ONE-PARAMETER GROUPS OF VOLUME-PRESERVING AUTOMORPHISMS OF $\mathbf{C}^2$ (*)

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SOMMARIO. - *Si esaminano i gruppi ad un parametro nel gruppo degli automorfismi polinomiali di* $\mathbf{C}^2$ *e nel gruppo degli shears, provando che sono coniugati a gruppi a un parametro nel gruppo degli automorfismi affini di* $\mathbf{C}^2$ *o nel gruppo degli automorfismi elementari; da ciò si deducono risultati sul comportamento asintotico del gruppo ad un parametro, sui suoi punti periodici e sui suoi punti fissi.*

SUMMARY. - *In this work we study the one-parameter groups in the group of all polynomial automorphisms of* $\mathbf{C}^2$ *and in the group of all shears. We prove that any such one-parameter group is conjugated to a one-parameter group contained either in the group of all affine automorphisms of* $\mathbf{C}^2$ *or in the group of elementary automorphisms. This implies some results on the asymptotic behaviour of the one-parameter group, on its periodic points and on its fixed points.*

0. Introduction.

In this work we investigate the structure of the one-parameter groups in $Aut_P\mathbf{C}^2$, the group of all polynomial automorphisms of $\mathbf{C}^2$, and in the group of all shears, G_1, introduced by J.-P. Rosay and W. Rudin in [12]. A one-parameter group in $Aut_P\mathbf{C}^2$ (G_1) is a continuous homomorphism from $\mathbf{R}$ to $Aut_P\mathbf{C}^2$ (G_1), where these last two groups are both endowed with the compact-open topology.

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In particular we will prove that, both in the case of $Aut_P\mathbf{C}^2$ and in the case of G_1, each one-parameter group is conjugated to a one-parameter group in two suitable subgroups: the subgroup of affine automorphisms of $\mathbf{C}^2$ and the subgroup of "elementary transformations" (to be defined in §1). This gives us the possibility to study the asymptotic behaviour of the one-parameter group, its set of common fixed points and other qualitative results on the behavior of its orbits. We will show, in particular, the lack of any chaotic phenomena, in contrast with the discrete case of iterates (see [4]).

In the first section we give the definition of the shear group, G_1, as well as the definition of the subgroup of all elementary automorphisms of the group $Aut_P\mathbf{C}^2$ and G_1, and present the main results.

In the second section we prove a structure theorem both for $Aut_P\mathbf{C}^2$ and G_1, *i.e.*, we prove the fact that there exist two subgroups $A, E \subset Aut_P\mathbf{C}^2$ such that $Aut_P\mathbf{C}^2$ is the free product of A and E amalgamated over their intersection. Similarly for G_1, we prove that there exist two subgroups $A_1, E_1 \subset G_1$ such that G_1 is the free product of A_1 and E_1, amalgamated over their intersection.

In the third section we prove that any one-parameter group in $Aut_P\mathbf{C}^2$ (G_1) is conjugated to a one-parameter group in A or E (A_1 or E_1, respectively).

1. Definitions and Main Results.

Let us first consider the group of polynomial automorphisms of $\mathbf{C}^2$, which we denote by $Aut_P\mathbf{C}^2$. An *elementary automorphism* of $Aut_P\mathbf{C}^2$ is a transformation of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha x + p(y) \\ \beta y + \gamma \end{pmatrix}, \tag{1.1}$$

where $\alpha, \beta \in \mathbf{C}^*$, $\gamma \in \mathbf{C}$ and p is a polynomial with coefficients in $\mathbf{C}$. Let A be the group of affine automorphisms of $\mathbf{C}^2$ and E be the group consisting of all elementary automorphisms of $\mathbf{C}^2$. Obviously A and E are subgroups of $Aut_P\mathbf{C}^2$. E is said to be the subgroup of elementary automorphisms in $Aut_P\mathbf{C}^2$.

The structure of the group $Aut_P\mathbf{C}^2$ is given by the following

THEOREM 1.1. *$Aut_P\mathbf{C}^2$ is the free product of A and E amalga-*

mated over $A \cap E$.

The structure theorem (whose proof is postponed, together with the proof of Theorem 1.3, to §2) will be useful in the following to understand the behaviour of one-parameter groups in $Aut_P\mathbf{C}^2$.

Together with the group of polynomial automorphisms of $\mathbf{C}^2$, we want to consider also the group of all shears, *i.e.*, the group generated by the automorphisms of $\mathbf{C}^2$ of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + f\left(\Lambda \begin{pmatrix} x \\ y \end{pmatrix}\right) e,$$

where $f \in \mathrm{Hol}(\mathbf{C}, \mathbf{C})$, $e \in \mathbf{C}^2$ and Λ is a linear form on $\mathbf{C}^2$ with $\Lambda e = 0$.

These automorphisms-which were introduced by J-P. Rosay and W. Rudin in [12] (see also [2])–are called *shears* and the group generated by them will be denoted by G_1 (by a slight modification of the notation introduced in [2]).

Let $Aut_1\mathbf{C}^2$ be the group of all holomorphic automorphisms of $\mathbf{C}^2$ whose Jacobian is equal to 1. E. Andersen proved in [2] that G_1 is a proper subgroup of $Aut_1\mathbf{C}^2$and that G_1 is dense in $Aut_1\mathbf{C}^2$ for the topology of uniform convergence on compact sets. Since any polynomial automorphism of $\mathbf{C}^2$ has constant Jacobian, the group G_1 can be seen as a generalization of $Aut_P\mathbf{C}^2$. This is also confirmed by the following Proposition, which is proved by Andersen in [2].

PROPOSITION 1.2. *The special linear group on* $\mathbf{C}^2$*,* $SL(2, \mathbf{C})$*, is contained in* G_1*.*

Let A_1 be the subgroup of all affine automorphisms of $\mathbf{C}^2$ with Jacobian equal to 1. A_1 is the semidirect product between $SL(2, \mathbf{C})$ and the group $\mathbf{C}^2$ of translations with the left action of $SL(2, \mathbf{C})$ on $\mathbf{C}^2$. In particular it is contained in G_1. Let E_1 be the subgroup of G_1 given by

$$E_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha x + f(y) \\ \alpha^{-1} y + \beta \end{pmatrix}, \;\middle|\; \alpha \in \mathbf{C}^*, \beta \in \mathbf{C}, \quad f \in \mathrm{Hol}(\mathbf{C}, \mathbf{C}) \right\}.$$

Since each element of E_1 is the composition of a shear such that $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, with an element in $SL(2, \mathbf{C})$ and a translation, it follows that $E_1 \subset G_1$. The group E_1 is said to be the group of elementary automorphisms of G_1.

The intersection $A_1 \cap E_1$ is given by

$$A_1 \cap E_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha x + ay + b \\ \alpha^{-1} y + \beta \end{pmatrix}, \quad \Big| \quad \alpha \in \mathbf{C}^*, \ \beta, a, b \in \mathbf{C} \right\}.$$

THEOREM 1.3. *G_1 is the free product of A_1 and E_1 amalgamated over $A_1 \cap E_1$.*

As we already said, we postpone the proof of Theorem 1.3 to §2. We endow $Aut_P\mathbf{C}^2$ and G_1 with the compact-open topology.

DEFINITION 1.1. A *one-parameter polynomial group* or a *one-parameter shear group* is a continuous homomorphism of $\mathbf{R}$ into $Aut_P\mathbf{C}^2$ or G_1 respectively.

Using Theorem 1.1 and Theorem 1.3 we shall prove the following

THEOREM 1.4. *A one-parameter polynomial group Φ (or a one-parameter shear group) is conjugated in $Aut_P\mathbf{C}^2$ (or G_1) to a one-parameter group in E or A (or E_1 or A_1), i.e. there exists $X \in Aut_P\mathbf{C}^2$ (G_1) such that for all $t \in \mathbf{R}$, $X \circ \Phi_t \circ X^{-1} \in A$ or E (A_1 or E_1).*

The proof of Theorem 1.4 is postponed to §3. In order to describe the possible conjugates, i.e. the one-parameter groups in A, A_1, E and E_1, we start by considering the one-parameter groups in A and A_1, looking for fixed points, periodic points and the behavior of the one-parameter groups as $t \to +\infty$ or $t \to -\infty$.

DEFINITION 1.2. A *fixed point* for a one-parameter group Φ is a fixed point for Φ_t for all $t \in \mathbf{R}$.

DEFINITION 1.3. A *periodic point* x for a one-parameter group Φ is a point such there exists $t_0 \in \mathbf{R}^*$ so that x is a fixed point for Φ_{t_0} (hence for Φ_{nt_0} for all $n \in \mathbf{Z}$).

DEFINITION 1.4. A *limit point* x for a one-parameter group Φ is a point x such there exists $\lim_{t\to+\infty} \Phi(x)$.

The following proposition collects the results of our investigation for the case of the group A. It can be also found in [3], together with

Proposition 1.7.

Proposition 1.5. *All the one-parameter groups in A are given, up to conjugation in A, by the following expressions*

$$a)\ \Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}e^{tc_1}x\\e^{tc_2}y\end{pmatrix},\ b)\ \Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+ts_1\\e^{tc_2}y\end{pmatrix},$$

$$c)\ \Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}e^{tc_1}x\\e^{tc_1}(tx+y)\end{pmatrix}$$

$$d)\ \Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+ts_1\\tx+y+ts_2+t^2s_1/2\end{pmatrix},\ e)\ \Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+ts_1\\y+ts_2\end{pmatrix},$$

where $c_1,\ c_2,\ s_1, s_2\in\mathbf{C}$.

