still holds under our assumptions.

LEMMA 2.8. *Let $v \in C(\overline{\Omega} \times [0, T]) \cap C^{0,1}(\Omega \times (0, T])$ be nonnegative and $\varphi(v) \in C^{2,0}(\Omega \times [0, T])$. i) If $v_t - \Delta\varphi(v) - f(v) \leq 0$ on $\Omega \times (0, T]$, $v \leq U$ on $\partial\Omega \times (0, T]$ and $v(\cdot, 0) \leq u_0$ on Ω , then v is a weak subsolution of (DP) in $[0, T]$ with data u_0, U and f . ii) If for some $C > 0$ it happens that $0 \leq v_t - \Delta\varphi(v) - f(v) \leq C$ on $\Omega \times (0, T]$, $v \geq$*

U on $\partial\Omega \times (0, T]$ and $v(\cdot, 0) \geq u_0$ on Ω , then v is a generalized supersolution of (DP) in $[0, T]$ with data u_0, U and f .

CONTINUOUS DEPENDENCE

THEOREM 2.9. *Assuming hypotheses A, let u, v be solutions of (DP) on $[0, T]$ with data u_0, U, f and v_0, U, f respectively. Then, for any $0 \leq t \leq T$,*

$$\|u(t) - v(t)\|_{L^1(\Omega)} \leq e^{KT} \|u_0 - v_0\|_{L^1(\Omega)},$$

where K is defined in hypothesis A2).

Proof. Define $w_0 := \max(u_0, v_0)$, $z_0 := \min(u_0, v_0)$ and respectively w, z the solutions of (DP) with these initial data. As, by comparison, we can show that $z \leq u \leq w$ and $z \leq v \leq w$ a.e. on Ω , we apply Proposition 2.3 to z and w in order to obtain:

$$\begin{aligned} \|u(t) - v(t)\|_{L^1(\Omega)} &\leq \|w(t) - z(t)\|_{L^1(\Omega)} \leq \\ &\leq e^{KT} \|w_0 - z_0\|_{L^1(\Omega)} = e^{KT} \|u_0 - v_0\|_{L^1(\Omega)}. \end{aligned}$$

3. The Cauchy Problem.

We will deal with the Cauchy problem:

$$\begin{aligned} \text{(CP)} \quad u_t &= \Delta\varphi(u) + f(u) && \text{on } \mathbf{R}^N \times (0, T] \\ u(x, 0) &= u_0(x) && \text{on } \mathbf{R}^N. \end{aligned}$$

We will assume that φ and f satisfy hypotheses A1) and A2) as prescribed in Section 2, while u_0 satisfies the following:

A3') $u_0 : \mathbf{R}^N \rightarrow \mathbf{R}$ is continuous, bounded and nonnegative.

For the uniqueness and comparison results we will also assume that φ is concave (at least in a right neighbourhood of the origin).

DEFINITION 3.1. *A nonnegative function u , defined on $\mathbf{R}^N \times [0, T]$ is said to be a continuous weak solution of problem (CP) on $[0, T]$ with data f and u_0 if:*

- i) u is continuous on $\mathbf{R}^N \times [0, T]$, nonnegative and bounded;
- ii) for any regular domain Ω and for any test function ζ as prescribed in Section 2 such that $\zeta \geq 0$ on $\overline{\Omega} \times [0, T]$ and $\zeta = 0$ on $\partial\Omega \times (0, T]$, u satisfies the integral identity:

$$(3.1) \quad \int_{\Omega} u(t)\zeta(t) = \int_{\Omega} u_0\zeta(0) - \int_0^t \int_{\partial\Omega} \varphi(u)\partial_{\nu}\zeta + \int_0^t \int_{\Omega} [u\zeta_t + \varphi(u)\Delta\zeta + f(u)\zeta],$$

for any $0 \leq t \leq T$. Replacing the equality into (3.1) by \leq , u is said to be a weak subsolution of problem (CP) on $[0, T]$ with data f and u_0 .

As in Section 2 we define supersolutions in this way: let's consider the problem:

$$(CS) \quad \begin{aligned} u_t &= \Delta\varphi(u) + f(u) + h(x, t) && \text{on } \mathbf{R}^N \times (0, T] \\ u(x, 0) &= u_0(x) && \text{on } \mathbf{R}^N, \end{aligned}$$

where $h \in L^\infty(\mathbf{R}^N \times (0, T])$, $h \geq 0$ a.e. on $\mathbf{R}^N \times (0, T]$.

A weak (sub-)solution of the problem (CS) and a generalized supersolution of the problem (CP) are defined according with Definitions 2.2 and 2.3 with obvious changes.

EXISTENCE

Let K be the constant defined in A2) and $M := \|u_0\|_{L^\infty(\mathbf{R}^N)}$.

THEOREM 3.1. *Assuming hypotheses A, the problem (CP) admits a continuous weak solution u on $[0, T]$ with data f and u_0 , which satisfies $u \leq Me^{Kt}$.*

Proof. Let's define f_n and A_n as in Theorem 2.1. Following [O], for any $n > 2$ define:

$$u_{0,n}(x) := \begin{cases} u_0(x) + 1/n & \|x\| < n - 2 \\ y_{0,n}(x) & n - 2 \leq \|x\| \leq n - 1 \\ M + 1/n & n - 1 < \|x\| < n, \end{cases}$$

where $\{y_{0,n}\}_{n>2}$ is a suitable sequence of continuous functions such that $u_{0,n}$ is continuous on $\overline{B(0, n)}$ and

$$u_0(x) + 1/n \leq y_{0,n}(x) \leq M + 1/n.$$

Consequently, using the ideas of Section 2, the standard theory guarantees that for any $n > 2$ there exists a unique classical solution u_n to the Dirichlet problems:

$$(CP)_n \quad \begin{array}{ll} u_t = \Delta \varphi(u) + f_n(u) + A_n & \text{on } \overline{B(0, n)} \times (0, T] \\ u(x, t) = (M + 1/n)e^{Kt} & \text{on } \partial B(0, n) \times (0, T] \\ u(x, 0) = u_{0,n}(x) & \text{on } \overline{B(0, n)}. \end{array}$$

It is clear that the sequence $\{u_n\}_{n>2}$ is positive, not increasing and uniformly bounded. Defining $u := \lim_{n \rightarrow \infty} u_n$, one shows that the u_n 's converges to a weak solution of (CP). To prove that u is continuous we may use the estimates:

- i) $\int_0^T \int_{\Omega} \|\nabla \varphi(u_n)\|^2 < C_1$
- ii) $\int_0^T \int_{\Omega} \|\nabla u_n\|^2 < C_2$
- iii) $\sup_{[0, T]} \|u_n(\cdot, t)\|_{L^2(\Omega)} < C_3$

where $\Omega \subset \mathbf{R}^N$ is any bounded domain and $C_i (1 \leq i \leq 3)$ are some positive constants, $C_i = C_i(\Omega)$, independent on n and t . Therefore we can apply the continuity results of [DB, Thm. 6.1, Thm. 7.1].

