THE GEOMETRY OF CERTAIN THREE-FOLDS (*)

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Sommario. - *Si considera un caso particolare di* 3*-folds compatte* M *diffeomorfi alla somma connessa di* n *copie di* $S^3 \times S^3$*. Se* $n \geq 2$*, la varietà non-Kähleriana* M *ha una struttura complessa con* $c_1 = 0$*. Si dimostra che non ci sono fibrati lineari non-banali su* M *e quindi si deduce che il fibrato tangente di* M *è stabile rispeto ad ogni metrica di Gauduchon. Dal teorema di Li e Yau si conclude che su* M *existe una metrica di Hermite-Einstein.*

Summary. - *We consider a special case of compact* 3*-folds* M *which are diffeomorphic to the connected sum of* n *copies of* $S^3 \times S^3$*. If* $n \geq 2$*, the non-Kähler manifold* M *has a complex structure with* $c_1 = 0$*. We prove that there are no non-trivial line bundles on* M *and hence we deduce that its tangent bundle is stable with respect to any Gauduchon metric. By a theorem of Li and Yau we conclude that there is an Hermitian-Einstein metric on* M*.*

Introduction.

In this paper we consider a class of compact simply connected 3-folds M with trivial canonical bundle and the following Hodge numbers:

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$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Namely, M has $h^{1,0} = h^{0,1} = 0, \quad h^{2,0} = h^{0,2} = h^{1,1} = 0, \quad h^{3,0} = h^{0,3} = 1$ and $c_1(M) = 0$.

The first question which could be raised is whether such M exist. R.Friedman [7] answered afirmatively this question proposing an algebraic-geometrical construction. Moreover, he showed that C.T.C.Wall's classification of 6-manifolds [28] implies that M is diffeomorphic to the connected sum of n copies of $S^3 \times S^3$, where $n \geq 103$ [7]. Recently, P.Lu and G.Tian [20] proved that for any $n \geq 2$ the connected sum of n copies of $S^3 \times S^3$ posseses a complex structure with trivial canonical class.

As it is seen, the algebraic-geometrical structure of M is well investigated and understood ([6, 7, 23, 20]), while its differential-geometrical structure does not seem to be satisfactorily clarified. Especially, on such non-Kählerian manifolds there is a lack of analytic instruments. Recalling the case of K3 surfaces and more general Calabi-Yau manifolds, we observe that the Calabi-Yau metric provides an important tool for the investigation of the moduli space. See [26, 25, 12]. Moreover, on K3 fixing a complex structure and a cohomology class, namely the class of the Kähler form, determines uniquely the Calabi-Yau metric.

Now it is an open problem to have a Calabi-Yau substitute for non-Kähler manifolds. With this paper we initiate a search for different sorts of conditions in order to have a rigidity theorem (as on K3), which could suggest the existence of a canonical metric. We propose one such candidate, using the notion of stability. Indeed, Uhlenbeck and Yau [27] proved the existence of Hermitian-Einstein metrics on stable bundles over compact Kähler manifolds. The theorem of Uhlenbeck and Yau [27] was generalized by Li and Yau [17] for non-Kähler manifolds. The Kähler condition is replaced by the Gauduchon condition which holds for a large class of Hermitian metrics. Therefore our first aim will be to prove the stability of the tangent bundle of M. Before doing this, in section 2 we prove that $h^{2,1} = n - 1$ and discuss the moduli space of M. Further we briefly describe the above mentioned constructions of R.Friedman [7] and Lu-Tian [20]. For completeness of the exposition, in section 3 we recall the definition of stability, Hermitian-Einstein metrics, Gauduchon's condition and Li-Yau's theorem.

In section 4 we prove that there are no non-trivial line bundles on M. This enables us to obtain the main result in the paper:

PROPOSITION. *The holomorphic tangent bundle of M is stable with respect to any Gauduchon metric.*

Then we are in position to apply the theorem of Li and Yau [17], from which we conclude that there is an Hermitian-Einstein metric on M. As far as is known to the author, the only previous application of the Li-Yau theorem can be found in [18], where Yau et al. suggest a short proof of a famous theorem of Bogomolov.

2. Complex Geometry.

At this stage, we have at our disposal only the Hodge diamond of M, a complex structure J with $c_1(J) = 0$ and the compactness. Here are some direct consequences of these facts.