Proof. Set $\Phi_t\begin{pmatrix}x\\y\end{pmatrix}=R_t\begin{pmatrix}x\\y\end{pmatrix}+S_t$, where $S_t\in\mathbf{C}^2$ and $R_t\in GL(2,\mathbf{C})$ if $\Phi_t\in A$ and $R_t\in SL(2,\mathbf{C})$ if $\Phi_t\in A_1$. The fact that $\Phi_{t+\tau}=\Phi_t\circ\Phi_\tau$ is equivalent to

$$i)\ R_{t+\tau}=R_tR_\tau \quad\text{and}\quad ii)\ S_{t+\tau}=R_tS_\tau+S_t.$$

The first equation implies that R is a one-parameter group in $GL(2,\mathbf{C})$ or in $SL(2,\mathbf{C})$. Hence $R_t=\exp tW$, where W is an element of $M(2,\mathbf{C})$ which, up to conjugation by a suitable element $M\in SL(2,\mathbf{C})$, may be assumed to be

$$W=\begin{pmatrix}c_1&0\\0&c_2\end{pmatrix}\quad\text{or}\quad W=\begin{pmatrix}c_1&0\\1&c_1\end{pmatrix},$$

where $\Phi_t\in A_1$ iff $\mathrm{tr}W=0$. Equation ii) implies that $R_tS_\tau+S_t=R_\tau S_t+S_\tau$.

If there is a t_0 such that $R_{t_0}-I$ is invertible, then we obtain $S_t=(R_t-I)S$, where $S=(R_{t_0}-I)^{-1}S_{t_0}\in\mathbf{C}^2$. Otherwise $c_1c_2=0$, if $\Phi_t\in A$ and $c_1=0$, if $\Phi_t\in A_1$. If $c_1=0$ and $c_2\neq 0$, then a simple computation in *ii*) yields $S_t=\begin{pmatrix}ts_1\\(e^{tc_2}-1)s_2\end{pmatrix}$. If $c_1=0$ and $c_2=0$, then $S_t=\begin{pmatrix}ts_1\\ts_2\end{pmatrix}$, in the case in which we have $R_t=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, or $S_t=\begin{pmatrix}ts_1\\ts_2+t^2s_1/2\end{pmatrix}$, if we have $R_t=\begin{pmatrix}1&0\\t&1\end{pmatrix}$.

We begin by examining the case in which

$\Phi_t\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}e^{tc_1}(x+s_1)-s_1\\e^{tc_2}(y+s_2)-s_2\end{pmatrix}$, where $c_1c_2 \neq 0$; conjugating by a suitable translation we obtain $\Phi_t\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}e^{tc_1}x\\e^{tc_2}y\end{pmatrix}$, which is case a).

Now we examine the case $\Phi_t\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+ts_1\\e^{tc_2}(y+s_2)-s_2\end{pmatrix}$. Again by conjugating by a suitable translation, we obtain $\Phi_t\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+ts_1\\e^{tc_2}y\end{pmatrix}$, which is case b).

We turn to consider $\Phi_t\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}e^{tc_1}(x+s_1)-s_1\\e^{tc_1}(tx+y+s_2)-s_2\end{pmatrix}$: by conjugating by a suitable translation we obtain

$$\Phi_t\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}e^{tc_1}x\\e^{tc_1}(tx+y)\end{pmatrix}$$

that is case c).

We are left with

$$\Phi_t\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+ts_1\\tx+y+ts_2+t^2s_1/2\end{pmatrix}$$

or

$$\Phi_t\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+ts_1\\y+ts_2\end{pmatrix},$$

i.e. cases d) and e). $\diamondsuit$

Now we investigate fixed points, periodic points and limit points in the different cases listed in Proposition 1.5. We begin with case a): if $x \neq 0$ and $y \neq 0$, then $\Phi_t\begin{pmatrix}x\\y\end{pmatrix}$ converges for $t \to +\infty$ iff $\mathrm{Re}c_1 < 0$ and $\mathrm{Re}c_2 < 0$. If $x \neq 0$ and $y = 0$ (or $x = 0$ and $y \neq 0$), then $\Phi_t\begin{pmatrix}x\\y\end{pmatrix}$ has limit for $t \to +\infty$ iff $\mathrm{Re}c_1 < 0$ (or $\mathrm{Re}c_2 < 0$); moreover $\begin{pmatrix}0\\0\end{pmatrix}$ is a fixed point for all Φ_t, hence the only case in which the limit of $\Phi_t(z)$ exists both for $t \to \pm\infty$ is given by $z = \begin{pmatrix}0\\0\end{pmatrix}$, which is a fixed point for the one-parameter group Φ.

If $x \neq 0$ and $y \neq 0$, the periodic points of Φ are given by $\mathrm{Re}c_1 = 0$ and $\mathrm{Re}c_2 = 0$, with c_1 and c_2 linearly dependent over $\mathbf{Q}$, if $x = 0$ (or $y = 0$), the periodic points are given by $\mathrm{Re}c_2 = 0$ (or $\mathrm{Re}c_1 = 0$) respectively.

Gathering all the results established so far, we obtain that for the one-parameter group Φ of case a) the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is always a fixed point, Φ has periodic points iff $\mathrm{Re}c_1 = 0$ or $\mathrm{Re}c_2 = 0$; moreover Φ_t converges for $t \to +\infty$ iff $\mathrm{Re}c_1 < 0$ and $\mathrm{Re}c_2 < 0$.

For case b) a simple computation shows that, if $s_1 \neq 0$, there are neither limit points nor periodic or fixed points; if $s_1 = 0$, $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is always a fixed point and the condition for $\Phi_t \begin{pmatrix} x \\ y \end{pmatrix}$ to converge, as $t \to +\infty$, for all $\begin{pmatrix} x \\ y \end{pmatrix}$, is $\mathrm{Re}c_2 < 0$. The condition for the existence of periodic points (in this case every point in $\mathbf{C}^2$ becomes a periodic one) is $\mathrm{Re}c_2 = 0$.

In that same way we obtain that, for case c), $\Phi_t \begin{pmatrix} x \\ y \end{pmatrix}$ has limit for $t \to +\infty$ iff $\mathrm{Re}c_1 < 0$. Periodic points are given by $x = 0$ and $\mathrm{Re}c_1 = 0$; the unique fixed point for the whole group is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

In case d) there are limit points iff $s_1 = 0$ and $x = -s_2$ and these are the only fixed points. Periodic points never exist. In case e) it is easily seen that, if $s_1 \neq 0$ or $s_2 \neq 0$, there are neither limit nor periodic or fixed points.

Corollary 1.6. *Let Φ be a one-parameter group in A or A_1 such that Φ has a fixed point. Then Φ can be expressed, up to conjugation in A, in one of the following ways:*

$$\Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{tc_1} x \\ e^{tc_2} y \end{pmatrix}, \quad \Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{tc_1} x \\ e^{tc_1}(tx + y) \end{pmatrix},$$

where $c_1, c_2 \in \mathbf{C}$. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is always a fixed points in these three cases.

Proof. In fact the proof comes down to showing that, if there are fixed points in case b), then we can pass to case a), and, if there are fixed points in case d), then we can pass to case c). In fact, as the presence of fixed points in cases b) and d) is equivalent to $s_1 = 0$, if this happens, then case b) reduces to case a) (with $c_1 = 0$), and, by conjugating with a translation $x \mapsto x + s_2$ on the first component we obtain that case d) reduces to case c). $\diamondsuit$

Turning our attention to the one-parameter groups in E and E_1 we consider separately the two subgroups.

PROPOSITION 1.7. *All the one-parameter groups in E are expressed, up to conjugation in $A \cap E$, by*

a) $\Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{tc_1} x + p_t(y) \\ e^{tc_2} y \end{pmatrix}$,

where p_t satisfies $p_{t+\tau}(y) = e^{\tau c_1} p_t(y) + p_\tau(e^{tc_2} y)$

b) $\Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{tc_1} x + p_t(y) \\ y + ts_2 \end{pmatrix}$,

where p_t satisfies $p_{t+\tau}(y) = e^{\tau c_1} p_t(y) + p_\tau(y + ts_2)$.

Proof. Let $\Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_t x + p_t(y) \\ \beta_t y + \gamma_t \end{pmatrix}$ be a one-parameter group in E. Then the condition $\Phi_{t+\tau} = \Phi_t \circ \Phi_\tau$ yields, up to a conjugation with a translation on the second variable, $\alpha_t = e^{tc_1}$, $\beta_t = e^{tc_2}$ and $\gamma_t = (e^{tc_2} - 1)c$, if $c_2 \neq 0$ (in which case by a suitable translation we obtain case a) of Proposition 1.7). Otherwise $\gamma_t = ts_2$ if $c_2 = 0$ (which yields b) of Proposition 1.7). $\diamondsuit$

A direct inspection shows that Φ_t has a limit for $t \to +\infty$ in case a) iff $\mathrm{Re} c_1 < 0$, $\mathrm{Re} c_2 < 0$ and there exists $\lim_{t\to+\infty} p_t(y)$; in case b) iff $\mathrm{Re} c_1 < 0$ and there exists $\lim_{t\to+\infty} p_t(y)$.

