APPROXIMATE SEQUENCES VERSUS INVERSE SEQUENCES (*)

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Sommario. - *In questa nota si costruisce una sequenza inversa approssimata* $\mathcal{X} = (P_n, \varepsilon_n, P_{nn'}, \mathbb{N})$ *di continui planari poliedrali* P_n *in maniera tale che* $\mathcal{X}$ *e la sequenza (commutativa) inversa corrispondente* $\underline{X} = (P_n, p_{n,n+1}, \mathbb{N})$ *abbiano limiti non omeomorfi. Si ha così un miglioramento essenziale di un precedente esempio del medesimo autore relativo a continui planari non poliedrali.*

Summary. - *An approximate inverse sequence* $\mathcal{X} = (P_n, \varepsilon_n, P_{nn'}, \mathbb{N})$ *of polyhedral planar continua* P_n *is constructed, such that* $\mathcal{X}$ *and the corresponding (commutative) inverse sequence* $\underline{X} = (P_n, p_{n,n+1}, \mathbb{N})$ *have non-homeomorphic limits. This is an essential improvement of the author's previous example, which consisted of non-polyhedral planar continua.*

1. Introduction.

S. Mardešić and L.R. Rubin [4] introduced the notion of an approximate inverse system of metric compacta $\mathcal{X} = (X_a, \varepsilon_a, p_{aa'}, A)$. They replaced (weakened) the commutativity condition $p_{aa'}p_{a'a''} = p_{aa''}$, $a \leq a' \leq a''$, by the following three requirements:

A1) $d(p_{aa'}p_{a'a''}, p_{aa''}) \leq \varepsilon_a$ whenever $a \leq a' \leq a''$;

A2) $(\forall a \in A)(\forall \eta > 0)\ (\exists a' \geq a)\ (\forall a_2 \geq a_1 \geq a')$
$d(p_{aa_1}p_{a_1a_2}, p_{aa_2}) \leq \eta$;

A3) $(\forall a' \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a'' \geq a')$
$(\forall x, x' \in X_{a''})\ d(x, x') \leq \varepsilon_{a''} \Rightarrow d(p_{aa''}(x), p_{aa''}(x')) \leq \eta$.

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An approximate map q of a space Y into an approximate system $\mathcal{X} = (X_a, \varepsilon_a, p_{aa'}, A)$, $q : Y \to \mathcal{X}$, is a collection $q = \{q_a | a \in A\} = (q_a)$ of mappings $q_a : Y \to X_a$ satisfying the following condition:

(AS) $(\forall a \in A)$ $(\forall \eta > 0)$ $(\exists a' \geq a)(\forall a'' \geq a')$ $d(q_a, p_{aa''}q_{a''}) \leq \eta$.

An approximate map $p = (p_a) : X \to \mathcal{X}$ is called a limit of $\mathcal{X}$ provided it has the following universal property:

(UL) For any approximate map $q : Y \to \mathcal{X}$ there exists a unique mapping $g : Y \to X$ satisfying $p_a g = q_a$, for every $a \in A$.

Since a limit space X is determined up to a unique homeomorphism, we often speak of the limit X of $\mathcal{X}$ and we write $X = \lim \mathcal{X}$.

Moreover, Mardešić and Rubin analogously defined the notion of an approximate inverse system of compact Hausdorff spaces as well as of its limit. Here, of course, the role of the numbers $\varepsilon_a > 0$ and $\eta > 0$ is taken over by open coverings $\mathcal{U}_a$ and $\mathcal{U}$ of X_a, $a \in A$, respectively. They established a very important theorem (which does not hold in the commutative case), [4]: A compact Hausdorff space X has covering dimension $\dim X \leq n$ if and only if X is the limit of an approximate inverse system of compact polyhedra X_a with $\dim X_a \leq n$.

M.G. Charalambous [1] was the first to consider (nongauged) approximate systems satisfying only condition (A2). Subsequently S. Mardešić [2] and the author [16] showed that these systems are equivalent to the (gauged) approximate systems.

The theory of approximate systems, as well as their applications, has been further intensively developed by S. Mardešić and J. Segal [6], [7], [8], S. Mardešić and L.R. Rubin [5], S. Mardešić and T. Watanabe [11], T. Watanabe [17], S. Mardešić [2], S. Mardešić and V. Matijević [3], V. Matijević [12], V. Matijević and N. Uglešić [13], N. Uglešić [15], [16], S. Mardešić and N. Uglešić [9], [10] and others.