If we look at the Hodge diamond, we can see at first sight that M is not a Kähler manifold. Indeed, the inequalities

$$0 \leq b_r \leq \sum_{p+q=r} h^{p,q},$$

which hold for any compact Hermitian manifold [8] imply that $b_2 = 0$. Note that in the Kähler case the second inequality is actually an equality which follows from the Hodge decomposition of Kähler manifolds.

Another natural question concerns the relationship between $h^{2,1}$ and $n-$the number of copies of $S^3 \times S^3$ in the connected sum. For any compact complex manifold the Euler characteristic can be calculated by

$$\chi(M) = \sum_{r=0}^{dim M} (-1)^r b_r = \sum_{p,q=0}^{dim M/2} (-1)^{p+q} h^{p,q},$$

where b_r is the r-th Betti number and $h^{p,q}$ is the respective Hodge number. The second equality is well-known for Kähler manifolds for the same reason we pointed out above$-$the particular Hodge decomposition of such manifolds. In the general Hermitian case this formula can be obtained by considering the Frölicher spectral sequences [8] which relate the cohomology groups of Dolbeault as invariants of the complex structure and the cohomology groups of de Rham as topological invariants. It can also be obtained by the Atiyah-Singer index theorem. For M it gives

$$\chi = -2h^{2,1}.$$

On the other hand, taking a connected sum of two even-dimensional manifolds N_1 and N_2, the Euler characteristic behaves as follows:

$$\chi(N_1 \# N_2) = \chi(N_1) + \chi(N_2) - 2.$$

See [1]. Thus

$$\chi(\#_n S^3 \times S^3) = \chi(\#_{n-1} S^3 \times S^3) - 2 = ... = -2(n-1),$$

since

$$\chi(S^3 \times S^3) = \chi(S^3)\chi(S^3) = 0.$$

Therefore

$$h^{2,1} = n - 1.$$

Further we note that

$$H^1(M, \Theta) = H^1(M, \Omega^2) = H_{\bar{\partial}}^{2,1}(M),$$

where Θ is the sheaf of germs of the holomorphic vector fields over M, $\Omega^p-$the sheaf of the holomorphic p-forms, and we have used the triviality of the canonical bundle which implies $\Theta \cong \Omega^2$. Moreover,

$$H^2(M, \Theta) = H^2(M, \Omega^2) = H_{\bar{\partial}}^{2,2}(M) = 0.$$

The spaces $H^1(\Theta)$ and $H^2(\Theta)$ play an important role in the theory of deformations of complex structures advanced by Kodaira and Spencer [15, 16]. A subsequent theorem of Kodaira, Nirenberg and Spencer [14] states that if a manifold has a complex structure J such that $H^2(\Theta) = 0$, then for any infinitesimal deformation I of J there exists an actual deformation of J which infinitesimally coincides with I. As we saw above, the obstruction space $H^2(\Theta)$ vanishes for M, so we can apply directly the Kodaira-Nirenberg-Spencer theorem [14], from which we conclude that the local moduli space of M is smooth, that is, the first order deformations are unobstructed.

It is worthwhile to note that the K3 surfaces provided one of the earliest examples which illustrate the Kodaira-Spencer theory. From Yau's proof of the Calabi's conjecture [29] and the fact that every K3 surface is Kähler [24], it is well known that the K3 surfaces admit non-trivial Kähler-Einstein-Calabi-Yau metrics. The Calabi-Yau manifolds give other examples for the theory of deformations of complex structures. G.Tian [25] proved that the local moduli space of a Calabi-Yau manifold is smooth of dimension $\dim H^1(\Theta) = \dim H^1(\Omega^{m-1})$. Since for such manifolds $\Theta \cong \Omega^{m-1}$, the obstruction space $H^2(\Theta)$ is $H^2(\Omega^{m-1})$. The latter one need not be zero for Kähler manifolds. Indeed, for Kähler three-folds $H^2(\Omega^2)$ is never zero in the contrast to our situation on M.

Before concluding this section, we shall give a brief description of the Friedman [7] and Lu-Tian [20] constructions of complex structures with trivial canonical bundle on the connected sum of n copies of $S^3 \times S^3$. The details and references can be found in [6, 7, 20, 23].

One begins with a smooth quintic three-fold N in CP^4 which contains infinitely many $(-1,-1)$ smooth rational pairwise disjoint curves C_i, one of which is a line. Recall that a $(-1,-1)$ curve C in N is a rational curve, such that the normal bundle of C in N splits into $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The existence of such quintic threefolds N is due to Clemens ([5]). In [7], p. 130, Friedman describes a modification of Clemens' construction which provides a simply connected N such that $[C_i]$ span $H^2(N, \Omega^2)$ and there is a relation $\sum_i \lambda_i [C_i] = 0$ in $H^2(N, \Omega^2)$, where $\lambda_i \neq 0$ for every i.