This latter limit may exist or may not exist: for instance take $\gamma_t \equiv 0$, $c_1 \neq c_2$, and $p_t(y) = (e^{tc_1} - e^{tc_2})y$: it is easily seen that this gives a one-parameter group in E and that, if $\mathrm{Re} c_1 < 0$ and $\mathrm{Re} c_2 < 0$, then $p_t(y)$ has always limit for $t \to +\infty$. If $\mathrm{Re} c_1 > 0$ and $\mathrm{Re} c_2 < 0$, then $p_t(y)$ has no limit for $t \to +\infty$ or $t \to -\infty$. Notice that in certain cases there are values of t different from 0 such that p_t is identically 0 : in the above example, if $c_1 = c_2 + 2\pi i$ and $t \in \mathbf{Z}$, then $p_t \equiv 0$. Hence it is not true that p_t has always the same degree.

Going back to the general case, periodic points do exist only if $\mathrm{Re}c_2 = 0$, in which case we must solve $e^{tc_1}x + p_t(y) = x$. If we want every point in $\mathbf{C}^2$ to be a periodic point, then we must require $\mathrm{Re}c_1 = 0$ and c_1 and c_2 must be linearly dependent over $\mathbf{Q}$. Moreover p_t must be zero for suitable values of t. If we want periodic points to exist, then it is enough that $(e^{tc_1} - 1)x + p_t(y) = 0$. For fixed t, the last equation defines hypersurfaces in $\mathbf{C}^2$.

We look for the solutions of equations

i) $p_{t+\tau}(y) = e^{\tau c_1}p_t(y) + p_\tau(e^{tc_2}y)$ and

ii) $p_{t+\tau}(y) = e^{\tau c_1}p_t(y) + p_\tau(y + ts_2)$,

where p_t is a polynomial and $t \mapsto p_t$ is a C^1 map.

Notice that by integrating *i)* and *ii)* in τ between 0 and 1 we can prove that the flow depends smoothly on t.

In case *i)* write $p_t(y) = \sum_{n\in\mathbf{N}} a_n(t)y^n$, where, for any fixed t, $a_n(t)$ vanishes when $n >> 0$. Then p satisfies *i)* iff

$$\sum_{n\in\mathbf{N}} a_n(t+\tau)y^n = \sum_{n\in\mathbf{N}} (e^{\tau c_1}a_n(t) + e^{ntc_2}a_n(\tau))y^n \quad \forall y \in \mathbf{C},$$

hence $a_n(t+\tau) = e^{\tau c_1}a_n(t) + e^{ntc_2}a_n(\tau) \ \ \forall n \in \mathbf{N}$.

We subtract $a_n(\tau)$ from both members, divide by t and let t go to 0; then, by recalling that $a_n(0) = 0 \ \forall n \in \mathbf{N}$ because $\Phi_0 = id_{\mathbf{C}}$, we obtain that

$$\frac{da_n}{d\tau}(\tau) = e^{\tau c_1}\frac{da_n}{d\tau}(0) + nc_2a_n(\tau),$$

which gives, together with the condition $a_n(0) = 0$,

$$a_n(t) = \begin{cases} \alpha_n(e^{c_1t} - e^{nc_2t}), & \text{if } c_1 \neq nc_2 \\ \alpha_n te^{c_1t} & \text{if } c_1 = nc_2, \end{cases}$$

where $\alpha_n \in \mathbf{C}$. It is easily seen that these functions give the solutions for *i)*. This proves that there is an upper bound, independent on t, for the degree of p_t.

In case *ii)* write again $p_t(y) = \sum_{n\in\mathbf{N}} a_n(t)y^n$, where, for any fixed t, $a_n(t)$ vanishes when $n >> 0$. Then p satisfies *ii)* iff

$$\sum_{n\in\mathbf{N}} a_n(t+\tau)y^n =$$

$$\sum_{n\in\mathbf{N}}\left(e^{\tau c_1}a_n(t)+\sum_{j\geq n}a_j(\tau)\binom{j}{n}(ts_2)^{j-n}\right)y^n \quad \forall y\in\mathbf{C},$$

whence

$$a_n(t+\tau)=e^{\tau c_1}a_n(t)+\sum_{j\geq n}a_j(\tau)\binom{j}{n}(ts_2)^{j-n} \quad \forall n\in\mathbf{N}.$$

By subtracting $a_n(\tau)$ from both members, dividing by t, letting t go to 0 and recalling that $a_n(0)=0$ $\forall n\in\mathbf{N}$, we obtain

$$\frac{da_n}{d\tau}(\tau)=e^{\tau c_1}\frac{da_n}{d\tau}(0)+(n+1)s_2a_{n+1}(\tau).$$

Choosing a_0, then we can find a_n by a recursive step obtaining that $a_k\in C^\infty$ for all $k\in\mathbf{N}$.

Now we consider one-parameter groups in E_1 :

Proposition 1.8. *All the one-parameter groups in E_1 are expressed (up to conjugation in $E_1\cap A_1$) by*

i) $\Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}e^{ta}x+f_t(y)\\e^{-ta}y\end{pmatrix}$

ii) $\Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+f_t(y)\\y+tb\end{pmatrix}$,

where $a\in\mathbf{C}$ and in case i) f_t satisfies $f_{t+\tau}(y)=e^{\tau a}f_t(y)+f_\tau(e^{-ta}y)$, while in case ii) it satisfies $f_{t+\tau}(y)=f_t(y+\tau b)+f_\tau(y)$.

Proof. The fact that $\Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\alpha_tx+f_t(y)\\\alpha_t^{-1}y+\beta_t\end{pmatrix}$ satisfies the composition rule is equivalent to the fact that $\alpha_t^{-1}y+\beta_t$ is a one parameter group of affine transformations of $\mathbf{C}$, hence it can be conjugated with a translation to obtain $y\mapsto e^{ta}y$ or $y\mapsto y+tb$; the relation on f follows immediately. $\diamondsuit$

In case *i)*, write $f_t(y)=\sum a_n(t)y^n$, where $a_n(0)=0$ for all $n\in\mathbf{N}$. The relation $f_{t+\tau}(y)=e^{\tau a}f_t(y)+f_\tau(e^{-ta}y)$ implies

$$a_n(t+\tau)=e^{\tau a}a_n(t)+e^{-nta}a_n(\tau).$$

If $a=0$, then there exists an entire function g such that $f_t(y)=tg(y)$ for all $t\in\mathbf{R}$ and $y\in\mathbf{C}$. It is easily seen that the point $\begin{pmatrix}x\\y\end{pmatrix}$ is fixed point for Φ iff $g(y) = 0$ and that it is a limit point iff it is a fixed point.

If $a \neq 0$, then $a_n(t+\tau) = e^{\tau a} a_n(t) + e^{-nta} a_n(\tau)$ must be equal to $e^{ta} a_n(\tau) + e^{-n\tau a} a_n(t)$, therefore we obtain that

$$(e^{-nta} - e^{ta}) a_n(\tau) = (e^{-n\tau a} - e^{\tau a}) a_n(t).$$

Since $a \neq 0$, then there exists τ_0 such that $e^{-n\tau_0 a} - e^{\tau_0 a} \neq 0$; hence we obtain that

$$a_n(t) = (e^{-nta} - e^{ta}) a_n(\tau_0)(e^{-n\tau_0 a} - e^{\tau_0 a})^{-1} = c_n(e^{-nta} - e^{ta}),$$

where $c_n \in \mathbf{C}$, and in this way we can recover the function f_t.

If we look for a limit point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ with $y_0 \neq 0$, then we must have $\mathrm{Re}a > 0$ and $x_0 + \sum c_n y_0^n = 0$. If we look for a limit point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ with $y_0 = 0$, then we must have $x_0 + c_0 = 0$.

To study periodic points we have to split up our investigation in two cases: the first case, in which $\mathrm{Re}a \neq 0$, and the second case, in which $\mathrm{Re}a = 0$. If $\mathrm{Re}a \neq 0$ and $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is a periodic point of period t_0, then $y_0 = 0$ and $e^{-t_0 a} x_0 + f_{t_0}(0) = x_0$, that is $e^{-t_0 a} x_0 + c_0(1 - e^{-t_0 a}) = x_0$ and therefore $x_0 = c_0$.

If $\mathrm{Re}a = 0$ and $t_0 a$ is an integer multiple of $2\pi i$, then it is easily seen that $f_{t_0} \equiv 0$, and therefore $\Phi_{t_0} = Id_{\mathbf{C}^2}$.

When we look for a fixed point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, it is easily seen that we must have $x_0 = y_0 = 0$ and $f_t(0) = 0$ for all $t \in \mathbf{R}$, that is $c_0 = 0$.