2. Example.

When S. Mardešić started studying approximate systems, he asked the following question:

Let $\mathcal{X} = (X_n, \varepsilon_n, p_{nn'}, \mathbb{N})$ be an approximate inverse sequence with limit $\lim \mathcal{X} = X$. Let $\underline{X} = (X_n, p_{n,n+1}, \mathbb{N})$ be the usual (commutative) inverse sequence obtained by replacing in $\mathcal{X}$ each $p_{nn'}$, $n' - n \geq 1$, by the composition $p'_{nn'} = p_{n,n+1} \circ \ldots \circ p_{n'-1,n'}$. Is $\lim\ \underline{X}$ homeomorphic to X?

The next result of M.G. Charalambous ([1], Proposition 8) addresses directly the above question:

Let $X = (X_n, p_{nn'}, \mathbb{N})$ be an approximate sequence of complete metric spaces (in particular, metric compacta) with limit $\lim \mathrm{X} = X$. Then X is uniformly isomorphic (hence, homeomorphic) to the limit $\lim \underline{X}'$, where $\underline{X}' = (X_m, p'_{mm'}, M)$ is the usual inverse sequence over some cofinal subset $M \subseteq \mathbb{N}$, where $p'_{mm'} = p_{mm'}$, whenever m' is an immediate successor of m in M.

Although this result suggests to answer the above question affirmatively, the author showed that it is not the case ([15], Example (2.3)). The counterexample, having its roots in [6], Example 1, has been constructed out of terms which are all equal to the same planar continuum - the Hawaiian earring H. Here we will obtain an improvement of that example, which consists in replacing H by compact connected planar polyhedra P_n, $n \in \mathbb{N}$, where P_n is the wedge of a compact polyhedral disc and n polyhedral circles.

Example 2.3. Let $S^1 = \{z \in \mathbb{C} | \; |z| = 1\}$ be the standard unit circle in $\mathbb{R}^2$. Consider for each $k \in \mathbb{N}$, the following four points $(\xi_k, \eta_k)_j \in \mathbb{R}^2$, $j = 1, 2, 3, 4$:

$$((k-1)/k, 0),\ (k/(k+1), -1/k(k+1)),$$

$$(1, 0) = x_0,\ (k/(k+1), 1/k(k+1))\ .$$

Let $D_k \subseteq \mathbb{R}^2$, $k \in \mathbb{N}$, be the convex hull of these four points. Then each D_k is a polyhedral disc, $D_1 \supseteq D_2 \supseteq D_k \supseteq \dots$ and $\bigcap_{k \in \mathbb{N}} D_k = \{x_0\}$. Let C_k be the boundary of D_k, $k \in \mathbb{N}$. Notice that $\mathrm{diam}(C_k) = \mathrm{diam}(D_k) = 1/k$. For every $k \in \mathbb{N}$, choose the homeomorphism $h_k : C_k \to S^1$ defined by the radial projection from the interior point $((2k-1)/2k, 0)$ of D_k. Let

$$P_n = (\bigcup_{k=1}^{n} C_k) \cup D_{n+1} \subset \mathbb{R}^2, \quad n \in \mathbb{N}\ ,$$

with the euclidean metric d. Obviously, $P_1 \supseteq P_2 \supseteq \dots \supseteq P_n \supseteq \dots$ is a sequence of compact connected polyhedra in $\mathbb{R}^2$. Let us define the sequence of mappings $\varphi_n : P_{n+1} \to P_n$, $n \in \mathbb{N}$, by putting

$$\varphi_n(x) = \begin{cases} x, & x \in P_{n+1} \backslash C_{n+1}\ , \\ h_n^{-1}(h_{n+1}(x)^2), & x \in C_{n+1}\ . \end{cases}$$

Observe that $\varphi_n(C_n \cup C_{n+1}) = \varphi_n(C_{n+1}) = C_n$, $n \in \mathbb{N}$. If $n' \geq n+2$, denote by $i_{nn'}$ the inclusion mapping of $P_{n'}$ into P_n. Now, we define mappings

$p_{nn'}: P_{n'} \to P_n$, $n' \geq n$, by

$$p_{nn'} = \begin{cases} id, & n' = n, \\ \varphi_n, & n' = n+1, \\ i_{nn'}, & n' \geq n+2. \end{cases}$$

Finally, let $\varepsilon_n = 1/n$, $n \in \mathbb{N}$.

LEMMA 2.4. *$\mathcal{X} = (P_n, \varepsilon_n, p_{nn'}, \mathbb{N})$ is an approximate sequence.*

Proof. We have to verify conditions (A1), (A2) and (A3).