UNIQUENESS

Let us first remark that the weak continuous solution u constructed in Theorem 3.1 is *maximal* in the set of the weak continuous solutions bounded above by $\|u_0\|_{L^\infty} e^{Kt}$. This can be shown by the following:

PROPOSITION 3.2. *Assuming hypotheses A, let u be the continuous weak solution constructed in Theorem 3.1 and let v be any weak*

subsolution of (CP) on $[0, T]$ with the same data f and u_0 such that $v \leq Me^{Kt}$. Then $u \geq v$.

Proof. We will follow the same ideas exposed in the proofs of Lemma 2.2 and Proposition 2.3. Integrating $(CP)_n$ and defining a as in (2.2) we obtain:

$$\begin{aligned} & \int_{B(0,n)} [v(t) - u_n(t)]\zeta(t) - \int_0^t \int_{B(0,n)} [\varphi(v) - \varphi(u_n)](a\zeta_s + \Delta\zeta) = \\ & = \int_{B(0,n)} (u_0 - u_{0,n})\zeta(0) + \int_0^t \int_{B(0,n)} (f(v) - f_n(u_n) - A_n)\zeta + \\ & \quad - \int_0^t \int_{\partial B(0,n)} [\varphi(v) - \varphi((M+1/n)e^{Kt})]\partial_\nu\zeta. \end{aligned}$$

Since by hypothesis the last term of the above inequality is non positive, then we can proceed as in Lemma 2.2 getting

$$\begin{aligned} e^{\lambda t} \int_{B(0,n)} [v(t) - u_n(t)]^+ & \leq \int_{B(0,n)} [u_0 - u_{0,n}]^+ + \\ & + \int_0^t \int_{B(0,n)} e^{\lambda s} [f(v) - f(u_n) + \lambda(v - u_n)]^+, \end{aligned}$$

and now we can argue as in Proposition 2.3 to conclude that:

$$\int_{B(0,n)} [v(t) - u_n(t)]^+ \leq 0.$$

Therefore we have that $v \leq u_n$ in $B(0, n) \times [0, T]$ and letting n tend to infinity we conclude the proof.

For general initial data u_0 as in A3') we can give uniqueness results when φ is concave.

LEMMA 3.3. *Assuming hypotheses A with φ concave, let u, v be two weak continuous solutions of (CP) on $[0, T]$ with the same data f and u_0 . If $u \geq v$ then for any $y \in \mathbf{R}^N$, $R > 0$ and $0 \leq s \leq t$ there exists a constant $C = C(N, M, K, T) > 0$ such that*

$$\int_{B(y,R)} [u(t) - v(t)] \leq$$

$$\begin{aligned} &\leq \text{meas}(B(y, 2R))e^{Kt}\Phi^{-1}\left[\Phi\left(\frac{e^{-Ks}}{\text{meas}(B(y, 2R))}\int_{B(y, 2R)}(u(s) - v(s))\right)\right] + \\ &\quad + CR^{-2}\frac{2}{K}(e^{-\frac{Ks}{2}} - e^{-\frac{Kt}{2}}), \end{aligned}$$

where $\Phi(s) := \int_0^s \frac{dz}{\sqrt{\varphi(z)}}$.

Proof. The proof is very similar to the one of Lemma 3.1 of [He-Pi]. We choose a test function $\zeta = \mu\beta$, where $\mu \in C_0^\infty(\mathbf{R}^N)$, $\beta \in C_0^\infty([0, T])$, $\beta \geq 0$, $0 \leq \mu \leq 1$ and $\mu = 1$ on $B(y, R)$, $\mu = 0$ outside $B(y, 2R)$ for any $y \in \mathbf{R}^N$; using the assumption $u \geq v$ and f Lipschitz from above, we get the following differential inequality in $\mathcal{D}'([0, T])$:

$$\begin{aligned} (3.2) \quad &\frac{d}{dt} \int_{\mathbf{R}^N} \mu(u(t) - v(t)) \leq \\ &\leq \int_{\mathbf{R}^N} \Delta\mu(\varphi(u(t)) - \varphi(v(t))) + K \int_{\mathbf{R}^N} \mu(u(t) - v(t)). \end{aligned}$$

By means of Hölder inequality we get:

$$\begin{aligned} &\int_{\mathbf{R}^N} \Delta\mu(\varphi(u(t)) - \varphi(v(t))) \leq \\ &\leq \sqrt{\varphi(Me^{KT})}C_1(\mu)\sqrt{\int_{\mathbf{R}^N} \mu(\varphi(u(t)) - \varphi(v(t)))}, \end{aligned}$$

where, as in [He-Pi]:

$$C_1(\mu) := \sqrt{\int_{\mathbf{R}^N} \frac{|\Delta\mu|^2}{\mu}} \leq C_2(N)R^{-2+\frac{N}{2}}.$$

Since φ is concave we can use Jensen inequality to get:

$$\begin{aligned} &\int_{\mathbf{R}^N} \mu(\varphi(u) - \varphi(v)) \leq \int_{\mathbf{R}^N} \varphi(\mu(u - v)) \leq \\ &\leq \text{meas}(B(y, 2R))\varphi\left(\frac{1}{\text{meas}(B(y, 2R))}\int_{\mathbf{R}^N} \mu(u - v)\right). \end{aligned}$$

Substituting the above inequality into (3.2) and denoting by $w := \int_{\mathbf{R}^N} \mu(u - v)(t)$, we get in $\mathcal{D}'([0, T])$ that:

$$(3.3) \quad \frac{dw}{dt} \leq Kw + C_3R^{N-2}\sqrt{\varphi\left(\frac{1}{\text{meas}(B(y, 2R))}w\right)},$$

where $C_3 = C_3(M, K, T, N)$. Integrating (3.3) we conclude the proof setting $C := C_3/\text{meas}(B(y, 2R))$.

THEOREM 3.4. *Assuming hypotheses A with φ concave, then there exists a unique continuous weak solution of (CP) on $[0, T]$ bounded above by Me^{Kt} with data f and u_0 .*

Proof. Let u be the solution constructed in Theorem 3.1 and let v be any other weak continuous solution bounded above by Me^{Kt} . Proposition 3.2 states that $u \geq v$, so we can apply Lemma 3.3. Then the proof is almost the same of the one of [He-Pi, Thm. 2.3]; the main difference is that they use the fact that $\int_0^t (u^m - v^m)$ is superharmonic in $\mathcal{D}'([0, T])$. Here multiplying (3.2) for e^{-Kt} and integrating in time we get:

$$0 \leq e^{-Kt} \int_{\mathbf{R}^N} \mu(u(t) - v(t)) \leq \int_{\mathbf{R}^N} \Delta \mu \int_0^t e^{-K\tau} (\varphi(u(\tau)) - \varphi(v(\tau))),$$

i.e. the function $\omega(t) := \int_0^t e^{-K\tau} [\varphi(u(\tau)) - \varphi(v(\tau))] d\tau$ is superharmonic in $\mathcal{D}'([0, T])$. Therefore, proceeding as in [He-Pi] and using again Jensen inequality, we have that for any $R > 0$:

$$\begin{aligned} \omega(y, t) &\leq C_4(N) R^{-N} \int_{B(y, R)} \omega(x, t) dx \leq \\ &\leq C_5(N) \int_0^t e^{-K\tau} \varphi\left(\frac{1}{\text{meas}(B(y, R))} \int_{B(y, R)} (u(\tau) - v(\tau))\right). \end{aligned}$$

Using now Lemma 3.3 with $s = 0$ and the above inequality we get that:

$$\begin{aligned} 0 &\leq e^{-Kt} \int_0^t (\varphi(u(y, \tau)) - \varphi(v(y, \tau))) \leq \omega(y, t) \leq \\ &\leq \frac{C_5}{K} (1 - e^{-Kt}) \varphi(C_6(N) e^{KT} \Phi^{-1}(CR^{-2} \frac{2}{K})). \end{aligned}$$

Letting R tend to infinity, one ends the proof.