In case $ii)$, we write again $f_t(x) = \sum_{n \geq 0} a_n(t) x^n$ and we obtain

$$a_n(t+\tau) = a_n(\tau) + \sum_{k \geq n} a_k(t)(\tau b)^{k-n} \begin{pmatrix} k \\ n \end{pmatrix}.$$

We subtract $a_n(\tau)$ from both members, we divide by t and let t go to 0, obtaining

$$\frac{da_n}{d\tau}(\tau) = \sum_{k \geq n} \frac{da_k}{d\tau}(0)(\tau b)^{k-n} \begin{pmatrix} k \\ n \end{pmatrix}.$$

In this way we can recover the form of f. It is easily seen that there is no possibility of having fixed points, periodic points or limit points.

The above considerations indicate that, in the continuous case, there is no chaotic behaviour (such as having many different periods), in sharp contrast with the discrete case of iterates (see [4] and [5]). The behaviour of all one-parameter groups in $Aut_P\mathbf{C}^2$ or in G_1 is clarified by the above models.

In particular the regularity in t is a consequence of the well-known theorem for continuous one-parameter groups on a complex manifold.

THEOREM 1.9. *Let X be a complex domain in $\mathbf{C}^n$ and Φ a one-parameter semigroup (i.e. a continuous homomorphism from $\mathbf{R}^+$ to $Hol(X,X)$). The map $t \mapsto \Phi_t(x)$ is analytic on $\mathbf{R}^+$ for all $x \in X$; moreover there exists a holomorphic map F from X to $\mathbf{C}^n$ such that*

$$\frac{\partial \Phi}{\partial t} = F \circ \Phi.$$

The proof of this theorem can be found, e.g., in [1] (see, p.296).

Fix $x \in \mathbf{C}^2$, let K be a compact neighborhood of x, then, by a corollary to Cauchy's theorem, we can solve the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,z) = F(u(t,z)), \\ u(0,z) = z \quad \text{on U} \end{cases}$$

on $(-a,a) \times U$ (where $a > 0$ and U is a neighborhood of K) with u analytic.

For the uniqueness of the solution we have $\Phi(t,z) = u(t,z)$ if $t \geq 0$ and $z \in K$; moreover, if $t, s, t+s \in (-a,a)$ we have

$$u(s, u(t,z)) = u(t+s, z)$$

on K.

Let $t \in [0,a)$ and $s = -t$, then last equality implies that, $u(-t, u(t,z)) = z$ if $z \in K$: then $u(-t,\cdot)$ is a local inverse of $\Phi(t,\cdot)$, hence $u(-t,z) = \Phi(-t,z)$ if $z \in K_1$ and $t \in [0,a)$ (where K_1 is a suitable neighborhood of x).

Therefore $\Phi(t,x)$ is analytic on $(-a,a)$, adding the fact that it is analytic on the two half-lines we obtain that the dependence is analytic on the whole line and by analytic continuation

$$\frac{\partial \Phi}{\partial t} = F \circ \Phi.$$

2. Group Structures.

In this section we prove Theorem 1.1 and Theorem 1.3, *i.e.*, we prove that $Aut_P\mathbf{C}^2$ is the free product of A and E amalgamated over their intersection $A \cap E$ and that G_1 is the free product of A_1 and E_1 amalgamated over their intersection $A_1 \cap E_1$.

To prove Theorem 1.1 and Theorem 1.3 we first prove that A and E (in the case of Theorem 1.1), and A_1 and E_1 (in the case of Theorem 1.3) generate $Aut_P\mathbf{C}^2$ and G_1, respectively.

THEOREM 2.1. (Jung) *The group $Aut_P\mathbf{C}^2$ of polynomial automorphisms of $\mathbf{C}^2$ is generated by A and E.*

The proof of Theorem 2.1 can be found in [10].

REMARK 2.1. If $g \in E \cap A$, then

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + ay + b \\ \beta y + \gamma \end{pmatrix},$$

where $\alpha, \beta \in \mathbf{C}^*$, $a, b, \gamma \in \mathbf{C}$.

LEMMA 2.2. *A_1 and E_1 generate G_1.*

Proof. In order to prove that E_1 and A_1 generate G_1, it is enough to prove that E_1 and A_1 generate all shears. First of all notice that, if $e \in \mathbf{C}^2 - \{0\}$, then there exists $T \in SL(2, \mathbf{C}) \subset A_1$ such that $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e$; moreover, if

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + f(y) \\ y \end{pmatrix},$$

then

$$T \circ S \circ T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + f\left(\Lambda\begin{pmatrix} x \\ y \end{pmatrix}\right)e,$$

where Λ is a linear form on $\mathbf{C}^2$ with $\Lambda e = 0$. Hence, conjugating $S \in E_1$ by a suitable element $T \in A_1$, we obtain every shear. That proves that G_1 is generated by A_1 and E_1. $\diamondsuit$

These two lemmas give us the possibility to prove a first, very simple result on the structure of $Aut_P\mathbf{C}^2$ and G_1.

LEMMA 2.3. *Both $Aut_P\mathbf{C}^2$ and G_1 are arcwise connected.*

Proof. In fact it is easily seen that both A and A_1 are arcwise connected. Moreover for an element $g \in E$ as in (1.1) it is enough to consider two continuous paths $\alpha, \beta : [0,1] \to \mathbf{C}^*$ such that $\alpha(1) = \beta(1) = 1$ and $\alpha(0) = \alpha$, $\beta(0) = \beta$; then

$$g_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha(t)x + (1-t)p(y) \\ \beta(t)y + (1-t)\gamma \end{pmatrix}$$

is a continuous path in E such that $g_0 = g$ and $g_1 = id_{\mathbf{C}^2}$.

In the same way we can connect any element g in E_1 to the identity map with a continuous path in E_1.

Now, for $h \in Aut_P\mathbf{C}^2$, (or $g \in G_1$) we choose a representation of h as $h = h_n \circ \ldots \circ h_1$ with $h_j \in A \cup E$ (respectively a representation of $g \in G_1$ as $g = g_n \circ \ldots \circ g_1$ with $g_j \in A_1 \cup E_1$) and we take n continuous paths $h_j(t)$ in A or E (or n continuous paths $g_j(t)$ in A_1 or E_1) such that $h_j(0) = h_j$ and $h_j(1) = id_{\mathbf{C}^2}$. In this way we obtain that $h(t) = h_n(t) \circ \ldots \circ h_1(t)$ is a continuous path in $Aut_P\mathbf{C}^2$ which connects h with the identity (and the same for G_1). $\diamondsuit$

We start by proving Theorem 1.1, whose proof is much simpler than the proof of Theorem 1.3, due to the fact that the degree induces a partial ordering on polynomials.

DEFINITION 2.1. A sequence $(g_n, \ldots, g_1)$ of length $n \geq 1$ is called a *reduced word* with respect to the subgroups A and E if, for each $i = 1, \ldots, n$, $g_i \in (A \cup E)/(A \cap E)$ and g_i, g_{i+1} do not belong both to the same of the two subgroups.

Let $g = g_n \circ \ldots \circ g_2 \circ g_1 \in Aut_P\mathbf{C}^2$, where $(g_n, \ldots, g_1)$ is a reduced word; we shall prove that this representation is "unique" up to products in $A \cap E$. For this we need the following

THEOREM 2.4. *If $g = g_n \circ \ldots \circ g_2 \circ g_1$ and $(g_n, \ldots, g_1)$ is a reduced word, then $g \neq id_{\mathbf{C}^2}$.*

DEFINITION 2.2. For $h \in Aut_P\mathbf{C}^2$, we define the *degree* of h to be the maximum between the degrees of its two scalar components.

The following theorem implies Theorem 2.4 and therefore Theorem 1.1.

THEOREM 2.5. *If $g = g_n \circ \ldots \circ g_2 \circ g_1$ and $(g_n, \ldots, g_1)$ is a reduced word, then the degree of g is the product of the degrees of its factors.*

Proof. Since for $n = 1$ the statement is trivial, we proceed by induction on n.

Let w_1 and w_2 be the two scalar components of $g_k \circ \ldots \circ g_2 \circ g_1$, where g_k is an element in E : by the induction step we can suppose that $\deg g_k \cdot \ldots \cdot \deg g_1 = \deg w_1 > \deg w_2$ (the relation $\deg w_1 > \deg w_2$ is a part of the inductive step).

As $g_{k+1} \in A \setminus E$, then $g_{k+1} \circ g_k \circ \ldots \circ g_2 \circ g_1$ has the same degree as $g_k \circ \ldots \circ g_2 \circ g_1$. Moreover if u_1 and u_2 are the two scalar components of $g_{k+1} \circ g_k \ldots \circ g_2 \circ g_1$, then $\deg u_2 \geq \deg u_1$.

Now we consider $g_{k+2} \in E \setminus A$. If v_1 and v_2 are the scalar components of $g_{k+2} \circ g_{k+1} \circ g_k \circ \ldots \circ g_2 \circ g_1$, then $\deg v_1 = \deg(\alpha u_1 + p(u_2))$, where $\alpha \in \mathbf{C}^*$ and p is a polynomial of degree > 1. Hence

$$\deg v_1 = \deg p \cdot \deg u_2 = \deg g_{k+2} \cdot \deg g_{k+1} \ldots \deg g_1 > \deg v_2 =$$

$$\deg g_{k+1} \ldots \deg g_1,$$

and that completes the proof. ◊

COROLLARY 2.6. *If $(g_n, \ldots, g_1)$ is a reduced word, then its length is an invariant of the element $g = g_n \circ \ldots \circ g_1 \in Aut_P \mathbf{C}^2$. Moreover the representation $g = g_n \circ \ldots \circ g_1$ is unique up to replacing g_k by hg_k and g_{k+1} by $g_{k+1}h^{-1}$, for some $h \in A \cap E$.*

In the same way, to clarify the structure of G_1, we prove now that G_1 is the free product of A_1 and E_1 amalgamated over the intersection $E_1 \cap A_1$.