(A1). Let $n \leq n' \leq n''$ in $\mathbb{N}$ be given. All the non-trivial cases are the following three:

$$d(p_{nn''}, p_{nn'}, p_{n'n''}) = \begin{cases} d(i_{n,n+2}, \varphi_n \varphi_{n+1}), & n'' = n'+1 = n+2, \\ d(i_{nn''}, \varphi_n | P_{n''}), & n'' > n'+1 = n+2, \\ d(i_{n,n'+1}, i_{nn'} \varphi_{n'}), & n'' = n'+1 > n+2\ . \end{cases}$$

In the first case, only the points of $(C_{n+1} \cup C_{n+2}) \backslash \{x_0\} \subset P_{n+2}$ are moving. Because $\varphi_n \varphi_{n+1}(C_{n+1} \cup C_{n+2}) = C_n$, $i_{n,n+2}(C_{n+1} \cup C_{n+2}) = C_{n+1} \cup C_{n+2}$ and $\operatorname{diam}(C_n \cup C_{n+1} \cup C_{n+2}) = \operatorname{diam}(C_n) = 1/n = \varepsilon_n$, condition (A1) for $\mathcal{X}$ is satisfied. In the second (third) case, only the points of $C_{n+1} \backslash \{x_0\}$ $(C_{n'+1} \backslash \{x_0\})$ are moving. Thus, the same argument applies.

(A2). Let $n \in \mathbb{N}$ and $\eta > 0$ be given. Choose an $n_0 \in \mathbb{N}$ such that $1/n_0 \leq \eta$. Then $\varepsilon_{n'} \leq \eta$ whenever $n' \geq n_0$. Now take $n' = \max\{n_0, n+2\}$, and let $n_2 \geq n_1 \geq n'$ be given. Observe that p_{nn_1} and p_{nn_2} are the inclusion mappings i_{nn_1} and i_{nn_2} respectively. Therefore,

$$d(p_{nn_2}, p_{nn_1} p_{n_1 n_2}) = \begin{cases} d(i_{nn_2}, i_{nn_2}), & n_2 - n_1 \neq 1, \\ d(i_{n,n_1+1}, i_{nn_1} \varphi_{n_1}), & n_2 - n_1 = 1\ . \end{cases}$$

Only the second case restricted to $C_{n_1+1} \backslash \{x_0\} \subseteq P_{n_1+1}$ is non-trivial. Because of $\varphi_{n_1}(C_{n_1+1}) = C_{n_1}, i_{n,n_1+1}(C_{n_1+1}) = C_{n_1+1}$ and diam $(C_{n_1} \cup C_{n_1+1}) = diam(C_{n_1}) = 1/n_1 \leq 1/n' = \varepsilon_{n'} \leq \eta$, $d(i_{n,n_1+1}, i_{nn_1} \varphi_{n_1}) \leq \eta$ holds true. This verifies condition (A2) for $\mathcal{X}$.

(A3). Let $n \in \mathbb{N}$ and $\eta > 0$ be given. Choose and $n_0 \in \mathbb{N}$ and n' as above, and let $n'' \geq n'$ be given. Then $p_{nn''} = i_{nn''}$. Therefore $d(x, x') \leq \varepsilon_{n''} = 1/n''$ implies $d(p_{nn''}(x) p_{nn''}(x')) = d(x, x') \leq 1/n'' \leq 1/n' = \varepsilon_{n'} \leq \eta$, which establishes condition (A3) for $\mathcal{X}$.

Let $X = \bigcap_{n \in \mathbb{N}} P_n \subseteq \mathbb{R}^2$ and let $p_n : X \to P_n$, $n \in \mathbb{N}$, be the inclusion mappings. Then one easily verifies that $p = (p_n) : X \to \mathcal{X}$ is the limit of $\mathcal{X}$

(see [11], (1.19) Theorem). Notice that $X \approx H$. Let $\underline{X} = (P_n, p'_{n,n'}, \mathbb{N})$ be the corresponding (commutative) inverse sequence associated with $\mathcal{X}$, i.e.

$$p'_{nn'} = \begin{cases} id, & n' = n, \\ \varphi_n \circ \ldots \circ \varphi_{n'-1}, & n' > n. \end{cases}$$

As in [6], Example 1, or [15], (2.3) Example, the limit space lim $\underline{X}$ contains a copy of the diadic solenoid. Therefore, lim $\underline{X}$ cannot be homeomorphic to lim $\mathcal{X}$. This answers our question in the negative.

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