Then one takes $k \geq 2$ such curves C_i of degrees d_i, one of which is chosen to be a line. Since the C_i are $(-1,-1)$ disjoint curves, they can be contracted to k ordinary double points P_i. In this way a three-fold $\tilde{N}$ is obtained. By contraction, we mean an isomorphism

$N \setminus C_i \to \tilde{N} \setminus P_i$ between complex analytic varieties. By [6] $\tilde{N}$ has small deformations M in which the singularities disappear and if $H^1(N, \mathcal{O}) = 0$, then all smoothings M have trivial canonical bundle. From Lemma 8.1, [7], p. 126, $\pi_1(N) = \pi_1(M)$ and hence M is simply connected. Moreover, since by the construction of N the curves C_i satisfy the above mentioned relation in $H^2(N, \Omega^2)$, the Corollary 8.8, p. 129 in Friedman's paper [7] implies that $H_2(M, Z) = Z/dZ$, where d is the greatest common divisor of the d_i. But $d = 1$ because one of the contracted curves is a line. Thus $H_2(M, Z) = 0$.

The Betti numbers of M and N are related by

$$b_2(M) = b_2(N) - s$$

and

$$b_3(M) = b_3(N) + 2k - 2s,$$

where $k - s$ is the rank of the kernel of $\bigoplus Z[C_i] \to H_2(N, Z)$ (see [7] or [23]). As we saw $b_2(M) = 0$. From the last formula and from the special construction of the "generic" quintic manifold N, the third Betti number of M is in fact $b_3(M) = 2(k + 101)$. It can also be seen that $H_3(M, Z)$ is torsion-free.

Summarizing, this complicated algebraic-geometrical procedure provides a compact simply connected 6-manifold M with $H_2(M, Z) = 0$, $H_3(M, Z)$ - torsion-free and which possesses a complex structure J with trivial canonical class.

On the other hand, according to the classification of C.T.C.Wall [28] any compact oriented 6-manifold which is simply connected and whose second Stiefel-Whitney class $w_2 = 0$, is classified up to diffeomorphism by the third Betti number b_3, $H^2(Z)$, first Pontrjagin class p_1 and a trilinear map $H^2(Z) \times H^2(Z) \times H^2(Z) \to Z$ given by cup product. Restricting to the case $H^2(Z) = 0$, this implies that any simply connected manifold with $H^2(Z) = 0$ and $H_3(Z)$ a torsion-free Z module of rank $2n$ is diffeomorphic to a connected sum of n copies of $S^3 \times S^3$ ([28]). Hence, since $w_2(M) = p_1(M) = 0$, M is diffeomorphic to $\#_n S^3 \times S^3$, where $n = 101 + k \geq 103$ and there is a complex structure J on M, such that its first Chern class $c_1(J) = 0$.

In [20] P.Lu and G.Tian start with the singular three-fold $N \subset CP^4$ defined by

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4 = 0,$$

$(z_0 : z_1 : z_2 : z_3 : z_4)$ being the homogeneous co-ordinates of CP^4. N has 125 ordinary double points. Then they consider (any) small resolution M_1 of N, which afterwards is contracted to a singular variety $\tilde{M}_1$. The latter has 26 double points. The whole construction provides data which satisfy the conditions of Corollary 8.8, p. 129, in [7]. Hence, there are smoothings $M = M_t$ of $\tilde{M}_1$ such that $c_1(M) = 0$, $\pi_1(M) = 0$, $H_2(M) = 0$ $b_3 = 4$ and by C.T.C.Wall's classification M is diffeomorphic to $(S^3 \times S^3)\#(S^3 \times S^3)$. Then one easily obtains complex structures of $c_1(M) = 0$ on $\#_n S^3 \times S^3$, $2 \leq n \leq 102$. It remains to combine this with the previously described R.Friedman's construction.