First of all we introduce the notion of rosary, due to Andersen (see [2]).

For any $r \geq 1$ set $\mathcal{H}_0(\mathbf{C}^r) = \{f \in \mathrm{Hol}(\mathbf{C}^r, \mathbf{C}) \ : \ f(0) = 0\}$.

DEFINITION 2.3. A *rosary* is a sequence $L = \{L_1, \ldots, L_n\}$ of linear subspaces of $\mathcal{H}_0(\mathbf{C}^2)$ such that

i) $\dim L_i = 2$,

ii) $\dim L_i \cap L_{i+1} \geq 1$,

iii) there are $u, v \in L_i$ such that (u, v) has Jacobian equal to 1 on $\mathbf{C}^2$.

DEFINITION 2.4. A rosary is said to be *non-tautological* if $L_i \cap L_{i+2} = \{0\}$ for $i = 1, \ldots, n-2$ and $L_i \neq L_{i+1}$ for $i = 1, \ldots, n-1$.

Notice that the second condition is necessary only if $n = 2$, because, if $n > 2$ and $L_i = L_{i+1}$ for some $i = 1, \ldots, n-1$, then dim $L_i \cap L_{i+2} \geq 1$ if $i < n-1$ or dim $L_{i-1} \cap L_{i+1} \geq 1$ if $i = n-1$.

We denote by $\langle v_1, \ldots, v_j \rangle$ the complex vector space spanned by the vectors $v_1, \ldots, v_j$.

DEFINITION 2.5. A sequence $\mathcal{U} = \{u_0, u_1, \ldots, u_n\} \subset \mathcal{H}_0(\mathbf{C}^2)$ is called a *basis* for the rosary L if

1) $L_i = \langle u_{i-1}, u_i \rangle$ and
2) $u_{i+1} = f_{i+1}(u_i) + u_{i-1}$, where $f_{i+1} \in \mathcal{H}_0(\mathbf{C})$.

Notice that, if f_{i+1} is linear, then $L_i = L_{i+1}$, and therefore the rosary L is tautological.

LEMMA 2.7. *Let $g_1, \ldots, g_n$ be a sequence of shears such that $g_j \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all $j = 1, \ldots, n$, and set*

$$L_1 = \langle x, y \rangle, \quad L_2 = \langle g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_1, g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_2 \rangle, \ldots,$$

$$L_i = \langle g_i \circ \ldots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_1, g_i \circ \ldots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_2 \rangle,$$

where the dot indicates the canonical hermitian product in $\mathbf{C}^2$ and $\{e_1, e_2\}$ is the canonical basis in $\mathbf{C}^2$. Then $L = \{L_1, \ldots, L_n\}$ is a rosary.

Proof. Conditions i) and iii) of Definition 2.3 are trivial.

If, with the same notation as before,

$$g_{j+1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + f\left(\Lambda \begin{pmatrix} x \\ y \end{pmatrix}\right) e,$$

then taking e and another suitable vector ε as a basis of $\mathbf{C}^2$ we can find a basis u, v of L_j such that $g_{j+1}(u, v) = u\varepsilon + (v + f(u))e$. Thus L_{j+1} is spanned by u and $v + f(u)$, which yields ii). $\diamondsuit$

The proof of the following proposition essentially follows an argument given in [2] for a more restrictive case.

PROPOSITION 2.8. *A non tautological rosary has a basis.*

Proof. We proceed by induction on the length n of the rosary, the case $n = 1$ being trivial.

We first consider $n = 2$, to clarify notations. Let $L = \{L_1, L_2\}$ be a rosary. We have, by definition, $L_1 = \langle u_0, u_1 \rangle$ and $L_2 = \langle u_1, v \rangle$, for some u_0, u_1, v. By property iii) of Definition 2.3, $J(u_0, u_1) = cJ(v, u_1)$, for some $c \in \mathbf{C}^*$, where J denotes the Jacobian. Hence $J(u_0 - cv, u_1) = 0$, which gives $cv = f(u_1) + u_0$, where f is an entire function. The choice $u_2 = cv$ completes the proof for $n = 2$.

By induction we can suppose we have found $u_1, \ldots, u_k$ with $L_k = \langle u_k, u_{k-1} \rangle$, $L_{k-1} = \langle u_{k-1}, u_{k-2} \rangle$ and $u_k = f_k(u_{k-1}) + u_{k-2}$. By definition of rosary there is $0 \neq v \in L_{k+1} \cap L_k$, $v = au_k + bu_{k-1}$. As L is non- tautological, $a \neq 0$. Hence we can replace u_k by $u_k' = u_k + a^{-1}bu_{k-1} = a^{-1}v \in L_k \cap L_{k+1}$ and then we get u_{k+1} with the same procedure as above. ♢

REMARK 2.2. In the choice of a basis of a non-tautological rosary $u_{k+1} = f_{k+1}(u_k) + u_{k-1}$, hence f_{k+1} cannot be linear.

We introduce an ordering on $\mathcal{H}_0(\mathbf{C})$ which will be useful in the following. This ordering is provided by Nevanlinna's value distribution theory; as our use of this tool is almost incidental we refer to [2] for a more exhaustive treatment of the subject and further references.

Let $f \in \mathcal{H}_0(\mathbf{C})$ and set

$$m(f, r) = \frac{1}{2\pi} \int_{S^1} \ln^+ |f(r\xi)| \, |d\xi|,$$

where S^1 is the unit circle in $\mathbf{C}$ oriented counterclockwise, and $|d\xi|$ is the standard Lebesgue measure on S^1.

The following proposition is obtained gathering Lemma 3.2.1 and Corollary 3.2.3 in [2], together with the trivial remark that, if $f = id_{\mathbf{C}}$, then $m(f, r) = \ln^+ r$. If $f = id_{\mathbf{C}}$ we denote $m(f, r)$ by $m(z, r)$.

PROPOSITION 2.9. *Let $f \in \mathcal{H}_0(\mathbf{C})$.*

If f is a polynomial of degree $d \geq 1$, then $\limsup_{r\to+\infty} \frac{m(f,r)}{m(z,r)} = d$ *;*

if f is a trascendental function, then $\limsup_{r\to+\infty} \frac{m(f,r)}{m(z,r)} = +\infty$.

Definition 2.5. If u and v are entire functions on $\mathbf{C}$ we say that

$$u \succ v \quad \text{if} \quad \limsup_{r\to+\infty} \frac{m(u,r)}{m(v,r)} > 1.$$

Then we have the following proposition which has been established in [2]; see Theorem 3.3.1 of [2], where it was stated in a slightly more restrictive form than in Proposition 2.10. The proof given by Andersen extends almost *verbatim* to our more general setting.

Proposition 2.10. *Let $p, q \in \mathcal{H}_0(\mathbf{C})$, let $u_0, \ldots, u_n$ be a basis of a non tautological rosary and let $u_j(s) = u_j(p(s), q(s))$. Then $u_j(s) \in \mathcal{H}_0(\mathbf{C})$, and, if $u_2(s)$ is non-zero and $u_2(s) \succ u_1(s)$, then $u_{k+1}(s) \succ u_k(s)$ for all $k \geq 1$.*

Our proof of Theorem 1.3 makes use of the ordering $\succ$ on $\mathcal{H}_0(\mathbf{C})$ introduced above to show that, if $(g_n, \ldots, g_1)$ is a reduced word in G_1, then $g_n \circ \ldots \circ g_1 \neq id_{\mathbf{C}^2}$. In fact, given a rosary L, we show that we can find a suitable basis $u_1, \ldots, u_k$ which is naturally ordered by $\succ$. This will imply that, given any reduced word $(g_n, \ldots, g_1)$ in G_1, then $g_n \circ \ldots \circ g_1 \neq id_{\mathbf{C}^2}$.

Take $g \notin A_1 \cap E_1$, and let $(g_n, \ldots, g_1)$ be a reduced word of length n with respect to the subgroups A_1 and E_1 such that $g = g_n \circ \ldots \circ g_1$. We prove that the length n is an invariant of the element $g \in G_1$. As in the case of $g \in Aut_P\mathbf{C}^2$, all we need follows from

Theorem 2.11. *If $g = g_n \circ \ldots \circ g_1$, where $(g_n, \ldots, g_1)$ is a reduced word, then $g \neq id_{\mathbf{C}^2}$.*

If $g = g_n \circ \ldots \circ g_1 = id_{\mathbf{C}^2}$, then the two scalar components of $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto g\begin{pmatrix} x \\ y \end{pmatrix}$ are both linear in x and y; moreover they both belong to $\mathcal{H}_0(\mathbf{C}^2)$.

The first step in the proof of Theorem 2.11 consists in showing that we can replace $g = g_n \circ \ldots \circ g_1$ by $g = \tilde{g}_n \circ \ldots \circ \tilde{g}_1$, where

$\tilde{g}_k\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, g_k and $\tilde{g}_k$ are in the same subgroup, and $\tilde{g}_k \notin A \cap E$, for $k = 1, \ldots, n$.

