THE WORK OF
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SOME IDEAS ON STRUCTURES (*)

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SOMMARIO. - *Viene ripresentata una nozione di struttura (per un insieme provvisto di un sistema di dati) introdotta dall'autore molti anni fa. Si mostra che la nozione proposta da Bourbaki (basata sulla "specie" di strutture, mirante alla classificazione di tutte le teorie matematiche astratte) è carente dal punto di vista logico; essa viene qui sostituita con una basata sul gruppo di automorfismi di un insieme strutturato. Si danno esempi ed una considerazione sull'assioma della scelta.*

SUMMARY. - *Here is re-presented a notion of structure (for a set endowed with a system of data) introduced by the author many years ago. It is shown that Bourbaki's notion (based on "espèce de structure", aiming at the classification of all abstract mathematical theories) is logically lacking; so, it is here replaced by one based on automorphism group of structured sets. Some examples and considerations on the axion of choice are added.*

I am very grateful to my colleagues and friends, to have inserted among the speeches of this meeting one devoted to my modest work. I acknowledge to be quite unworthy of such an honour, due to personal friendship and estimation towards me. What I can admit is only to have laid stress, many years ago, to the interest for General Topology in this University, supported by the important contribution of foreign mathematicians, among the most valid in our domain.

Since Prof. Tironi has given account of some works and results I brought in Topology, let me add to his exposition a mention of another subject I dealt with in my early years, even if my elaboration has been largely incomplete and somehow utopian. It is matter only of ideas, without any claim of well-formulated results.

As it is well known, Bourbaki's monumental treatise has been intended as an unification of mathematical abstract theories; as it is presented in the synthetic first edition [1] (sets and essential related notions), such an aim is based on the notion of **structure**.

Briefly, the Bourbaki's presentation of structures is the following. Starting with some sets $A, B, C, \ldots$ two rules are given for the generation of other sets: the product $A \times B$ and the power set $\mathcal{P}(A)$. This enables us to the construction of an **échelle d'ensembles** by iteration of the operations: so we get sets like $A \times \mathcal{P}(B)$, $\mathcal{P}(\mathcal{P}(A \times B)), \ldots$ Each M of these sets is an **échelon** in the échelle d'ensembles having as base $A, B, C, \ldots$. A relation between échelons M, N, P defines an element of $\mathcal{P}(M \times N \times P)$.

If we fix some properties of an échelon of M, we can consider the intersection T of the subsets of elements satisfying them and call it **espèce de structure**, which is then characterized by the base sets and by the properties defining T (**axioms of the structure**).

So, giving some elements of an échelle, some relations between their elements and mappings between some of such sets, we are led to one element of a set of the échelle.

Among the most familiar species, we mention the order structures: i.e. those consisting in the choice of one element C of $\mathcal{P}(E \times E)$, provided the condition $C \circ C \subset C$ is fulfilled and the diagonal of $E \times E$ is in C.

According to Bourbaki, the fact that a system of data on a set or on a class of sets belongs to one espèce is considered as characteristic of a mathematical theory. Indeed, it is not so: a theory can be expressed without changing its proper contents, in different ways.

I give two samples.

(1) Lattices can be conceived as sets endowed with an order relation or as sets where two binary compositions are defined. In the first case a lattice turns to be a subset of $\mathcal{P}(E \times E)$, i.e. as an element of $\mathcal{P}(\mathcal{P}(E \times E))$; in the second case, as a subset of $\mathcal{P}(E \times E \times E) \times \mathcal{P}(E \times E \times E)$.

(2) Topological spaces can be regarded as sets of the échelle of base $\{E\}$, either as subsets of $\mathcal{P}(E)$ (if defined by open or by closed sets), or as a subset of $E \times \mathcal{P}(\mathcal{P}(E))$ (if by neighbourhoods) or else as a subset of $\mathcal{P}(E) \times \mathcal{P}(E)$ (if by closure of sets).

Hence I feel able to conclude that Bourbaki's approach to structures has merely methodological character, lacking in theoretical contents. We could even say that it depends on the notations we choose.

