AN APPLICATION OF THE THEORY OF SELECTIONS IN ANALYSIS (*)

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Sommario. - *Usando uno dei teoremi di selezione di E. Michael si prova il seguente risultato: siano* (X, d) *e* (Y, ρ) *spazi metrici e sia* X *localmente compatto. Sia* $\mathcal{C}(X, Y)$ *l'insieme di tutte le mappe continue da* X *a* Y*, dotato della topologia della convergenza uniforme. Allora esiste una funzione continua ad un valore* $\hat{\delta} : \mathcal{C}(X, Y) \times X \times (0, \infty) \to (0, \infty)$ *tale che per ogni* $(f, x, \varepsilon) \in \mathcal{C}(X, Y) \times X \times (0, \infty)$ *e per ogni* $x' \in X : d(x, x') < \hat{\delta}(f, x, \varepsilon) \Rightarrow \rho(f(x), f(x')) < \varepsilon$. *Come corollario, si ottiene un'altra dimostrazione del fatto che il teorema di Cantor sulla uniforme continuità implica il Teorema di Weierstrass sulla limitatezza delle funzioni continue sui compatti.*

Summary. - *Using one of E. Michael's selection theorems we prove the following result: Let* (X, d) *and* (Y, ρ) *be metric spaces and suppose that* X *is locally compact. Let* $\mathcal{C}(X, Y)$ *be the set of all continuous maps from* X *to* Y*, endowed with the topology of uniform convergence. Then there exists a continuous singlevalued function* $\hat{\delta} : \mathcal{C}(X, Y) \times X \times (0, \infty) \to (0, \infty)$ *such that for every* $(f, x, \varepsilon) \in \mathcal{C}(X, Y) \times X \times (0, \infty)$ *and for every* $x' \in X : d(x, x') < \hat{\delta}(f, x, \varepsilon) \Rightarrow \rho(f(x), f(x')) < \varepsilon$. *As a corollary, we obtain another proof that the Cantor theorem on uniform continuity implies the Weierstrass theorem on boundedness of continuous functions on compacta.*

1. Introduction.

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Let (X,d) and (Y,ρ) be metric spaces and let $\mathcal{C}(X,Y)$ be the set of all continuous maps from X into Y, endowed with the topology of *uniform convergence:* (i.e. the ε–neighbourhood of a map $f \in \mathcal{C}(X,Y)$ is the set $\{g \in \mathcal{C}(X,Y) \mid \rho(f(x),g(x)) < \varepsilon \text{ for all } x \in X\}$.) For every triple $z = (f,x,\varepsilon)$ from the Cartesian product $Z = \mathcal{C}(X,Y) \times X \times (0,\infty)$ there exists, by the definition of continuity, $\delta > 0$ such that for every $x' \in X$: $d(x,x') < \delta \Longrightarrow \rho(f(x),f(x')) < \varepsilon$. The purpose of this note is to show that whenever X is a locally compact space, it is possible to choose $\delta > 0$ which *continuously* depends on $z = (f,x,\varepsilon)$.

THEOREM 1.1. *Let (X,d) and (Y,ρ) be metric spaces and suppose that X is locally compact. Then there exists a continuous singlevalued function*

$$\hat{\delta} : \mathcal{C}(X,Y) \times X \times (0,\infty) \longrightarrow (0,\infty)$$

such that for every $(f,x,\varepsilon) \in \mathcal{C}(X,Y) \times X \times (0,\infty)$ and for every $x' \in X$ the following implication holds:

$$d(x,x') < \hat{\delta}(f,x,\varepsilon) \Longrightarrow \rho(f(x),f(x')) < \varepsilon \ .$$

The function $\hat{\delta}$ will be constructed as a selection of some lower semicontinuous multivalued map $\Delta : \mathcal{C}(X,Y) \times X \times (0,\infty) \to (0,\infty)$ with convex values. Recall, that a singlevalued map $\varphi : A \to B$ is said to be a *selection* of a multivalued map $\Phi : A \to B$ if for every point $a \in A$, we have that $\varphi(a) \in \Phi(a)$. A multivalued map $\Phi : A \to B$ between topological spaces A and B is called *lower semicontinuous* if for every open nonempty subset $G \subset B$, the following subset $\Phi^{-1}(G) = \{a \in A \mid \Phi(a) \cap G \neq \emptyset\}$ is open in A.