Proposition 2.12. *Suppose $(g_n, \ldots, g_1)$ is a reduced word in G_1 with—letting as before $g = g_n \circ \ldots \circ g_1$— $g\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$. We can find a representation of g as a reduced word in G_1, $g = \tilde{g}_n \circ \ldots \circ \tilde{g}_1$, where $\tilde{g}_k\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, for all $k = 1, \ldots, n$.*

Proof. Set $B_1 = g_1\begin{pmatrix}0\\0\end{pmatrix}$ and $\tilde{g}_1(z) = g_1(z) - B_1$, that is $\tilde{g}_1 = r_1 \circ g_1$, where r_1 is the translation of vector $-B_1$.

Set again $B_{k+1} = g_{k+1}(B_k)$ and $\tilde{g}_{k+1}(z) = g_{k+1}(z + B_k) - B_{k+1}$. Hence $\tilde{g}_{k+1} = r_{k+1} \circ g_{k+1} \circ r_k^{-1}$, where r_k is the translation of vector $-B_k$. Then $\tilde{g}_k\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, and, as the translations are contained in $A_1 \cap E_1$, $\tilde{g}_k$ is contained in the same subgroup as g_k (i.e. $g_k \in A_1$ iff $\tilde{g}_k \in A_1$ and similarly $g_k \in E_1$ iff $\tilde{g}_k \in E_1$); moreover $\tilde{g}_k \notin A_1 \cap E_1$.

Since $B_n = \begin{pmatrix}0\\0\end{pmatrix}$, then $g_n \circ \ldots \circ g_1 = \tilde{g}_n \circ \ldots \circ \tilde{g}_1$, and we are done. $\diamondsuit$

By Proposition 2.12 we can suppose that, if $g = g_n \circ \ldots \circ g_1 = id_{\mathbf{C}^2}$, then $g_k\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, for $k = 1, \ldots, n$. Now we can come to the proof of Theorem 2.11.

Proof. (of Theorem 2.11) By Proposition 2.12 there is no restriction in assuming that each g_k maps $\begin{pmatrix}0\\0\end{pmatrix}$ to $\begin{pmatrix}0\\0\end{pmatrix}$. Since the case $n = 1$ is obvious, we can suppose $n \geq 2$ and proceed by induction on n.

If g_n and g_1 are both in A_1 or both in E_1, then we can replace $g_n \circ \ldots \circ g_2 \circ g_1$ by $g_1 \circ g_n \circ \ldots \circ g_2$: this is still equal to the identity, but, as a reduced word, has length $n - 1$. So we can suppose that n is even, in fact we have seen that we can suppose that g_n and g_1 do not belong to the same subgroup; hence, as each g_j does not belong to the same subgroup of g_{j+1}, we can suppose that n is even.

If $(g_n,\dots,g_1)$ is a reduced word such that $g_n\circ\dots\circ g_1=id_{\mathbf{C}^2}$, we can suppose that $g_1\in E_1$. In fact if $g_1\in A_1$ we have $g_2\in E_1$ and $g_1\circ g_n\circ\dots\circ g_2$ is still equal to the identity map. If $n=2$ we get $g_1=g_2^{-1}$, whence $g_1\in A_1\cap E_1$, which is a contradiction. Thus we can suppose $n\geq 4$.

Now we prove that $L_1,\dots,L_m$ given by

$$L_1=\langle x,y\rangle,\ L_2=\langle g_1\begin{pmatrix}x\\ y\end{pmatrix}\cdot e_1, g_1\begin{pmatrix}x\\ y\end{pmatrix}\cdot e_2\rangle,$$

$$L_3=\langle g_3\circ g_2\circ g_1\begin{pmatrix}x\\ y\end{pmatrix}\cdot e_1, g_3\circ g_2\circ g_1\begin{pmatrix}x\\ y\end{pmatrix}\cdot e_2\rangle,\dots,$$

$$L_m=\langle g_{n-1}\circ\dots\circ g_1\begin{pmatrix}x\\ y\end{pmatrix}\cdot e_1, g_{n-1}\circ\dots\circ g_1\begin{pmatrix}x\\ y\end{pmatrix}\cdot e_2\rangle,$$

where $m=n/2+1$, is a non tautological rosary.

First of all $L=\{L_1,L_2,\dots,L_m\}$ is a rosary because, if we set

$$M_1=\langle x,y\rangle,\dots,$$

$$M_{j+1}=\langle g_j\circ\dots\circ g_2\circ g_1\begin{pmatrix}x\\ y\end{pmatrix}\cdot e_1, g_j\circ\dots\circ g_2\circ g_1\begin{pmatrix}x\\ y\end{pmatrix}\cdot e_2\rangle,\dots,$$

then $M=\{M_1,M_2,\dots,M_n\}$ is a rosary by Lemma 2.7 and so, $L_1=M_1$, $L_2=M_2=M_3$, $L_3=M_4=M_5$, $\dots$, is a rosary.

We now prove that L is a non tautological rosary. Note first that $L_1\neq L_2$ because $L_2=\langle x+f_1(y),y\rangle$, where f_1 is non-linear (because $g_1\in E_1\setminus A_1$ and $f_1(0)=0$).

Suppose that $L_1\cap L_3\neq\{0\}$, and write

$$g_1\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\alpha_1x+f_1(y)\\ \alpha_1^{-1}y\end{pmatrix},\quad g_2\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}a_1x+b_1y\\ c_1x+d_1y\end{pmatrix}\quad\text{and}$$

$$g_3\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\alpha_3x+f_3(y)\\ \alpha_3^{-1}y\end{pmatrix},$$

where f_1 and f_3 are on-linear elements of $\mathcal{H}_0(\mathbf{C})$ and $c_1\neq 0$. Then

$$g_3\circ g_2\circ g_1\begin{pmatrix}x\\ y\end{pmatrix}=$$

$$\begin{pmatrix}\alpha_3(a_1(\alpha_1x+f_1(y))+b\alpha_1^{-1}y)+f_3(c_1(\alpha_1x+f_1(y))+d\alpha_1^{-1}y)\\ \alpha_3^{-1}(c_1(\alpha_1x+f_1(y))+d_1\alpha_1^{-1}y)\end{pmatrix}.$$

If $\langle x, y\rangle \cap \langle g_3 \circ g_2 \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_1, g_3 \circ g_2 \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_2\rangle \neq \{0\}$ there exist $\gamma, \delta \in \mathbf{C}$ such that $|\gamma| + |\delta| > 0$ and

$$\gamma[\alpha_3(a_1(\alpha_1 x + f_1(y)) + b\alpha_1^{-1} y) + f_3(c_1(\alpha_1 x + f_1(y)) + d\alpha_1^{-1} y)] +$$
$$\delta[\alpha_3^{-1}(c_1(\alpha_1 x + f_1(y)) + d_1\alpha_1^{-1} y)]$$

is linear in x and y. Then $\gamma(\alpha_3 f_1(y) + f_3(c_1(\alpha_1 x + f_1(y)) + d_1\alpha_1^{-1} y)) + \delta c_1\alpha_3^{-1} f_1(y)$ is linear in x and y, and therefore taking the derivative with respect of x we find that $\alpha_1\gamma c_1 f_3'(c_1(\alpha_1 x + f_1(y)) + d_1\alpha_1^{-1} y)$ is constant. As f_3 is non-linear and $c_1(\alpha_1 x + f_1(y)) + d_1\alpha_1^{-1} y$ is non-constant, then $\alpha_1\gamma c_1 = 0$, and since $\alpha_1 c_1 \neq 0$, we obtain that $\gamma = 0$. Thus $\delta c_1 f_1(y)$ is linear in x and y in contrast with the fact that $\delta c_1 \neq 0$ and f_1 is non-linear.

Suppose now that $L_k \cap L_{k+2} \neq \{0\}$, with $k > 1$ and set $w_j = g_{2k-2} \circ \ldots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_j$, $j = 1, 2$. Then

$$L_k = \langle g_{2k-3} \circ \ldots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_1, g_{2k-3} \circ \ldots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_2\rangle$$
$$= \langle g_{2k-2} \circ \ldots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_1, g_{2k-2} \circ \ldots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_2\rangle = \langle w_1, w_2\rangle,$$

because g_{2k-2} is linear. Moreover

$$L_{k+2} = \langle g_{2k+1} \circ \ldots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_1, g_{2k+1} \circ \ldots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_2\rangle$$
$$= \langle g_{2k+1} \circ g_{2k} \circ g_{2k-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \cdot e_1, g_{2k+1} \circ g_{2k} \circ g_{2k-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \cdot e_2\rangle.$$

If $L_k \cap L_{k+2} \neq \{0\}$, then there are $\gamma, \delta \in \mathbf{C}$ such that $|\gamma| + |\delta| > 0$ and the holomorphic function

$$\gamma g_{2k+1} g_{2k} g_{2k-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \cdot e_1 + \delta g_{2k+1} \circ g_{2k} \circ g_{2k-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \cdot e_2$$

is linear in w_1 and w_2. Hence, arguing as in the case $k = 1$, we get a contradiction, proving that L is a non tautological rosary.