3. Hermitian geometry.

To study an Hermitian manifold it is always useful to pick a metric with some special properties. Such metrics could be Kähler, balanced, Einstein, etc. However, to admit a special metric the Hermitian manifold must satisfy some conditions, generally of a topological nature. First of all, one seeks a metric within a given conformal class. For instance, to obtain a balanced metric, i.e. a metric whose fundamental form F satisfies $\partial(F^{m-1}) = 0$, in general is impossible. In many cases such metrics simply do not exist. What one can always achieve is due to the following result of Gauduchon [9, 10]:

THEOREM. [9, 10] *Given any Hermitian metric on a compact complex manifold of dimension at least* 2, *there is a conformal metric unique up to homothety, such that its fundamental form* F *satisfies*

$$\partial\bar{\partial}(F^{m-1}) = 0. \tag{1}$$

We shall call *Gauduchon metric* a metric for which the condition (1) holds. In his own terminology such metrics are said to be *standard* or of *null eccentricity.* In fact, there are many of them–one within each conformal class.

N. Hitchin observed in [11] that the Gauduchon metrics enable us to extend the notion of stability to holomorphic bundles on an arbitrary Hermitian manifold M. Namely, if L is a holomorphic line bundle on M, its degree with respect to a given Gauduchon metric F is

$$\deg(L) = \deg(L, F) = \frac{i}{2\pi} \int_M l \wedge F^{m-1},$$

where l is the curvature of any Hermitian connection on L compatible with $\bar{\partial}_L$, or more generally for a torsion-free coherent sheaf $\mathcal{S}$ on M

$$\deg(\mathcal{S}) = \int_M c_1(\mathcal{S}) \wedge F^{m-1}$$

is well-defined for the Gauduchon condition since any two first Chern forms differ by a $\partial\bar{\partial}$ - exact form. If $c_1 = 0$ then the degree vanishes.

Denote

$$\mu(\mathcal{S}) = \deg(\mathcal{S})/\mathrm{rank}(\mathcal{S}).$$

DEFINITION. *$\mathcal{S}$ is called stable if and only if*

$$\mu(\mathcal{S}') < \mu(\mathcal{S})$$

for any torsion-free subsheaf $\mathcal{S}'$ of $\mathcal{S}$.

On the other hand there is the notion of Hermitian-Einstein metric. Let (E, h) be a holomorphic vector bundle over (M, g), where h is an Hermitian metric in $E \longrightarrow M$ and g is an Hermitian metric on M. The Chern connection D of h is the unique metric connection which preserves the complex structure and whose torsion is a vector valued (2,0) form. The latter condition is equivalent to $D'' = \bar{\partial}$ ([13]). D is also called the (standard) Hermitian connection, or in the terminology of A.Lichnerovicz "second canonical Hermitian connection" ([19]). For the Chern connection, in any local holomorphic frame, the corresponding connection forms are of type (1,0) with values in $End(E)$. In local complex coordinates adapted to the complex structure, D has the well-known components:

$$\Gamma_{\alpha\ \beta}^{\ \lambda} = h^{\lambda\bar{\mu}} \frac{\partial}{\partial z^{\alpha}} h_{\beta\bar{\mu}}$$

and its curvature is given by

$$R_{\alpha\ \lambda\bar{\mu}}^{\ \beta} = \frac{\partial}{\partial \bar{z}^{\mu}} \Gamma_{\lambda\ \alpha}^{\ \beta}.$$

Then the Chern connection D of h is said to be Hermitian-Einstein with respect to g if and only if

$$g^{\lambda\bar{\mu}} R(h)_{\alpha\ \lambda\bar{\mu}}^{\ \beta} = k.\delta_{\alpha}^{\beta}, \tag{2}$$

where $R(h)$ is the curvature of D and k is a function. We shall also call h an Hermitian-Einstein metric, when its Chern connection is Hermitian-Einstein. If $E = \mathcal{T}-$the tangent bundle of M, and if $g = h$, then (2) reads

$$r_{\alpha}{}^{\beta} = k.\delta_{\alpha}^{\beta}, \tag{3}$$

with $r-$the second Ricci form of $g = h$. In the latter case if g is Kähler, (3) is the well-known Kähler-Einstein condition.