We shall use the following selection theorem for convex-valued but non-closed valued maps (cf. [1, Theorem (3.1)''']):

THEOREM 1.2. (E. Michael) *For every Hausdorff space X the following conditions are equivalent:*

a) *X is perfectly normal; and*

b) *Every lower semicontinuous map from X into convex D–type subsets of a separable Banach space has a continuous singlevalued selection.* $\diamondsuit$

Recall, that a convex subset of a Banach space is said to be of D–type if it contains all interior (in the convex sense) points of its closure. A point of a closed convex subset of a Banach space is said to be *interior* (in the convex sense) if it isn't contained in any supporting hypersubspace. It is easy to see that all finite–dimensional convex sets are examples of convex D–type sets. In this case the set of all interior, in the convex sense, points coincides with the set of all interior, in the usual sense, points. Next, the space $\mathcal{C}(X,Y)$ is metrizable and the space $Z = \mathcal{C}(X,Y) \times X \times (0,\infty)$ is metrizable, too. Hence the space Z is perfectly normal and so we may indeed use Theorem (1.2) for lower semicontinuous maps with finite–dimensional convex values.

2. Proof of Theorem 1.1.

We shall denote by $V(t;\delta)$ the open neighborhood of radius δ and by $\overline{A}$ the closure of a subset $A \subset X$ in X. Define the multivalued map $\Delta : Z \to (0,\infty)$ as follows:

$$\Delta(z) = \Delta(f,x,\varepsilon) = \Big\{\delta \in \mathbf{R}^*_+ \mid \overline{\mathbf{V}(\mathbf{x};\delta)} \quad \text{is compact; and} \tag{1}$$

$$\text{for every} \quad x' \in X,\ \left(d(x,x') < \delta \Longrightarrow \rho(f(x),f(x')) < \varepsilon\right)\Big\} \ . \tag{2}$$

The set $\Delta(z)$ is a nonempty subset of $(0,\infty) \subset \mathbf{R}$ because X is locally compact and $f \in \mathcal{C}(X,Y)$. From the obvious inclusion $V(x,t\delta) \subset V(x,\delta)$, $0 < t < 1$, the convexity of the set $\Delta(z)$ follows.

So, in order to prove Theorem (1.1) we only need to check the lower semicontinuity of the multivalued map $\Delta : Z \to \mathbf{R}^*_+$. Suppose that, to the contrary, there exists

i) a point $z_0 = (f_0, x_0, \varepsilon_0) \in Z$;

ii) a point $\delta_0 \in \Delta(z_0) \subset \mathbf{R}^*_+$;

iii) a number $0 < \sigma < \delta_0$; and

iv) a sequence $\{z_n = (f_n, x_n, \varepsilon_n) \in Z\}$, such that $z_n \to z_0$; and

$$\Delta(z_n) \cap V(\delta_0;\sigma) = \emptyset \ . \tag{3}$$

If $\delta \in \Delta(z_n)$ then $t\delta \in \Delta(z_n)$, for every $0 < t < 1$. Hence the condition (3) is equivalent to

$$\sup \Delta(z_n) \le \delta_0 - \sigma \ . \tag{4}$$

Since $x_n \to x_0$ in X we may assume that for every point x_n there exists a δ_n–neighborhood in X such that

$$\delta_n > \delta_0 - \sigma \tag{5}$$

and

$$V(x_n, \delta_n) \subset V(x_0, \delta_0 - \sigma/2) \ . \tag{6}$$

In particular, $\overline{V(x_n, \delta_n)}$ will automatically be compact.