REMARK 3.5. *With the same method used in Lemma 3.3 and Theorem 3.4 one can prove a uniqueness result also if φ is concave only near the origin. To be more precise, one can assume that there exists $s_0 > 0$ such that $\varphi(s)$ is concave for $0 < s < s_0$. We omit the proof of this result for sake of brevity; let us only remark that in Lemma 3.3 the function Φ must be substituted by*

$\Phi^* := \int_0^s \frac{dz}{\sqrt{K^*z + \varphi(z)}}$, where K^* is the Lipschitz constant of φ for $s_0 \leq s \leq Me^{KT}$.

In the case that $\varphi(u) := u^m$, $0 < m < 1$ one can obtain an estimate similar to the one of [He-Pi]. In fact, in the assumptions of Lemma 3.3 one get that there exists a constant $C = C(N, m) > 0$ such that:

$$\begin{aligned} \int_{B(y,R)} [u(t) - v(t)] &\leq C \{ \int_{B(y,2R)} (u(s) - v(s)) e^{K(t-s)} \} + \\ &+ [K^{-1} e^{K(1-m)(t-s)} - 1]^{\frac{1}{1-m}} R^{N - \frac{2}{1-m}} \}. \end{aligned}$$

COMPARISON

This comparison criterion is the same of that seen in Section 2. As its validity depends also from the uniqueness of the solutions of (CP), we consider φ concave.

THEOREM 3.6. *Assuming hypotheses A, let u be the continuous weak solution of (CP) on $[0, T]$ of data u_0 and f . i) If v is a weak subsolution of (CP) on $[0, T]$ with same data, then $v \leq u$ a.e. on \mathbf{R}^N , for any $0 \leq t \leq T$. ii) If w is a generalized supersolution of (CP) on $[0, T]$ with same data, then $w \geq u$ a.e. on \mathbf{R}^N , for any $0 \leq t \leq T$.*

For the proof of this Theorem hold the same considerations exposed in Section 2 and particularly in Lemma 2.7; moreover one has the following:

LEMMA 3.7. *For any domain $\Omega \subset \mathbf{R}^N$ let $v \in C(\overline{\Omega} \times [0, T]) \cap C^{0,1}(\Omega \times (0, T])$ be nonnegative and $\varphi(v) \in C^{2,0}(\Omega \times (0, T])$. i) If $v_t - \Delta\varphi(v) - f(v) \leq 0$ on $\Omega \times (0, T]$ and $v(\cdot, 0) \leq u_0$ on Ω , then v is a weak subsolution of (CP) in $[0, T]$ with data u_0 and f . ii) If for some $C > 0$ it happens that $0 \leq v_t - \Delta\varphi(v) - f(v) \leq C$ on $\Omega \times (0, T]$ and $v(\cdot, 0) \geq u_0$ on Ω , then v is a generalized supersolution of (CP) in $[0, T]$ with data u_0 and f .*

For general φ satisfying hypothesis A1), the first part of Theorem 3.6 holds if we denote by u the solution constructed in the Theorem

3.1 (see Proposition 3.2). The second part holds if we denote by w the generalized supersolution constructed by the same method of Theorem 3.1, i.e. if $w := \lim_n w_n, w_{n,0} \geq u_{n,0}$.

Let us also remark that in the absorption case one could also consider unbounded solutions. In fact the solutions of the absorption equation are smaller than the ones of the equation of pure diffusion and for the latter it is only required that $u \in L^1_{loc}$ (see [He-Pi]). Moreover in Section 4 it is given an explicit unbounded stationary solution (see (4.4),(4.5)).

4. Instantaneous Shrinking.

We will deal with the Cauchy problem in the strong absorption case

$$(AP) \quad \begin{aligned} u_t &= \Delta \varphi(u) - F(u) && \text{on } \mathbf{R}^N \times (0, T] \\ u(x, 0) &= u_0(x) && \text{on } \mathbf{R}^N \end{aligned}$$

i.e. the problem (CP) with $f \equiv -F$ under hypotheses B.

Let us make some preliminar remarks. First of all, hypotheses B1) + B2) imply that η is a nondecreasing function, $\eta(0) = 0$ and hence (see [Ev-Kn])

$$\frac{s}{2} \eta\left(\frac{s}{2}\right) \leq \int_0^s \eta \leq s \eta(s).$$

Therefore B3) implies that

$$(4.1) \quad \int_0^1 \left[\int_0^s \eta(s) \right]^{-1/2} < \infty \quad (\text{i.e. localization})$$

and

$$\frac{1}{2} \sqrt{\frac{s}{\eta(s)}} < \int_{s/2}^s \frac{dz}{\sqrt{z\eta(z)}} < C < \infty,$$

therefore:

$$(4.2) \quad \int_0^1 \frac{dz}{\eta(z)} < \int_0^1 \frac{2C dz}{\sqrt{z\eta(z)}} < \infty.$$

Since assumption B1) holds, the last inequality implies that $\int_0^1 [F(s)]^{-1} ds < \infty$ and hence a condition of extinction in finite time is ensured.

In the power nonlinearity case, B1) and B3) implies respectively $m < 1$ and $\frac{p}{m} < 1$, and so one can restate hypotheses B simply as $0 < p < m < 1$. Clearly, assumption B2) implies that also A2) is satisfied with $K = 0$.

In what follows we will denote by $M := \|u_0\|_{L^\infty(\mathbf{R}^N)}$, u will be the continuous weak solution constructed in Theorem 3.1 and $\psi(s) := \varphi^{-1}(s)$. Let us remark here that since u is maximal (see Proposition 3.2) all the result exposed in this Section still hold for any continuous solution of (AP) bounded by M . In the power nonlinearity diffusion case, when $\varphi(s) := s^m$, u is the unique weak solution (see Theorem 3.4).

LEMMA 4.1. *Assuming hypotheses B, let u_0 be a nonnegative real valued function defined on \mathbf{R}^N such that*

$$(4.3) \quad \lim_{\|x\| \rightarrow \infty} u_0(x) = 0.$$

Then the solution u of (AP) is such that

$$\lim_{\|x\| \rightarrow \infty} u(x, t) = 0,$$

uniformly in t .

Proof. The proof is obtained by comparison. As in [Ev-Kn], we will consider the stationary problem

$$(4.4) \quad \begin{aligned} y''(r) &= \kappa \eta(y(r)) & r > 0 \\ y(0) &= 0 \\ y'(0) &= 0, \end{aligned}$$

where κ is a positive constant which will be defined in the following. By hypotheses B3) (see 4.1) the problem (4.4) possesses a unique nonnegative strictly increasing solution

$$(4.5) \quad \int_0^{y(\pm r)} [(2\kappa \int_0^s \eta(z) dz)^{-1/2}] ds = r.$$

If we consider the power nonlinearity case then $\eta(s) \equiv \lambda s^{p/m}$, $\lambda > 0$ and $y(r) \equiv \tilde{c} r^{\frac{2m}{m-p}}$, $\tilde{c} := (\frac{\lambda \kappa (m-p)^2}{2m(m+p)})^{\frac{m}{m-p}}$. Recall also that in our assumptions $\frac{2m}{m-p} > 2$, as $0 < p < m$.