We now choose a basis $\{u_0, u_1, \ldots, u_m\}$ of the rosary L. As $L_1 = \langle x, y\rangle$, $L_2 = \langle x + f_1(y), y\rangle$, and so on, then we have $u_0 = x$, $u_1 = y$ and $u_2 = \mu(x + f_1(y)) + \nu y$, where $\mu \in \mathbf{C}^*$. Choosing $p(s) = q(s) = s$, and, using the notations of Proposition 2.10, we obtain that $u_0(s) = u_1(s) = s$, $u_2(s) = \mu f_1(s) + (\mu+\nu)s$, where $\mu \neq 0$. It is easily seen that $u_2(s)$ is non-zero. To apply Proposition 2.10 we only need to prove that $u_2(s) \succ u_1(s)$.

For this goal note that, if f_1 is a polynomial of degree $d > 1$ and $\mu \neq 0$, then $\mu f_1(s) + (\mu+\nu)s$ is still a polynomial of degree d, hence, by Proposition 2.9,

$$\limsup_{r\to+\infty} \frac{m(\mu f_1(s) + (\mu+\nu)s, r)}{m(s,r)} = d > 1;$$

whereas, if f_1 is trascendental, we have

$$\limsup_{r\to+\infty} \frac{m(\mu f_1(s) + (\mu+\nu)s, r)}{m(s,r)} = +\infty,$$

again by Proposition 2.9. Hence we can apply Proposition 2.10, obtaining $u_m(s) \succ u_{m-1}(s) \succ \ldots u_2(s) \succ u_1(s) = s$. As $m = n/2 + 1 \geq 3$, neither $u_m(s)$ nor $u_{m-1}(s)$ are linear in s.

If $g = g_n \circ \ldots \circ g_1$ were the identity map, then the components of $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto g\begin{pmatrix} x \\ y \end{pmatrix}$ were x and y. Thus L_m would be equal to $\langle u_m, u_{m-1} \rangle$, with $u_m(s)$ and $u_{m-1}(s)$ non-linear in s, whereas, replacing x by $p(s) = s$ and y by $q(s) = s$, we obtain two functions which are both linear in s. This contradiction shows that $g_n \circ \ldots \circ g_1$ is not the identity map and completes the proof of Theorem 2.11. $\diamondsuit$

Then we obtain, as a trivial consequence, the following

Corollary 2.13. *If $(g_n, \ldots, g_1)$ is a reduced word, then its length is an invariant of the element $g = g_n \ldots g_1 \in G_1$. Moreover the representation $g = g_n \circ \ldots \circ g_1$ is unique up to replacing g_k by hg_k and g_{k+1} by $g_{k+1}h^{-1}$, for some $h \in A_1 \cap E_1$.*

Theorem 1.1 and Theorem 1.3 are the keys of the forthcoming section, which contains the proof of the conjugacy theorem for the one-parameter groups in $Aut_P\mathbf{C}^2$ and G_1.

3. Proof of the Conjugacy Theorem.

This section contains the proof of Theorem 1.4. As the proof is equal in the case of the two groups $Aut_P\mathbf{C}^2$ and G_1, we introduce the following notations:

$$\mathcal{G} = Aut_P\mathbf{C}^2 \text{ (respectively } G_1),$$

$$\mathcal{E} = E \text{ (respectively } E_1),$$

$$\mathcal{A} = A \text{ (respectively } A_1).$$

Let $g \in \mathcal{G} \setminus (\mathcal{A} \cap \mathcal{E})$ and let $(g_n, \dots, g_1)$ be its representation as a reduced word in $\mathcal{G}$ (this representation is almost "unique", in the sense specified in Corollary 2.6 and Corollary 2.13).

Obviously $g_k \circ g_{k-1} \circ \dots \circ g_1 \circ g_n \circ \dots \circ g_{k+1}$ is conjugated to g in $\mathcal{G}$. Hence, if g_n and g_1 both belong to either $\mathcal{A}$ or $\mathcal{E}$, in the conjugacy class of g there is an element which has a representation as a reduced word whose length is strictly less than the length of g.

At this point only two cases can occur: either the element $\hat{g}$ of minimal length in the conjugacy class of g is of length 1, i.e., $g = X \circ h_1 \circ X^{-1}$, where $X \in \mathcal{G}$ and $h_1 \in (\mathcal{A} \cup \mathcal{E}) \setminus (\mathcal{A} \cap \mathcal{E})$, or $\hat{g}$ has a representation as a reduced word of even length and the first and last elements of this word do not both belong to the same of the two groups $\mathcal{A}$ or $\mathcal{E}$.

Our proof of Theorem 1.4 relies on the following estimate of the length of the powers of a word.

PROPOSITION 3.1. *If $g \in \mathcal{G} \setminus (\mathcal{A} \cap \mathcal{E})$ is conjugated in $\mathcal{G}$ to an element of minimal length $2r$, then the length of g^m is at least $2rm$.*

Proof. As g is conjugated in $\mathcal{G}$ to an element of minimal length $2r$, then $g = X \circ h_{2r} \circ \dots \circ h_1 \circ X^{-1}$, where $\hat{g} = h_{2r} \circ \dots \circ h_1$, $(h_{2r}, \dots, h_1)$ is a reduced word, $X \in \mathcal{G}$ and h_1 and h_{2r} do not both belong to the same of the two groups $\mathcal{A}$ or $\mathcal{E}$. Hence

$$g^m = X \circ (h_{2r} \circ \dots \circ h_1)^m \circ X^{-1}. \tag{3.1}$$

We remark that, if $X \in \mathcal{A} \cap \mathcal{E}$, then (3.1) implies that the length of g^m is equal to $2rm$. If $X \notin \mathcal{A} \cap \mathcal{E}$, let $X = \xi_1 \circ \dots \circ \xi_j$, where $(\xi_1, \dots, \xi_j)$ is a reduced word. We proceed by induction on j.

If $j = 1$, then $g = \xi_1 \circ h_{2r} \circ \ldots \circ h_1 \circ \xi_1^{-1}$ and, as h_{2r} and h_1 do not belong to the same of the two subgroup $\mathcal{A}$ and $\mathcal{E}$, this is not a representation as a reduced word. Suppose that h_{2r} and ξ_1 belong to the same subgroup (the case in which h_{2r} is replaced by h_1 can be dealt with exactly in the same way). If $\xi_1 \circ h_{2r} \notin \mathcal{A} \cap \mathcal{E}$, then $((\xi_1 \circ h_{2r}), h_{2r-1}, \ldots, h_1, \xi_1^{-1})$ is a reduced word and thus

$$g^m = (\xi_1 \circ h_{2r}) \circ h_{2r-1} \circ \ldots \circ h_1 \circ h_{2r} \circ \ldots \circ h_1 \circ \ldots \circ h_{2r} \circ \ldots \circ h_1 \circ \xi_1^{-1}.$$

Hence a representation of g^m as a reduced word is obviously given by

$$((\xi_1 \circ h_{2r}), h_{2r-1}, \ldots, h_1, h_{2r}, \ldots, h_1, \ldots, h_{2r}, \ldots, h_1, \xi_1^{-1})$$

and the length of g^m is bigger than $2rm$. If $\xi_1 \circ h_{2r} \in \mathcal{A} \cap \mathcal{E}$ we set $c = \xi_1 \circ h_{2r}$, $\tilde{h}_{2r-1} = c \circ h_{2r-1}$ and $\tilde{h}_{2r} = h_{2r} \circ c^{-1}$. Then $\tilde{h}_{2r-1}$ and $\tilde{h}_{2r}$ are not in the same of the two subgroups $\mathcal{A}$ and $\mathcal{E}$ (because h_{2r-1} and h_{2r} are not). Then we have $g = \tilde{h}_{2r-1} \circ h_{2r-2} \circ \ldots \circ h_1 \circ \tilde{h}_{2r}$, where $(\tilde{h}_{2r-1}, h_{2r-2}, \ldots, h_1, \tilde{h}_{2r})$ is a reduced word; then, using the above remark, we obtain that the length of g^m is $2rm$.

Proceeding by induction on j, we can suppose that the statement is true for $j-1$ and consider $X = \xi_j \circ \ldots \xi_1$, where $(\xi_j, \ldots, \xi_1)$ is a reduced word.

Then $g = \xi_1 \circ \ldots \circ \xi_j \circ h_{2r} \circ \ldots \circ h_1 \circ \xi_j^{-1} \circ \ldots \circ \xi_1^{-1}$ and, as h_{2r} and h_1 do not belong to the same of the two subgroups $\mathcal{A}$ and $\mathcal{E}$, this is not a representation as a reduced word. Suppose that h_{2r} and ξ_j belong to the same subgroup (the case in which h_{2r} is replaced by h_1 can be dealt with exactly in the same way). If $\xi_j \circ h_{2r} \notin \mathcal{A} \cap \mathcal{E}$, then

$$(\xi_1, \ldots, \xi_{j-1}, (\xi_j \circ h_{2r}), h_{2r-1}, \ldots, h_1, \xi_j^{-1}, \ldots, \xi_1^{-1})$$

is a reduced word and we obtain

$$g^m = \xi_1 \circ \ldots \circ (\xi_j \circ h_{2r}) \circ h_{2r-1} \circ \ldots \circ h_1 \circ \ldots \circ h_{2r} \circ \ldots \circ h_1 \circ \xi_j^{-1} \circ \ldots \circ \xi_1^{-1}.$$

Thus a representation of g^m as a reduced word is given by

$$(\xi_1, \ldots, \xi_{j-1}, (\xi_j \circ h_{2r}), h_{2r-1}, \ldots, h_1, \ldots h_{2r}, \ldots, h_1, \xi_j^{-1}, \ldots, \xi_1^{-1}),$$

showing that the length of g^m is $2rm + 2j - 1 \geq 2mr$.