In [21] Lübke proved if an indecomposible bundle E over a Kähler manifold M admits an Hermitian-Einstein metric, then it is stable. Later Uhlenbeck and Yau [27] proved the opposite statement. N.Hitchin suggested that the same relationship between stability and Hermitian-Einstein metrics should be also valid in the general Hermitian setting. Buchdahl [4] proved the theorem for surfaces and Li and Yau [17] generalized the work of Uhlenbeck-Yau to the non-Kähler case for all dimensions. Namely,

The theorem of Li and Yau.[17] *Let M be a compact Hermitian manifold with a Gauduchon metric, and E be a holomorphic vector bundle over M. Then E is stable if and only if it admits an Hermitian-Einstein metric .*

4. Stability of the tangent bundle.

Proposition 1. *There are no non-trivial line bundles on M.*

Proof. Consider the exponential exact sequence

$$0 \longrightarrow Z \longrightarrow \mathcal{O} \stackrel{\exp}{\longrightarrow} \mathcal{O}^* \longrightarrow 0$$

and a part of the corresponding long exact sequence

$$H^1(Z) \to H^1(\mathcal{O}) \to H^1(\mathcal{O}^*) \stackrel{\delta}{\to} H^2(Z) \to H^2(\mathcal{O}) \to \ldots$$

We have

1) $H^1(\mathcal{O}) \cong H^{0,1}_{\bar{\partial}}$ by the Dolbeault theorem. But $h^{0,1} = 0$. Therefore $H^1(\mathcal{O}) = 0$;

2) since M is simply connected $H^1(Z) = 0$;

3) $H^2(\mathcal{O}) \cong H^{0,2}_{\bar{\partial}}$ and $h^{0,2} = 0$. Thus $H^2(\mathcal{O}) = 0$.

Then we have the exact sequence

$$0 \to H^1(\mathcal{O}^*) \xrightarrow{\delta} H^2(Z) \to 0.$$

Thus δ is an isomorphism and therefore

$$H^1(M, \mathcal{O}^*) \cong H^2(M, Z).$$

Further

$$\dim H^1(M, \mathcal{O}^*) = \text{rank}_Z H^2(M, Z) = \text{rank}_Z H_4(M, Z) = h^{2,2} = h^{1,1}$$

from the form of the Hodge diamond in [23]. Hence

$$H^1(M, \mathcal{O}^*) = 0 \quad \text{if and only if} \quad h^{1,1} = 0.$$

The latter holds for M.

From this proposition we deduce that there are no subvarieties of M of co-dimension 1, that is, no divisors.

The main result in this paper is the following

Proposition 2. *The (holomorphic) tangent bundle $\mathcal{T}$ of M is stable with respect to any Gauduchon metric.*

Proof. By Proposition (7.6)(b′) in [13], p.169, (also valid in the Hermitian case) it is sufficient to check the stability condition only for such subsheaves S of $\mathcal{O}(\mathcal{T})$ for which the quotient sheaf $Q = \mathcal{O}(\mathcal{T})/\mathcal{S}$ is torsion-free.

Let S be a rank 1 subsheaf of $\mathcal{O}(\mathcal{T})$ with Q torsion-free. Since $\mathcal{T}$ is a vector bundle, $\mathcal{O}(\mathcal{T})$ is a locally free sheaf and therefore torsion-free. Hence, S is also torsion-free as subsheaf of $\mathcal{O}(\mathcal{T})$. Thus we have the following exact sequence

$$0 \longrightarrow S \longrightarrow \mathcal{O}(\mathcal{T}) \longrightarrow Q \longrightarrow 0.$$

Since Q is torsion-free and $\mathcal{O}(\mathcal{T})$ is reflexive, from Lemma 1.1.16 in [22] it follows that S is normal and being torsion-free, we get that S is a reflexive rank 1 sheaf. The latter means, equivalently, that S is a line bundle. But from the Proposition 1 we conclude that S is the trivial line bundle. Its non-vanishing section is therefore a non-zero section of $\mathcal{O}(\mathcal{T}) = \Theta$. On the other hand, since the canonical bundle K_M is trivial, that is,

$$K_M = \wedge^3 \mathcal{T}^* = \mathcal{O},$$

we have the pairing

$$\mathcal{T}^* \otimes \wedge^2 \mathcal{T}^* \longrightarrow \mathcal{O}$$

from which we obtain

$$\mathcal{T} \cong \wedge^2 \mathcal{T}^*$$

and

$$\Theta \cong \Omega^2.$$

So far, we have a non-zero holomorphic 2-form. This contradicts the fact that

$$\dim H^0(M, \Omega^2) = \dim H^{2,0}(M) = h^{2,0} = 0$$

which follows from the Dolbeault theorem and from the Hodge diamond. Therefore there are no rank 1 subsheaves of Θ with torsion-free quotient.