With the help of y we are going to construct a supersolution of (AP). For any $a > 0$ and $x_0 \in \mathbf{R}^N$, let's set $\kappa := \frac{1}{N}$ and define on $\mathbf{R}^N \times [0, T]$

$$g_{a,x_0}(x) := a + \sum_{i=1}^N y(|x^i - x_0^i|)$$

and then

$$W_{a,x_0}(x, t) := (\psi \circ g_{a,x_0})(x).$$

We have that, dropping for simplicity the indexes:

$$\begin{aligned} W_t - \Delta\varphi(W) + F(W) &= \\ &= - \sum_{i=1}^N y''(|x^i - x_0^i|) + \eta(a + \sum_{i=1}^N y(|x^i - x_0^i|)) \geq \\ &\geq - \sum_{i=1}^N \frac{1}{N} \eta(y) + \eta(\sum_{i=1}^N y) \geq 0, \end{aligned}$$

where the last statement makes use of a simple monotonicity argument, (i.e. $\eta(\sum a_i) \geq \frac{1}{N} \sum \eta(a_i)$, when $a_i \geq 0$). Now as (4.3) holds, for any $\varepsilon > 0$ there exists some $R(\varepsilon) > 0$ such that $0 \leq u_0 \leq \varepsilon$ if $\|x\| > R(\varepsilon)$ and $0 \leq u_0 \leq M$ if $\|x\| \leq R(\varepsilon)$. A similar thing can be made for the $u_{0,n}$'s introduced in Section 3: for any sufficiently large n we have that $u_{0,n} < \varepsilon + 1/n$ if $R(\varepsilon) \leq \|x\| \leq n - 2$ and $u_{0,n} < M + 1/n$ if $\|x\| < R(\varepsilon)$ or $n - 2 \leq \|x\| \leq n$. If we define the positive constant

$$(4.6) \quad d := y^{-1}(\varphi(M+1)) \equiv \int_0^{\varphi(M+1)} 2\kappa[\int_0^s \eta(z)dz]^{-1/2} ds,$$

it will be sufficient to take $a := \varphi(\varepsilon + \frac{1}{n})$ and to consider any x_0 such that $R(\varepsilon) + d\sqrt{N} < \|x_0\| < n - 2 - d\sqrt{N}$, in order to obtain that:

$$W(x, 0) \geq u_{0,n} \quad \text{on } \|x\| \leq n$$

$$W \geq M + 1 \geq M + \frac{1}{n} = u_n \quad \text{on } \{ \|x\| = n \} \times [0, T],$$

where u_n is the classical solution found in Section 3. Finally the comparison results (see Theorem 2.6 and Lemma 2.8) ensures that W is a generalized supersolution, and in particular for any large n :

$$\frac{1}{n} \leq u_n(x_0, t) \leq W(x_0, t) = \varepsilon + \frac{1}{n},$$

for $R(\varepsilon) + d\sqrt{N} < \|x_0\| < n - 2 - d\sqrt{N}$. Passing to the limit as n diverges to infinity, we obtain that $u(x_0, t) \leq \varepsilon$, for any $x_0 > R(\varepsilon) + d\sqrt{N}$. Since d is independent of t (and x_0), the Lemma remains completely proved.

REMARK 4.2. *The technique here exposed can be also used to prove a localization property: suppose that for some x_0 and some $c \geq 0$ it holds that $u_0(x) \leq c$ for every x in the cube $\|x - x_0\|_\infty < d$; then $u(x_0, t) \leq c$, for every $t > 0$. Let us also remark that since we use a “local” comparison function, the same result holds for the solution of a Dirichlet problem in a sufficiently large domain. Moreover the only assumption used is B3), i.e. that the absorption is stronger than the diffusion (see [Ev-Kn]).*

THEOREM 4.3. *In the same hypotheses of Lemma 4.1, the support $S(t)$ of the solution u is bounded for any $t > 0$.*

Proof. The idea is to use Lemma 4.1 and to show that a comparison function similar to the one of [Ev-Kn], is a supersolution of (AP) only in a certain given bounded parabolic neighborhood of arbitrary (x_0, t_0) . Note that this can't be made in the whole $\mathbf{R}^N \times [0, T]$ as in slow-diffusion case. First of all, let's define such comparison function; for any $C > 0, x_0 \in \mathbf{R}^N, t_0 \geq 0$ we set:

$$(4.7) \quad w(x, t) := \psi(C + h(t_0 - t) + \sum_{i=1}^N y(|x^i - x_0^i|)),$$

where y and h respectively solve the ordinary problems (4.4) and

$$(4.8) \quad \begin{aligned} h'(s) &= \kappa \eta(h(s)) \\ h(0) &= 0 \end{aligned}$$

with $\kappa := \frac{1}{N+1}$. Hypotheses B (see 4.2) ensure that h is a nonnegative function given by:

$$(4.9) \quad \int_0^{h(s)} [\eta(z)]^{-1} dz = \frac{s}{N+1}.$$

In the power nonlinearity case, when $\eta(s) \equiv \lambda s^{p/m}$, h is given by

$$h(s) \equiv [\lambda \kappa \frac{m-p}{m}]^{\frac{m}{m-p}} s^{\frac{m}{m-p}}.$$

The first step is to show that there exists a bounded parabolic neighbourhood of (x_0, t_0) which we denote by G such that if $(x, t) \in G$ then

$$\mathcal{L}w := w_t - \Delta \varphi(w) + F(w) \geq 0.$$

Let's define the constants:

$$\gamma := \sup\{s > 0 \mid \psi'(z) \leq 1, 0 \leq z \leq s\}$$

$$\delta := (N+1) \int_0^{\gamma/2(N+1)} [\eta(s)]^{-1} ds$$

$$\rho := \int_0^{\gamma/2(N+1)} \left[\int_0^s \eta(z) dz \right]^{-1/2} ds$$

in a way that $h(\delta) = g(\rho) = \gamma/2(N+1)$. Note that in the power nonlinearity case, $\gamma = m \frac{\gamma}{1-m}$. For any $0 \leq C \leq \gamma/2$ and for any (x_0, t_0) we define

$$G_{x_0, t_0} := \{(x, t) \mid t_0 - \delta < t \leq t_0; \|x - x_0\|_\infty < \rho\}$$

and set $v := \varphi(w)$, obtaining:

$$\begin{aligned} \mathcal{L}w &= \psi(v)_t - \Delta v + \eta(v) = \\ &= (\psi'(v) - 1)v_t + [-h' - \sum g'' + \eta(h + \sum g + C)] \geq \\ &\geq (\psi'(v) - 1)v_t, \end{aligned}$$

where the addendum in bracket square vanishes applying the same monotonicity argument used in the proof of Lemma 4.1. As for $0 < t < t_0$ the term v_t is negative, it is enough to show that $\psi'(v) \leq 1$ in order to prove that $\mathcal{L}w \geq 0$. As a matter of fact, for any $0 \leq C \leq \gamma/2$ and for any (x_0, t_0) , if we take any point $(x, t) \in G$ we have that:

$$0 \leq v = h + \sum g + C \leq \frac{\gamma}{2(N+1)} + \frac{N\gamma}{2(N+1)} + \frac{\gamma}{2} = \gamma,$$

and hence $\psi'(v) \leq 1$.