The notion of structure that I presented has been firmly opposed by J. Dieudonné [5] showing a case of very simple objects which, though of quite

different nature (a group and an ordered set), should have according to my definition, the same structure. Replying to such criticism, I observed that having the same structure does not mean to be the same thing; paradoxically I added that the set of ancient Rome's kings and that of the days of a week have the number 7 in common though they have quite different "nature". Unfortunately, my answer to such a criticism has not been taken into consideration for publication anywhere. Clearly, my idea of structure was intended just to mean a common property of set families, far from the purpose of identifying or even of strictly comparing the sets involved.

Quite different is, with regard to my criticism, the opinion of A. Bennet [6]. He writes:

"This paper examines critically the general concept of structures of an abstract mathematical system, on the ground that despite wide methodological use of the notion (as in mathematical fields cited by the author) exact formulation has hitherto been lacking. The author concludes that two mathematical systems constituted each by a set of postulates concerning a common object set should be said to have the same structure if and only if each automorphism of the object set which leaves either set of postulates valid leaves the other also valid ...".

I agree that my position may be judged as a utopian one, but I ought to mention the fact that, after Bourbaki's foundation exposed in three pages of their first booklet, nowhere more in their huge treatise it is explicitly dealt with "espèces de structures" in the sense of their premises.

Now, I summarize my position, based essentially on automorphisms associated with relations existing in a set. This is exposed in [1] (see also [3] and [4]). I confine myself, for simplicity, to the case of one set, though all can be extended to classes of sets (so, to espèces de structures of Bourbaki).

A **system of data** $\mathcal{S}$ on a set E is any class of propositions involving variable elements or subsets of E, whose truth values depend on the values given to the variables. We denote such an object by $E, \mathcal{S}$).

A bijective mapping of E into itself (permutation) preserving the truth values of all propositions of $\mathcal{S}$ is said an **automorphism** of $(E, \mathcal{S})$. Clearly, the set of all automorphisms of $(E, \mathcal{S})$ is a group with the composition law of permutations of E: the **autogroup** of $(E, \mathcal{S})$; we denote it by $\Gamma(E, \mathcal{S})$.

Here, having to do with groups of permutations, we define two such groups G and G' to be **strictly isomorphic** if there is a bijective mapping $\sigma : G \to G'$ such that for every $\gamma \in G$ we have $\sigma\gamma\sigma^{-1} \in G'$.

As an example, let E be a set of four elements with data consisting in order relations, namely

$$\mathcal{S}' : a < b, \; b < c, \; b < d \qquad \mathcal{S}'' : a < b, \; c < d \; .$$

The autogroups are isomorphic as abstract groups (being both of order 2), but they fail to be strictly isomorphic since in $\mathcal{S}'$ a and b are fix elements, while no element is fix in $\mathcal{S}''$.

Let E and E' be two sets of the same cardinality and let E be endowed with a system of data $\mathcal{S}$. Then we can transfer $\mathcal{S}$ into a system $\mathcal{S}'$ on E' by an arbitrary bijective mapping $\alpha : E \to E'$. Clearly, the autogroup of $(E, \mathcal{S})$ is carried by α into the autogroup of $(E', \alpha(\mathcal{S}))$. We then consider $(E, \mathcal{S})$ and $(E', \mathcal{S}')$ ($\mathcal{S}'$ being $\alpha(\mathcal{S})$) as **equivalent systems of data**. In particular, in the case $E = E'$, conjugate subgroups are equivalent.

Then, we assume each class of strictly isomorphic subgroups in the symmetric group Σ_E as a **structure** (related to the given cardinality of sets): $\mathcal{S}$ is a **representation** of the structure on E.

Various notions on permutation groups can be referred to structures, in particular transitivity and imprimitivity. So, the structures of a (classical) projective plane is 2-transitive; and it is not difficult to define the projective plane by means of its autogroup. The structure of $\mathcal{P}(E)$ ordered by inclusion is imprimitive, and so on.

An alternative notion of structure can be given through the following argument.