Now suppose E to be a rank 2 subsheaf of Θ and let $F = \Theta/E$. We have the exact sequence

$$0 \longrightarrow E \longrightarrow \Theta \longrightarrow F \longrightarrow 0$$

and also a part of the dual long sequence

$$0 \longrightarrow F^* \longrightarrow \Omega^1 \longrightarrow \ldots.$$

For an arbitrary coherent sheaf A its dual A^* is reflexive ([13], Proposition 5.18, p.160) and therefore F^* is a rank 1 reflexive sheaf, i.e. F^* is a line bundle. Again from Proposition 1, F^* has to be the trivial line bundle and to have a non-vanishing section, which, from the inclusion $F^* \longrightarrow \Omega^1$, provides a non-zero holomorphic 1-form. This is a contradiction since

$$\dim H^0(M, \Omega^1) = \dim H^{1,0}(M) = h^{1,0} = 0.$$

Therefore there are no rank 2 subsheaves of Θ.

In this way we have proved the stability of the holomorphic tangent bundle of M.

COROLLARY. *There are no holomorphic subbundles of* $\mathcal{T}$.

5. Concluding remarks.

As we proved in the previous section, the tangent bundle of M is stable with respect to any Gauduchon metric. Thus the theorem of Li and Yau applied to the tangent bundle implies that for any Gauduchon metric g there exists an Hermitian-Einstein metric h, that is,

$$g^{\lambda\bar{\mu}} R(h)_{\alpha}{}^{\beta}{}_{\lambda\bar{\mu}} = k.\delta^{\beta}_{\alpha}, \tag{4}$$

where $R(h)$ is the curvature of the Chern connection, determined by h. By a conformal change of h, we can always make the function k to be a constant [13]. Moreover, since $c_1(M) = 0$, the degree of the tangent bundle must be zero. Hence, an easy calculation gives $k = 0$ (see [13]), and therefore

$$g^{\lambda\bar{\mu}} R(h)_{\alpha}{}^{\beta}{}_{\lambda\bar{\mu}} = 0. \tag{5}$$

Up to this point the Gauduchon condition has been used only to have a definition of the degree which makes sense. Any Hermitian metric g_1 can be written as

$$g_1 = \varphi g, \tag{6}$$

where the smooth function $\varphi > 0$ is uniquely determined and g is the respective Gauduchon metric in the conformal class of g_1 [9]. And vice-versa, any Gauduchon metric can be obtained from some Hermitian metric by (6). Hence, inserting (6) into (5) gives

$$g_1^{\lambda\bar{\mu}} R(h)_{\alpha}{}^{\beta}{}_{\lambda\bar{\mu}} = 0.$$

Since the tangent bundle is stable with respect to any Gauduchon metric, we see from the above equation and from the Li-Yau theorem that any Hermitian metric g_1 determines a unique Hermitian metric h which is Hermitian-Einstein with respect to g_1. Of course, h is Hermitian-Einstein with respect to any Hermitian metric in the conformal equivalence class of g_1. This is not a surprise since the Hermitian-Einstein condition is not "differential" with respect to the Gauduchon metric.

Now let $\rho(h)$ be the first Ricci tensor of h, which has the components

$$\rho(h)_{\lambda\bar{\mu}} = \frac{\partial^2}{\partial z^\lambda \partial \bar{z}^\mu} \log\det(h). \tag{7}$$

From (5) we obtain

$$g^{\lambda\bar{\mu}}\rho(h)_{\lambda\bar{\mu}} = 0. \tag{8}$$

But $\rho(h)$ is given by (7). Therefore from (8) we conclude that

$$L(\log\det(h)) = 0, \tag{9}$$

where

$$L = g^{\lambda\bar{\mu}}\frac{\partial^2}{\partial z^\lambda \partial \bar{z}^\mu}.$$

L is an elliptic operator such that $L(1) = 0$. Hence (9) and the maximum principle of E.Hopf imply that

$$\det(h) = c = constant. \tag{10}$$

The last remark is in fact a tautology since we look for $U(3)$ connection and the first Chern class vanishes.

Then from (7) and (10) we also get

$$\rho(h) = 0.$$

As we pointed out in the Introduction, it is an open problem to have a substitute of the Calabi-Yau metric for non-Kähler manifolds. However, if we suppose that the Hermitian-Einstein metric h and the Gauduchon metric g coincide, this would be one possible candidate. In another paper we shall look for some consequences of the existence of such a metric. Here we would only like to note that the situation is similar to that on the K3 surfaces considered in [2, 3], where we also had at our disposal two metrics: the Eguchi-Hanson and the Euclidean ones.

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