The second step is to apply Lemma 4.1: given γ as above, for any $t \geq 0$ there exists $R(\gamma) > 0$ such that if $\|x\| > R(\gamma)$ then $u(x, t) < \psi(\gamma/2(N+1)) < \psi(\gamma/2)$. As δ is a well defined constant independent from t_0 , let's fix $0 < t_0 \leq \delta$ and, assuming that $S(t_0)$ is unbounded, choose any $x_0 \in S(t_0) \cap \{\xi \in \mathbf{R}^N \mid \|\xi\| > R(\gamma) + \rho\sqrt{N}\}$. Recalling the definition of w (4.7), let's fix C in a way that

$$w(x_0, t_0) \equiv \frac{1}{2}u(x_0, t_0) > 0.$$

Since $\|x_0\| > R(\gamma)$ then $u(x_0, t_0) < \psi(\frac{\gamma}{2(N+1)})$ and hence

$$C = \varphi(\frac{1}{2}u(x_0, t_0)) < \varphi(u(x_0, t_0)) < \frac{\gamma}{2}$$

(this choice of C still ensures that $\mathcal{L}w \geq 0$ on G). As a consequence, for any $(x, t) \in \Gamma := \partial_t G \equiv \{(x, t) | 0 < t \leq \delta; \|x - x_0\|_\infty = \rho\}$ we obtain that:

$$w(x, t) \geq \psi(g(\|x^i - x_0^i\|)) = \psi(g(\rho)) \geq \left(\frac{\gamma}{2(N+1)}\right) \geq u|_\Gamma(x, t).$$

Now it is possible to claim that there is a point $x^* \in \mathbf{R}^N$ with the property that $\|x^* - x_0\|_\infty < \rho$ such that:

$$(4.10) \quad w(x^*, 0) \leq u(x^*, 0) \equiv u_0(x^*).$$

In fact if this would not be the case and if it were $w(x, 0) > u_0(x)$ for any $x \in G \cap \{t = 0\}$, our choices should allow to apply Theorem 2.6 to obtain that $w \geq u$ on G , yielding the contradiction that:

$$\frac{1}{2}u(x_0, t_0) \equiv w(x_0, t_0) \geq u(x_0, t_0) > 0.$$

Finally, inequalities (4.10) states that:

$$u_0(x^*) \geq w(x^*, 0) \geq \psi(h(t_0)) > 0,$$

and so $x^* \in L_0 := \{\xi \in \mathbf{R}^N | u_0(\xi) > \psi(h(t_0))\}$, which is a bounded set by the hypothesis (4.3). But the quantity $\|x^* - x_0\|_\infty$ is bounded by ρ and so the same x_0 must belong to a bounded set. In conclusion we have shown that $S(t_0)$ is bounded for any $0 < t_0 \leq \delta$. For further references, let's be more precise: the support $S(t_0)$ is such that:

$$(4.11) \quad S(t_0) \subset B(0, R(\gamma) + \rho(\gamma)\sqrt{N}) \cup \{\xi | \xi \in B(\bar{x}, \rho\sqrt{N}); \bar{x} \in L_0\}.$$

Since ρ, δ depends only on γ , which is fixed once for all, we are able to repeat this reasoning step by step on the intervals $(\delta, 2\delta], (2\delta, 3\delta], \dots$ thus proving that $S(t)$ is bounded for any $0 < t \leq T$.

PROPOSITION 4.4. *Assuming hypotheses B, let u_0 be a nonnegative real valued function defined on \mathbf{R}^N with compact support. Then the solution u of (AP) possesses the localization property (i.e. the support of u is contained in a fixed ball for any time).*

Proof. Recalling what stated in Remark 4.2, it is sufficient to use the comparison function w defined in (4.7) after having set $C = 0$ and chosen x_0 rather far from $\text{Supp}(u_0)$.

THEOREM 4.5. *In the same hypotheses of Lemma 4.1, there exists an “extinction time” $T^* > 0$ such that $u(x, t) \equiv 0$ for any $t \geq T^*$.*

Proof. Set $T^* := \int_0^M [F(s)]^{-1} ds$ and consider for instance $t_0 := T^*/2$. Two cases are possible: either $u(x, t_0) \equiv 0$, and so there is nothing else to prove since $u \equiv 0$ for any $t > t_0$. Otherwise, if $u(x, t_0) \not\equiv 0$, we can argue this way: as $S(t_0)$ is bounded, using Proposition 4.4 we have that there exists some $r > 0$ such that for any $t > t_0$ if $|x| > r$ then $u(x, t) \equiv 0$. This implies that u can be regarded also as the unique solution of a Dirichlet problem with zero data on the lateral boundary $\partial B(0, r) \times [t_0, T]$. Again the idea is simply to compare u with the supersolution $y(x, t) := [q(t)]^+$, where $q(t)$ is defined by

$$\int_M^{q(t)} [F(s)]^{-1} ds = -t$$

and it solves the ordinary problem

$$\begin{aligned} q'(t) &= -F(q(t)) & r > 0 \\ q(0) &= M \end{aligned}$$

in order to have that $0 \leq u(x, t) \leq y(x, t) = 0$ for any $T^* \leq t \leq T$.

Let us remark that all the results of this Section also hold for the initial boundary values problem in $\mathbf{R}^N \setminus \Omega$, being Ω a bounded regular domain.

Let us also remark that hypothesis B3) is crucial to have I.S. (see also [Ev-Kn], Remark 2.4). In fact if we assume that $\int_0^1 [s\eta(s)]^{-1/2} ds = \infty$, then there exist a stationary solution which tends to 0 as $\|x\|$ diverges but it is everywhere positive. As an example, take $N = 1$ and consider the function $u := \psi(y)$, where y solves the problem

$$\begin{aligned} y''(r) &= \eta(y(r)) & r > 0 \\ y(0) &= y_0 > 0 \\ y'(0) &= v_0 := \sqrt{2 \int_0^{y_0} \eta(s) ds}. \end{aligned}$$

In this case $\int_0^{y_0} dz [\int_0^z \eta]^{-1/2} = \infty$ and hence $y \searrow 0$ as $\|x\| \rightarrow \infty$.