If $\xi_j \circ h_{2r} \in \mathcal{A} \cap \mathcal{E}$, setting $c = \xi_j \circ h_{2r}$, we obtain $g = \xi_1 \circ \ldots \circ \xi_{j-1} \circ c \circ h_{2r-1} \circ \ldots \circ h_1 \circ h_{2r} \circ c^{-1} \circ \xi_{j-1}^{-1} \circ \ldots \circ \xi_1^{-1}$. If we define $\tilde{h}_{2r-1} = c \circ h_{2r-1}$ and $\tilde{h}_{2r} = h_{2r} \circ c^{-1}$, then $\tilde{h}_{2r-1}$ and $\tilde{h}_{2r}$ are still in different subgroups and they both lie outside $\mathcal{A} \cap \mathcal{E}$. Hence $(\tilde{h}_{2r-1}, h_{2r-2}, \ldots, h_1, \tilde{h}_{2r})$ is a reduced word and we have found a representation of g

$$g = Y \circ \tilde{h}_{2r-1} \circ h_{2r-2} \circ \ldots \circ h_1 \circ \tilde{h}_{2r} \circ Y^{-1},$$

with $Y = \xi_1 \circ \ldots \circ \xi_{j-1}$. Then we can proceed by induction on j to obtain that g^m has length greater or equal than $2rm$. $\diamondsuit$

At last we come to the proof of Theorem 1.4, which will be given in the case of a one-parameter group $\Phi : \mathbf{R} \to \mathcal{G}$.

First of all we prove that $\mathcal{A}$ and $\mathcal{E}$ are closed in $\mathcal{G}$: in fact, let $\varphi_n \to \varphi$ in $G_1(Aut_P\mathbf{C}^2)$ and suppose that $\varphi_n \in E_1(E)$, then the first component of φ_n is affine in x and the second component of φ_n does not depend on x and is affine in y, and therefore also the first component of φ is affine in x (it depends on x because φ is a biholomorphism of $\mathbf{C}^2$) and the second component of φ does not depend on x and is affine in y (it depends on y because φ is a biholomorphism of $\mathbf{C}^2$). Moreover the Jacobian of φ is equal to the limit of the Jacobian of φ, therefore also E_1 is closed in G_1. If $\varphi_n \to \varphi$ in G_1 $(Aut_P\mathbf{C}^2)$ and φ_n is affine, then φ is affine too and the same reasoning on Jacobians implies that A is closed in $Aut_P\mathbf{C}^2$ and A_1 is closed in G_1. Therefore also $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$ is closed in $\mathcal{G}$.

Next we prove that, for any $t \in \mathbf{R}$, Φ_t is conjugated to an element in $\mathcal{A}$ or $\mathcal{E}$.

If $\Phi_t \in \mathcal{A} \cap \mathcal{E}$, this is obvious. If $\Phi_t \notin \mathcal{A} \cap \mathcal{E}$, let l be the length of Φ_t and choose $m_0 \in \mathbf{N}$ such that $l < 2m_0$. Consider Φ_{t/m_0}: if this were not conjugated to an element in $\mathcal{A}$ or $\mathcal{E}$, then it should be conjugated to an element of minimal length $2r$, therefore Proposition 2.2 implies that the length of $\Phi_t = (\Phi_{t/m_0})^{m_0}$ is greater than or equal to $2rm_0 \geq 2m_0 > l$, that is a contradiction. Then Φ_{t/m_0} is conjugated to an element of $\mathcal{A}$ or $\mathcal{E}$ and hence $\Phi_t = (\Phi_{t/m_0})^{m_0}$ is conjugated to an element of $\mathcal{A}$ or $\mathcal{E}$.

We recall that we proved that $\mathcal{G}$ is the free product of $\mathcal{A}$ and $\mathcal{E}$, amalgamated over $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$. We call in the following theorem, due to Moldavanski (see [13], Theorem 0.3).

THEOREM 3.2. *Let H be an abelian subgroup of $\mathcal{G}$ where $\mathcal{G}$ is* *the free product of $\mathcal{A}$ and $\mathcal{E}$ amalgamated over their intersection $\mathcal{B}$. Precisely one of the following situation holds:*

1) *H is conjugated in $\mathcal{G}$ to a subgroup of $\mathcal{A}$ or $\mathcal{E}$,*
2) *H is not conjugated in $\mathcal{G}$ to any subgroup of $\mathcal{A}$ or $\mathcal{E}$. There exists a nested chain of subgroups $H_0 \subset H_1 \subset \ldots \subset H_{i-1} \subset H_i \subset \ldots$ such that $H = \cup_{i=0}^{\infty} H_i$ and each H_i is conjugated in $\mathcal{G}$ to a subgroup of $\mathcal{B}$,*
3) *$H = F \times < g >$, where $< g >$ is the subgroup of $\mathcal{G}$ generated by g, F is conjugated to a subgroup of $\mathcal{B}$, g is not conjugated to any element of $\mathcal{A}$ or $\mathcal{E}$ (where $\times$ denotes the map $\mathcal{B} \times W \to \mathcal{G}$ given by multiplication and W denotes the set of reduced words in $\mathcal{G}$, see [13]).*

The subgroups of $\mathcal{G}$ are called of type 1), 2) or 3), according to the fact that they satisfy 1), 2) or 3).

Remark 3.1. If H is of type 3), in particular it contains the element g, which is not conjugated to any element in $\mathcal{A}$ or $\mathcal{E}$.

Now, let Φ be a one-parameter group in $\mathcal{G}$: we already proved that for all $t \in \mathbf{R}$, Φ_t is conjugated to an element of $\mathcal{A}$ or $\mathcal{E}$, hence $H = \{\Phi_t, t \in \mathbf{R}\}$ is an abelian subgroup of $\mathcal{G}$ which cannot be of type 3).

Let us prove that H cannot be of type 2).

If H is of type 2), we denote by C_i the set $\{t \in \mathbf{R} : \Phi_t \in H_i\}$, where the H_i's are the subgroups in the definition of subgroup of type 2).

Let $X_i \in \mathcal{G}$ be such that $X_i \circ h \circ X_i^{-1} \in \mathcal{B}$ for all $h \in H_i$, then $X_i \circ \Phi_t \circ X_i^{-1} \in \mathcal{B}$ for all $t \in C_i$. As $\mathcal{B}$ is closed and both conjugation and Φ are continuous mappings, then $X_i \circ \Phi_t \circ X_i^{-1} \in \mathcal{B}$ for all $t \in \bar{C}_i$.

Since $H = \cup_{i=0}^{\infty} H_i$, then $\cup_{i=0}^{\infty} C_i = \mathbf{R}$ and therefore $\cup_{i=0}^{\infty} \bar{C}_i = \mathbf{R}$. Baire's category theorem implies that there exists $i_0 \in \mathbf{N}$ such that the inner part of $\bar{C}_{i_0}$ is not empty, therefore there exist $\tau \in \mathbf{R}$ and a positive number ε such that $(\tau - \varepsilon, \tau + \varepsilon) \subset \bar{C}_{i_0}$.

Therefore $X_{i_0} \circ \Phi_t \circ X_{i_0}^{-1} \in \mathcal{B}$ for all $t \in (\tau - \varepsilon, \tau + \varepsilon)$.

Let us consider the one-parameter group $\Psi_t = X_{i_0} \circ \Phi_t \circ X_{i_0}^{-1}$, we already proved that $\Psi_t \in \mathcal{B}$ for all $t \in (\tau - \varepsilon, \tau + \varepsilon)$.

Let $\rho \in (-\varepsilon, \varepsilon)$, then $\Psi_\rho = \Psi_{\tau+\rho-\tau} = \Psi_{\tau+\rho} \circ \Psi_{-\tau} = \Psi_{\tau+\rho} \circ (\Psi_\tau)^{-1}$; as $\tau + \rho$ and τ belong to $(\tau - \varepsilon, \tau + \varepsilon)$ and $\mathcal{B}$ is a subgroup of $\mathcal{G}$, then $\Psi_\rho \in \mathcal{B}$ for all $\rho \in (-\varepsilon, \varepsilon)$. Taking all integer multiples of the interval $(-\varepsilon, \varepsilon)$, we obtain the $\Psi_t \in \mathcal{B}$ for all $t \in \mathbf{R}$.

Then H is conjugated to a subgroup of $\mathcal{B}$, as $\mathcal{B}$ is in particular a subgroup of $\mathcal{A}$, then H is conjugated to a subgroup of $\mathcal{A}$; this contradicts the fact that H is of type 2), therefore, by Theorem 3.2, H is of type 1), i.e., H is conjugated to a subgroup of $\mathcal{A}$ or $\mathcal{E}$ and this proves Theorem 1.4. $\diamondsuit$

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