UNIVERSAL COVERING CATEGORIES (*)

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SOMMARIO. - *La teoria dei limiti semplicemente connessi è ulteriormente sviluppata introducendo l'analogo categoriale dello spazio di ricoprimento universale. Questa categoria del ricoprimento universale è utilizzata per dare una dimostrazione concettuale del fatto che i limiti connessi possono essere calcolati utilizzando prodotti fibrati ed equalizzatori. Si dimostra anche una proprietà fondamentale di esattezza della riflessione di gruppoidi per categorie: che essa preserva certi oggetti comma.*

SUMMARY. - *The theory of simply connected limits is further developed by the introduction of the categorical analogue of the universal covering space. This universal covering category is used to give a conceptual proof that connected limits can be computed using fibred products and equalizers. Along the way we prove a fundamental exactness property of the groupoid reflection for categories: that it preserves certain comma objects.*

Introduction.

Category theory was issued from the marriage of algebra and topology, and it bears a strong resemblance to each of its parents. A category with one object is a monoid and a category may be thought of as a monoid with several identities. There is in fact a precise sense in which this is so. A *groupoid* is a category in which each morphism is an isomorphism, and a groupoid with one object is a group. Functors between such categories are exactly group homomorphisms. A functor from a group into the category of sets is a G-set and the Yoneda lemma becomes Cayley's representation theorem.

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But one of the important features of category theory is the ability to represent equations as (commutative) diagrams, giving shape to long strings of calculations. Thus topology enters; objects are represented as points, morphisms as arrows (directed line segments or paths), and equations as two-dimensional entities. Thus we can talk of connected categories, simply connected categories, fibrations and so on. In fact, a category gives rise to a simplicial set as suggested above, its nerve, which has a geometric realization. Thus to each category we can associate a topological space, and the above analogies have a precise mathematical meaning.

What is even more important is the interplay between the algebraic and the topological sides of category theory. It is in this direction that this paper is directed. We will present an application of topological ideas to pure category theory, namely to the theory of limits.

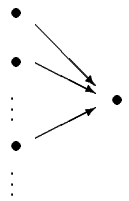
The concept of limit in a category **A** has been around since the beginning of the subject (see Notes on p. 76 of [5]). One of the basic facts is that the limit of an arbitrary diagram $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}$ can be computed as an equalizer of products (assuming that these exist)

$$\varprojlim \Gamma \longrightarrow \prod_{I} \Gamma I \rightrightarrows \prod_{I \to I'} \Gamma I' \ .$$

By a *connected* limit we mean the limit of a diagram $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}$ where the category **I** is connected, i.e. for any two objects I and I' there is a path of arrows, back and forth joining them

$$I \longrightarrow I_1 \longleftarrow I_2 \longrightarrow \cdots \longleftarrow I'.$$

Barr asked if every connected limit could be constructed using equalizers and fibered products, i.e. limits of diagrams of the form



The answer is yes but not as straightforward as with products. An *ad hoc* construction was published by Cockett in [3]. We propose to give the problem a conceptual solution inspired by ideas from topology. In fact the problem is merely an excuse to study the concept of universal covering category whose theory provides a nice blend of topology and algebra.

I would like to thank the organizers of the XI Convegno Internazionale di Topologia for inviting me to present these results and the Università degli Studi di Trieste for their kind hospitality. I would also like to thank G. Janelidze, K. Johnson, I. Moerdijk, and R. Wood for valuable discussions.

1. Simply Connected Categories.

In this section we give a brief review of simply connected categories and their relation to limits, as exposed in [6].

Let **Cat** be the category of small categories and **Set** the category of sets. There is a forgetful functor $Ob : \mathbf{Cat} \longrightarrow \mathbf{Set}$ which remembers only the set of objects of a category. Ob has right and left adjoints C and D respectively. C is the *chaotic* or *indiscrete* category on a set of objects, i.e. there is exactly one morphism between any pair of objects. D is the *discrete* category on a set of objects, i.e. the only morphisms are identities. **Set** is often viewed as a full subcategory of **Cat** via D. D itself has a left adjoint $\pi_0 : \mathbf{Cat} \longrightarrow \mathbf{Set}$, the set of *connected components* of a category. $\pi_0(\mathbf{I})$ is the set of equivalence classes of objects where two objects are related if there is a path of morphisms, back and forth joining them. This is the equivalence relation generated by $I \sim I'$ if $\mathbf{I}(I, I') \neq \emptyset$. We say that $\mathbf{I}$ is *connected* if $\pi_0(\mathbf{I}) \cong 1$.

Let **Gpd** be the category of small groupoids. It is a full subcategory of **Cat**. The inclusion has right and left adjoints, Iso and π_1. $Iso(\mathbf{I})$ is the subcategory of $\mathbf{I}$ consisting of all isomorphisms. On the other hand, the left adjoint, $\pi_1(\mathbf{I})$, is obtained by formally adding inverses for all morphisms of $\mathbf{I}$. More precisely, we first construct a graph with the objects of $\mathbf{I}$ as nodes and with $\mathbf{I}(I, I') + \mathbf{I}(I', I)$ as edges from I to I'. We denote the edge corresponding to $\alpha : I \longrightarrow I'$ by α^1 and the one corresponding to $\beta : I' \longrightarrow I$ by β^{-1}. We now take the free category on this graph and then the quotient by the congruence generated by the relations

i) $\alpha_1^1 \alpha_2^1 \sim (\alpha_1 \alpha_2)^1$

ii) $1_I^1 \sim 1_I$ (empty word)

iii) $\alpha^1 \alpha^{-1} \sim 1_{I'}$

iv) $\alpha^{-1} \alpha^1 \sim 1_I$.