5. Estimates of the Solution and of its Support.

Throughout this Section we suppose that there exists a continuous bounded positive radially symmetric function V_0 such that:

$$\text{B4) } \lim_{r \rightarrow \infty} V_0(r) = 0, \varphi(u_0(x)) \leq V_0(\|x\|) \text{ for } \|x\| > R_0 \\ \text{(e.g. } R_0 = 1)$$

and we will denote by hypotheses B^* the set of hypotheses B1), B2), B3) and B4). Under this assumptions we will give superior estimates on the support of u as $t \rightarrow 0$. First of all, we note that the proof of Theorem 4.3 (more precisely the estimate (4.11)) immediately provide the following superior estimate, whose proof is omitted:

PROPOSITION 5.1. *Assuming hypotheses B^* , for any $t > 0$ sufficiently small the support $S(t)$ of the solution u is such that $S(t) \subset B(0, \mathcal{R}_+(t))$, where*

$$(5.1) \quad \mathcal{R}_+(t) := V_0^{-1}(h(t))$$

and h is defined in (4.8), (4.9), i.e. we have:

$$\int_0^{V_0(\mathcal{R}_+(t))} [\eta(z)]^{-1} dz = \frac{t}{N+1}.$$

In the power nonlinearity case, when $\eta(t) = \lambda t^{p/m}$ and $h(t) = c_1 t^{\frac{m}{m-p}}$, if one chooses for instance $V_0(r) := Ar^{-\alpha}$, $A, \alpha > 0$ then

$$(5.2) \quad \mathcal{R}_+(t) = c_2 t^{\frac{-m}{\alpha(m-p)}}.$$

The estimate (5.2) is not sharp, as it can be shown with the following argument: let's suppose that for $\|x\| > R_1$, besides hypotheses B^* one also has that $\varphi(u_0(x)) = u_0^m \geq V_0(\|x\|) := a\|x\|^{-\alpha}$. Then it is possible to show that for $t > 0$ sufficiently small:

$$(5.3) \quad B(0, \mathcal{R}_-(t)) \subset S(t), \quad \mathcal{R}_-(t) := c_3 t^{\frac{-m}{\alpha(1-p)}}$$

where

$$\begin{aligned}
p + m &< 2 \\
p &< m \\
p &< 1 && \text{(e.g. } 0 < p < m < 1) \\
\frac{m}{1-p} &> \max(1, [N-2]^+) \\
[N-2]^+ &< \alpha < \frac{m}{1-p} \\
c_3 &= c_3(\alpha, a, m, p, \lambda).
\end{aligned}$$

The estimate (5.3) is obtained by comparison with the function

$$\mathcal{U} := [\mathcal{U}_0^{1-p} - (1-p)\lambda t]_+^{\frac{1}{1-p}}, \quad \mathcal{U}_0 = \frac{1}{2}V_0^{\frac{1}{m}}$$

which is a solution of the pure absorption problem

$$\begin{aligned}
\mathcal{U}_t &= -\lambda\mathcal{U}^p && \text{on } [R_1, \infty) \times (0, T] \\
\mathcal{U}(r, 0) &= \mathcal{U}_0 && \text{on } [R_1, \infty).
\end{aligned}$$

The next statement provides a “sharp” superior estimate.

PROPOSITION 5.2. *Under hypotheses B^* , if $\varphi(s) := s^m$, $F(s) := \lambda s^p$ and $V_0(r) := Ar^{-\alpha}$ for $r > R_0$ are such that $m > p > 0$, $p < 1$, $2 \geq m + p$ (e.g. $0 < p < m < 1$) and $0 < \alpha \leq \frac{m}{1-p}$, then, for any $t > 0$ sufficiently small, one has that $S(t) \subset B(0, \mathcal{R}(t))$, where*

$$(5.4) \quad \mathcal{R}(t) := c_4 t^{\frac{-m}{\alpha(1-p)}} \equiv c_4 t^{-\beta} \quad c_4 = c_4(N, \alpha, m, p, A, \lambda).$$

Proof. Let’s consider the comparison function

$$(5.5) \quad \mathcal{U}(r, t) := \varphi^{-1}(V_0 w(\frac{r}{\mathcal{R}(t)})),$$

where w is the solution of the problem (4.4) in the power nonlinearity case, i.e.

$$w(z) = \begin{cases} w_0(1-z)^{\frac{2m}{m-p}} & 0 \leq z < 1 \\ 0 & z \geq 1. \end{cases}$$

Let $R > R_0$ be fixed and define the constants w_0 and T_0 such that:

$$(5.6) \quad V_0(R + d\sqrt{N})w_0 = 2V_0(R)$$

$$(5.7) \quad \varphi(\mathcal{U}(R + d\sqrt{N}, T_0)) = V_0(R)$$

where d is the constant defined in (4.6). We will show that it is possible to fix $R = R(N, \alpha, A, m, p, \lambda)$ so that

$$(5.8) \quad \mathcal{U} \geq u \quad \text{on } D := \{(x, t) \mid \|x\| > R + d\sqrt{N}, 0 < t < T_0\}.$$

Let us remark that Lemma 4.1, under assumption B*, ensure that $\varphi(u(x, t)) \leq V_0(\|x\| - d\sqrt{N})$ if $\|x\| \geq R_0 + d\sqrt{N}, t \geq 0$. Therefore it is easy to check that the given definitions (5.5), (5.6) and (5.7) ensure that $\mathcal{U} \geq u$ on the parabolic boundary of the domain D, for any $R > R_0$. In conclusion it remains to prove that we can fix R so that $\mathcal{L}\mathcal{U} \geq 0$ in D.

Set $\mathcal{L}\mathcal{U} \equiv I_1 + I_2$ where

$$(5.9) \quad \begin{aligned} I_1 &:= \frac{3}{4}\eta(V_0 w) - \left[\frac{1}{\mathcal{R}^2} V_0 w'' + \frac{1}{\mathcal{R}} w' \left(V_0 \frac{N-1}{r} + 2V_0' \right) \right. \\ &\quad \left. + w(V_0'' + \frac{N-1}{r} V_0') \right] \\ I_2 &:= \frac{1}{4}\eta(V_0 w) + w' \frac{d}{dt} \left(\frac{1}{\mathcal{R}} \right) r V_0 \psi'(V_0 w). \end{aligned}$$

Since we have:

$$(5.10) \quad w' = \frac{-2m}{m-p} w_0^{\frac{m-p}{2m}} w^{\frac{m+p}{2m}},$$

we get that, for $0 < z := \frac{r}{\mathcal{R}} < 1$:

$$I_1 \geq \frac{\eta(V_0 w)}{\lambda} \left[\frac{3\lambda}{4} - \frac{1}{\sigma} (w_0 V_0(r))^{\frac{m-p}{m}} \tilde{V}(r) \right],$$

where

$$(5.11) \quad \begin{aligned} \sigma &:= [2m(m+p)]^{-1} (m-p)^2 \\ \tilde{V} &:= \frac{1}{r^2} + \frac{1}{r} \left[\frac{(N-1)}{r} + 2 \frac{V_0'}{V_0} \right]^- + \left[\frac{(N-1)}{r} \frac{V_0'}{V_0} + \frac{V_0''}{V_0} \right]^+ = \\ &= \frac{1}{r^2} (1 + [N-1-2\alpha]^- + \alpha[\alpha+2-N]^+) = \\ &= \frac{1}{r^2} c_5(\alpha, N). \end{aligned}$$

Therefore, for $r > R + d\sqrt{N}$,

$$I_1 \geq \frac{\eta(V_0 w)}{\lambda} \left[\frac{3\lambda}{4} - \frac{1}{\sigma} w_0^{\frac{m-p}{m}} (V_0^{\frac{m-p}{m}} \tilde{V})(R + d\sqrt{N}) \right].$$