From (5) we have that $\delta_n \notin \Delta(z_n)$, i.e. there exists $x'_n \in X$ such that

$$x'_n \in V(x_n, \delta_n) \tag{7}$$

and

$$\rho(f_n(x'_n), f_n(x_n)) \geq \varepsilon_n \ . \tag{8}$$

Now, the set $\overline{V(x_0, \delta_0 - \sigma/2)}$ is compact. Therefore, we may assume that

$$x'_n \longrightarrow x' \in \overline{V(x_0; \delta_0 - \sigma/2)} \subset V(x_0, \delta_0) \ .$$

Since $\delta_0 \in \Delta(z_0)$ we have that

$$\rho(f_0(x'), f_0(x_0)) < \varepsilon_0 \ . \tag{9}$$

On the other hand, if we pass in (8) to the limit (when $n \to \infty$), then we have that

$$\rho(f_0(x'), f_0(x_0)) \geq \varepsilon_0 \tag{10}$$

which contradicts (9).

To verify (10) it suffices to check that $f_n(x_n) \to f_0(x_0)$ and that $f_n(x'_n) \to f_0(x')$. But we have that

$$\rho(f_n(x_n), f_0(x_0)) \leq \rho(f_n(x_n), f_0(x_n)) + \rho(f_0(x_n), f_0(x_0)) \ . \tag{11}$$

The first term on the right hand side of (11) converges to zero because the sequence $\{f_n\}_{n\in\mathbf{N}}$ is uniformly converging to f_0. The second term on the right hand side of (11) converges to zero because f_0 is continuous. The convergence $f_n(x'_n) \to f_0(x')$ may be checked in an analogous manner. $\diamondsuit$

Example 2.1. Let $X = Y = (-1, 1)$, $f_0(x) = x$ and let $\Delta_1(f_0, x, \varepsilon) = \{\delta \in (0, \infty) \mid \text{for every } x' \in X, \ (d(x, x') < \delta \Longrightarrow \rho(f_0(x), f_0(x')) < \varepsilon)\}$ (i.e. in the definition (1) above we omit the condition that $\overline{V(x;\delta)}$ is compact. Then obviously, $\Delta_1(f_0, 0, 1) = \mathbf{R}^*_+$. However, for every nonzero $x \in (-1, 1)$ we have that $\Delta_1(f_0, x, 1) = (0, 1]$. Hence, the map Δ_1 is *not* lower semicontinuous. Note that in this example, $\overline{V(0,1)} = V(0,1)$ isn't compact. So the condition (1) from the definition of the map Δ above is indeed necessary for our application of Theorem (1.2). Clearly, if we had condition (1) added in this example then we would also obtain a lower semicontinuous map Δ which would be inscribed into the map Δ_1. So in this example our proof would also work.

Recall two results from classical analysis: let $f : X \to \mathbf{R}$ be a continuous real-valued function on a compact metric space X. Then the Weierstrass theorem asserts that f is bounded (above and below) and the Cantor theorem asserts that f is uniformly continuous on X. As an application of Theorem (1.1) we shall prove the following interesting observation:

COROLLARY 2.2. *The Cantor theorem on uniform continuity is a corollary of the Weierstrass theorem on boundedness of continuous functions on compacta.*

Proof. Suppose that X is a compact metric space. Pick any $f_0 \in \mathcal{C}(X, \mathbf{R})$ and $\varepsilon_0 > 0$ and consider the set $W = \{\hat{\delta}(f_0, x, \varepsilon_0) \mid x \in X$ and $\hat{\delta}(f_0, x, \varepsilon_0) : \mathcal{C}(X, \mathbf{R}) \times \mathbf{X} \times (\mathbf{0}, \infty) \to (\mathbf{0}, \infty)$ is continuous$\}$. Then by Theorem (1.1) the set W is nonempty, so pick any $\hat{\delta}_0 \in W$. Clearly, one can consider $\hat{\delta}_0$ as $\hat{\delta}_0 \in \mathcal{C}(X, \mathbf{R})$. Apply the Weierstrass theorem to obtain the minimum $\delta_0 = \min\{\hat{\delta}_0(x) \mid x \in X\}$. Then any $\delta \in (0, \delta_0)$ will provide the uniform continuity assertion of the Cantor theorem (for f_0 and ε_0). $\diamondsuit$

QUESTION 2.3. *Is the hypothesis about the local compactness of X in Theorem (1.1) necessary?* (It certainly is for *our* proof as Example (2.1) shows.)*

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