In the symmetric permutation group Σ_E, for every subgroup $\Gamma(E, \mathcal{S})$ we can consider its normalizer, i.e. the largest group Δ in which Γ is normal; its elements may be called the **displacements** of $\mathcal{S}$; they carry $\mathcal{S}$ into equivalent systems.

Replacing Γ by Δ we obtain a different notion of structure (the "weak structure" generated by $\mathcal{S}$ on E). An example: the group giving place to the structure of an euclidian plane (anyhow we define it by axioms) is usually considered that of congruences. But, since it is normal in the group of similitudes, we may have some preference to associate it with the (weak) structure of the plane endowed with similitudes. This is quite reasonable, since no proposition expressed in terms of congruences fails to be true if we replace the involved entities (points, lines, ...) by their images in a similitude.

One more example, from set theory. Let E be countable and let G be the group of its permutations moving only a finite number of elements. Since G is normal in Σ_E, it gives place, as weak structure, only to the

trivial one associated with Σ_E; the data vanish. This is the case of the "Fréchet filter".

By the way, we notice that if in Σ_E there is a normal subgroup of index 2, $(E, \mathcal{S})$ can be said to be **orientable**: the various cases of "orientability" fall under the above general definition. So is, in the simplest case, for the autogroup of a line before having fixed an arrow on it; the arrow will then make the line into an ordered set. Another sample: if E is finite, Σ_E is an orientable set, even permutations being a subgroup of index 2.

For the projective plane we can state that no subgroup of its autogroup is normal in it; hence its non-orientability.

I will spend some words on the problems rising from the notions introduced above; in particular as for relations with the familiar idea of a structure (without entering the "espèces" of Bourbaki).

The fact that two systems $\mathcal{S}$ and $\mathcal{S}'$ on a given set have their autogroups strictly isomorphic does insure us that they data are translatable from one to the other by a law given in advance? See, e.g., the case of lattices given above: if $\mathcal{S}$ is defined as an order relation and $\mathcal{S}'$ as algebraic operations, $x < y$ is translated into $x \wedge y = x$, etc.

Another problem: given a group on a set E, to find a system of data having it as autogroup.

Since these questions are not formally well expressed, we better may put us in the more restricted extent consisting in requiring in advance the desired kind of data. So, for instance, we ask for conditions to put to a permutation group in order it be the autogroup of a topological space; such an extremely difficult question is in its essential the so called Wiener's problem [7]. Less difficult would be the same question when referred to order relations; but, as far as I know, this problem has not yet been solved. Even in the very simple case consisting in an equivalence relation on sets, the characterization of their autogroup is not at all trivial when E is an infinite set.

Finally, let me add some words on Zermelo's axiom, in connection with automorphisms of structured sets.

Everybody knows how many disputes rose, chiefly in the period from 1920 to 1950 about "infinity of choices": some mathematicians rejecting it, some others finding it as a quite permissible act. Indeed, some confusion of ideas has been originated from the formulation "given a set of non-void sets, it is possible to choose...". It would have been less misleading to speak of existence, not of possibility to choose. At the time I was interested in

structures, the question was still alive.

What has to be particularly stressed is the different contents of the axiom according to the character of transitivity or not of the structure of sets whence we choose; in other words, according to whether the elements are or not distinguished by the data. If the structure of the set involved is transitive, the choice of one element seems to be fully legitimate; otherwise, it seems reasonable to exclude such legitimation.

For the sake of brevity, let me mention two samples.

(1) " Choose a point from each of a given sequence of (ordinary) planes" seems to be legitimate, since to ask for additional criteria in the choice cannot be drawn from the (transitive) structure of each set of the family; the question "what point did you choose?" has no sense. The same for the act "choose, in a topological plane, a sequence of distinct points going to infinity": the structure of the plane do not privilege any of such sequences (for reasons of transitivity).

(2) We consider the classical example given by Vitali in order to prove the existence of non-measurable sets in $\mathbb{R}$: (partition of $\mathbb{R}$ into classes of real numbers like $\{x : x - a \in \mathbb{Q}\}, a \in \mathbb{R}$); here, if we claim to choose one element from each class, a question like "the number 6 has been chosen?", though well formulated, cannot be answered.

References

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