Now recalling the definition of w_0 one has that $I_1 \geq 0$ when $r > R + d\sqrt{N}$ and R is such that

$$(5.12) \quad V_0^{\frac{m-p}{m}}(R)\tilde{V}(R+d\sqrt{N}) \leq \frac{3\lambda\sigma}{2^{3-p/m}}.$$

For instance, one can choose R such that

$$\begin{aligned} V_0^{\frac{m-p}{m}}(R)\tilde{V}(R+d\sqrt{N}) &\leq (V_0^{\frac{m-p}{m}}\tilde{V})(R) = \\ &= A^{\frac{m-p}{m}}c_6(\alpha, N)R^{-2-\frac{\alpha(m-p)}{m}} \leq \frac{3\lambda\sigma}{2^{3-p/m}}. \end{aligned}$$

Let us consider I_2 ; since

$$\frac{d}{dt}\left(\frac{1}{\mathcal{R}}\right) = \beta c_4^{-\frac{1}{\beta}}\left(\frac{1}{\mathcal{R}}\right)^{1-\frac{1}{\beta}} = g\left(\frac{1}{\mathcal{R}}\right),$$

where

$$c_4^{\frac{1}{\beta}} := 2^{2+\frac{1}{m}}\left(1 + \frac{d}{R_0}\right)^{\frac{\alpha(1-m)}{m}}\beta[\lambda(m-p)]^{-1}A^{\frac{1-p}{m}}$$

and $0 < z = \frac{r}{\mathcal{R}} < 1$, we have that, being $m+p < 2$ and $w < w_0$:

$$(5.13) \quad I_2 \geq -w'\psi'V_0(r)[q(w)\check{V}(r) - g\left(\frac{1}{r}\right)]$$

where

$$\begin{aligned} q(w) &:= \frac{\lambda(m-p)}{8}w_0^{\frac{(m-p)}{2m}}w^{\frac{(m+p-2)}{2m}} \geq \frac{\lambda(m-p)}{8}w_0^{\frac{m-1}{m}} \\ \check{V}(r) &:= r^{-1}V_0^{-\frac{1-p}{m}} = A^{-\frac{1-p}{m}}r^{-1+\alpha\frac{1-p}{m}}. \end{aligned}$$

Now recalling the definition of w_0 and c_4 one gets that $I_2 \geq 0$, thus concluding the proof.

Let us now remark that with the same method of Proposition 5.2 one can prove a priori estimates for more general functions $V_0(r)$.

If $V_0 = Ar^{-\alpha}$ and $\alpha > \frac{m}{1-p}$, then (5.4) and (5.8) still hold with $\beta = 1$. This is again a better estimate than the one given in Proposition 5.1 for $\alpha < \frac{m}{m-p}$. If V_0 is not a power function then (5.4), (5.8) hold with \mathcal{R} defined by

$$\int_{\mathcal{R}(t)}^{+\infty} s^{-1}V_0^{\frac{1-p}{m}}(s)ds = c_7t$$

provided V_0 satisfy the following assumptions:

- i) $V_0^{\frac{m-p}{m}} \tilde{V}$ is decreasing for $r > R_0$.
- ii) $V_0^{\frac{m-p}{m}}(r) \tilde{V}(r + d\sqrt{N})$ tends to 0 as r diverges to infinity.
- iii) \tilde{V} tends to 0 as r diverges to infinity.
- iv) $r^{-1} V_0^{\frac{m-p}{m}}(r) \in L^1(R_0, \infty)$
where \check{V} and \tilde{V} are defined in (5.11) and in (5.13).

As we already mentioned, the proof is the same of that of Proposition 5.2, with R as in (5.12). For example the method works for $V_0 = A(\log r)^{-q - \frac{m}{1-p}}$, $q > 0$, $r > 2$ and gives $\mathcal{R}(t) = \exp[(dt)^{-\frac{m}{q(1-p)}}]$. This estimate is better than (5.1) if $q > \frac{m(m-p)}{(1-m)(1-p)}$. We note that if the hypotheses iii) and iv) are substituted by iii') $\tilde{V} \geq v_0 > 0$ for $r > R_0$, then the comparison function of Proposition 5.2 still provides the superior estimate $\mathcal{R}(t) = c_8 t^{-1}$, where $c_8 = c_8(\lambda, d, p, m, V_0)$ is a suitable constant.

Let us now consider the general equation (AP).

PROPOSITION 5.3. *Under the assumptions B^* , if η is concave and $V_0 := Ar^{-\alpha}$ is such that $r^{-2}V_0^{-1}\eta(V_0)$ converges monotonically to zero as r diverges to $+\infty$, then the free boundary of the maximal solution u is bounded by above by $\mathcal{R}(t)$ defined by:*

$$(5.14) \quad \int_0^{w_0 V_0(\mathcal{R}(t))} [s\eta(s)]^{-\frac{1}{2}} ds = c_9 t,$$

where $w_0 := 2(1 + \frac{d\sqrt{N}}{R_0})^\alpha$ and $c_9 = c_9(A, \alpha, \lambda, p, m, d, V_0, \max_{[0,2]} \psi')$.

Proof. The idea is again to follow what has already been seen in Proposition 5.2 using the hypotheses η concave and ψ' bounded. The main difference is that now it is not possible to use directly the comparison criterion for u .

Define the sequences:

$$\mathcal{U}_n := \psi(V_n)$$

$$V_n := V_0(|x|) w(\frac{r}{R}) + b_n$$

$$b_n := \varphi(\frac{1}{n}) + V_0(n - 2 - 2\sqrt{N}),$$

where w is defined in a similar way of Proposition 5.2:

$$(5.15) \quad w(z) := \begin{cases} A^{-1}y(1-z) & 0 \leq z < 1 \\ 0 & z \geq 1 \end{cases}$$

and $y(r)$ is the solution of (4.4) with $\kappa := \frac{C^2}{4}$. $C = C(w_0)$ is chosen so that $w(0) = w_0$, that is:

$$C := \sqrt{2A} \int_0^{w_0} [\int_0^\xi \eta(As) ds]^{-\frac{1}{2}} d\xi.$$

We will compare \mathcal{U}_n with the approximating sequence u_n of Theorem 3.1 on the domain $\Omega_n \times [0, T_0]$, $\Omega_n := \{x \in \mathbf{R}^N \mid R + d\sqrt{N} < |x| < n - 2 - d\sqrt{N}\}$, R and T_0 being two suitable constants independent from n . Once proved that for sufficiently large n one has that $u_n \leq \mathcal{U}_n$ on $\Omega_n \times [0, T_0]$, then letting n tend to infinity it remains proved that for the maximal solution u one gets:

$$(5.16) \quad u(x, t) \leq \mathcal{U} := \lim_n \mathcal{U}_n \equiv \psi(V_0 w)$$

and so the desired estimate.

In order to prove that $u_n \leq \mathcal{U}_n$, we observe that from Lemma 4.1 and from B4), setting $v_{0,n} := \varphi(u_{0,n})$, $v_n := \varphi(u_n)$, we can say that for sufficiently large n ($n > n^*$, n^* depending only on the modulus of continuity of φ):

$$\begin{aligned} \text{i)} \quad & \varphi\left(\frac{1}{n}\right) \leq v_{0,n}(x) \leq \varphi\left(\frac{1}{n} + \psi(V_0(\|x\|))\right) \leq \\ & \leq V_0(\|x\|) + \varphi\left(\frac{1}{n}\right) \quad \text{when } R_0 < \|x\| < n - 2 \\ \text{ii)} \quad & \varphi\left(\frac{1}{n}\right) \leq v_n(x, t) \leq \varphi\left(\frac{1}{n} + \psi(V_0(\|x\| - d\sqrt{N}))\right) \leq \\ & \leq V_0(\|x\| - d\sqrt{N}) + \varphi\left(\frac{1}{n}\right) \quad \Omega_n \times [0, \infty). \end{aligned}$$

From the definition of V_n and setting T_0 as in (5.7) one can easily find that $u_n = \varphi(v_n) \leq \varphi(V_n) = \mathcal{U}_n$ on the parabolic boundary of $\Omega_n \times [0, T_0]$. As for $\mathcal{L}\mathcal{U}_n$ one has:

$$\begin{aligned} \mathcal{L}\mathcal{U}_n &= \psi(V_0 w + b_n)_t - \Delta(V_0 w) + \eta(V_0 w + b_n) \geq \\ & \psi(V_0 w + b_n)_t - \Delta(V_0 w) + \eta(V_0 w) \geq I_1 + I_{2n} \end{aligned}$$

with I_1 and I_{2n} as in (5.9) - with $\psi'(V_0w + b_n)$ in place of $\psi'(V_0w)$. Since η is concave we have that $\eta(\lambda s) \geq \lambda\eta(s)$, $0 < \lambda < 1$ and $\eta(\lambda s) \geq \sqrt{\lambda s\eta(\lambda)\eta(s)}$ for $0 < \lambda, s < 1$; therefore we have, in place of (5.10),

$$w' \geq -\frac{C}{\sqrt{2A}}\sqrt{w\eta(Aw)}.$$

Hence:

$$I_1 \geq \eta(V_0w) \left\{ \frac{3}{4} - \frac{1}{\mathcal{R}(t)^2} \frac{C^2}{4} V_0(r) - \frac{1}{\mathcal{R}(t)} \frac{C}{\sqrt{2}} \sqrt{\frac{w_0V_0}{\eta(w_0V_0)}} \left[\frac{N-1}{r} + \frac{2V_0'}{V_0} \right]^+ + \right. \\ \left. - \frac{w_0V_0}{\eta(w_0V_0)} \left[\frac{V_0''}{V_0} + \frac{N-1}{r} \frac{V_0'}{V_0} \right]^+ \right\} =: \tilde{I}_1(r, t).$$

Now we fix $R := \max(R_1, R_2)$, being R_1 and R_2 determined by the positions:

$$w_0V_0(R_1) \leq 1$$

$$\left[\frac{V_0''}{V_0} + \frac{N-1}{r} \frac{V_0'}{V_0} \right]^+(R_2) = \alpha R_2^{-2} [\alpha + 2 - N]^+ \leq \frac{\eta(1)}{4}$$

and we set T_0 sufficiently small in a way that, besides (5.7), it holds on D (see (5.8)):

$$\tilde{I}_1(r, t) \geq \tilde{I}_1(R + d\sqrt{N}, T_0) \geq 0.$$

As for I_{2n} , (5.13) still holds with the positions

$$q(w) := q_0 \equiv \frac{\sqrt{2}}{4C\Psi^*}$$

$$\check{V} := \frac{1}{r} \sqrt{\frac{\eta(w_0V_0(r))}{w_0V_0(r)}}$$

$$g\left(\frac{1}{r}\right) = \frac{d}{dt}\left(\frac{1}{\mathcal{R}}\right)|_{\mathcal{R}=r} = C^*\check{V}(r)$$

where $\Psi^* := \max_{[0,2]}\psi'$ and we have used the definition of \mathcal{R} (see (5.14)). Since \check{V} is decreasing when r diverges to infinity, it is possible to choose C^* independent from n such that $I_{2n} \geq 0$ on $\Omega_n \times (0, T_0]$, thus concluding the proof.

The comparison method outlined in Proposition 5.3 can be applied to more general function $V_0(r)$ and can be refined as soon as one has the precise form of the functions F and φ . We will not state the precise assumptions here since they are too cumbersome.

Let us also remark that Proposition 5.3 gives always a less accurate estimate on the free boundary than the one given in (5.2); on the other hand it provides an estimate on the maximal solution itself (see (5.15), (5.16)). For instance, let us consider $F(s) := \frac{\log(-\log s)^\vartheta}{(-\log s)}$, $\varphi(s) := \frac{1}{-\log s}$, $\vartheta > 2$, $0 < s < \frac{1}{2}$, $V_0 = Ar^{-\alpha}$. Then $\eta(s) = s(-\log s)^\vartheta$ and the estimates of Propositions 5.1 and 5.3 are respectively $\mathcal{R}(t) = A \exp(\frac{c}{\alpha} t^{-\frac{1}{\vartheta-1}})$ and $\mathcal{R}(t) = (Aw_0)^{\frac{1}{\alpha}} \exp(\frac{\tilde{c}}{\alpha} t^{-\frac{2}{\vartheta-2}})$.

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REFERENCES

- [Ar-Cr-Pe] ARONSON D.G., CRANDALL M.G. and PELETIER L.A., *Stabilization of Solutions of a Degenerate Nonlinear Diffusion Problem*, *Nonlinear Anal.* **6** (1982), 1001-1022.
- [Be] BERTSCH M., *A Class of Degenerate Diffusion Equations with a Singular Nonlinear Term*, *Nonlinear Anal.* **7** (1983), 117-127.
- [Be-Ke-Pe] BERTSCH M., KERSNER R. and PELETIER L.A., *Positivity versus Localization in Degenerate Diffusion Equation*, *Nonlinear Anal.* **9** (1985), 987-1008.
- [DB] DiBENEDETTO E., *Continuity of Weak Solutions to a General Porous Medium Equation*, *Indiana Univ. Math. J.* **32** (1983), 83-118.
- [Ev-Kn] EVANS L.C. and KNERR B.F., *Instantaneous Shrinking of the Support of Nonnegative Solutions to Certain Nonlinear Parabolic Equations and Variational Inequalities*, *Illinois J. Math.* **23** (1979), 153-166.
- [Fi] FILO J., *On Solutions of a Perturbed Fast Diffusion Equation*, *Aplikace Matematiky* **32** (1987), 364-380.
- [He-Pi] HERRERO M.A. and PIERRE M., *The Cauchy Problem for $u_t = \Delta u^m$ when $0 < m < 1$* , *Trans. Amer. Math. Soc.* **291** (1985), 145-158.

- [Ke-Ni] KERSHNER R. and NICOLosi F., *The Nonlinear Heat Equation with Absorption: Effects of Variable Coefficient*, J. Math. Anal. Appl. **170** (1992), 551-566.
- [La-So-Ur] LADYZHENSKAYA O.A., SOLONNIKOV V.A. and URAL'CEVA N.N., *Linear and Quasi-linear Equations of Parabolic Type*, Providence Rhode Island, American Mathematical Society, (1986).
- [Ol] OLEINIK O., *On Some Degenerate Quasilinear Parabolic Equations*, Seminari INDAM 1962-1963, Roma, Ed. Cremonese (1963), 355-371.