The quotient is the category $\pi_1(\mathbf{I})$. Thus the objects of $\pi_1(\mathbf{I})$ are the same as those of $\mathbf{I}$, and the morphisms are equivalence classes of words $a_n a_{n-1} \cdots a_1$

where a_i is either an α^1 or α^{-1}. Of course in such a word, the domains and codomains must match in the usual sense. Thus a morphism $I \longrightarrow I'$ in $\pi_1(\mathbf{I})$ can be represented as an equivalence class of paths

$$I = I_0 \longleftrightarrow I_1 \longleftrightarrow I_2 \cdots \longleftrightarrow I_n = I'$$

where $\longleftrightarrow$ represents either $\longrightarrow$ or $\longleftarrow$. Informally, two paths are equivalent if one can be deformed into the other by using commutative triangles.

In view of this discussion it seems fair to denote this category by $\pi_1(\mathbf{I})$ and call it *fundamental groupoid* of $\mathbf{I}$.

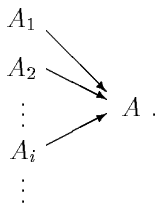
Because the equivalence relation is a bit unwieldy, it is best to avoid it when possible and rely on the universal property, i.e. that π_1 is left adjoint to the inclusion $\mathbf{Gpd} \rightarrowtail \mathbf{Cat}$. We shall denote the unit for the adjunction by

$$Q : \mathbf{I} \longrightarrow \pi_1(\mathbf{I}).$$

It is the identity on objects and $Q(\alpha) = \alpha^1$ on morphisms.

A category $\mathbf{I}$ is called *simply connected* if $\pi_1(\mathbf{I})$ is equivalent to the terminal category $\mathbf{1}$, $\pi_1(\mathbf{I}) \simeq \mathbf{1}$. In elementary terms, $\pi_1(\mathbf{I})$ has exactly one morphism between each pair of objects.

By a *fibered product* (sometimes called *wide pullback* or *infinite pullback*) we mean the limit of a diagram of the form



There can be any number of A_i, finite, infinite or even none. We reserve the term *pullback* for the case when there are two.

The main result of [6] is that $\mathbf{I}$ limits can be computed using fibered products if and only if $\mathbf{I}$ is simply connected.

REMARK 1. Carboni and Johnstone point out in [1] that if a functor on a category with terminal object preserves fibered products (pullbacks), then it preserves all (finite) connected limits. This is because a diagram $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}$ with a cocone $\gamma : \Gamma \longrightarrow A$ can be lifted to a simply connected diagram $\Gamma^+ : \mathbf{I}^+ \longrightarrow \mathbf{A}$ by adding a terminal object ∞ to $\mathbf{I}$ and defining

$\Gamma^+(\infty) = A$. If $\mathbf{I}$ is connected then $\varprojlim \Gamma^+ \cong \varprojlim \Gamma$ and a functor preserving fibered products will preserve the limit of Γ^+ as well as the cocone. This gives rise to the following curiosity. A category with pullbacks and coequalizers also has equalizers and any functor preserving pullbacks will also preserve the equalizers.

EXAMPLE 1. The categories represented by

and

are both simply connected. So is the monoid $\{1, e | e^2 = e\}$. It is an amusing exercise to give a construction of limits of these types using fibered products. On the other hand, for the category $\mathbf{E} = \bullet \rightrightarrows \bullet$, it is easily seen that $\pi_1(\mathbf{E}) \simeq \mathbf{Z}$, so $\mathbf{E}$ is not simply connected.

2. The Universal Covering Category.

Barr's problem, to construct all connected limits from fibered products and equalizers, suggests the following approach. From a connected category construct a simply connected one which has the given one as a quotient in some sense, i.e. construct a universal covering category. The topological construction is to take homotopy classes of paths starting at some base point. This suggests a well-known construction from category theory, the comma category construction.

The *comma category* was introduced by Lawvere in his thesis [4] in order to express adjointness $F \dashv U : \mathbf{B} \longrightarrow \mathbf{A}$ as an isomorphism of categories $(F, \mathbf{B}) \cong (\mathbf{A}, U)$, thus giving a precise meaning to the statement "a morphism $FA \longrightarrow B$ is the same as a morphism $A \longrightarrow UB$". The construction has since proved to be central. Given functors $\Phi : \mathbf{A} \longrightarrow \mathbf{C}$ and $\Psi : \mathbf{B} \longrightarrow \mathbf{C}$, the comma category comes with projections $P_1 : (\Phi, \Psi) \longrightarrow \mathbf{A}$

and $P_2 : (\Phi, \Psi) \longrightarrow \mathbf{B}$ and a natural transformation γ

$$\begin{array}{ccc} (\Phi,\Psi) & \xrightarrow{P_1} & \mathbf{A} \\ {\scriptstyle P_2}\downarrow & \Downarrow & \downarrow{\scriptstyle \Phi} \\ \mathbf{B} & \xrightarrow[\Psi]{} & \mathbf{C} \end{array}$$

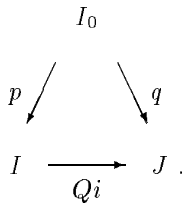
and is the universal such. This means that for any $T_1 : \mathbf{X} \longrightarrow \mathbf{A}$, $T_2 : \mathbf{X} \longrightarrow \mathbf{B}$ and natural transformation $\tau : \Phi T_1 \longrightarrow \Psi T_2$, there exists a unique functor $\Xi : \mathbf{X} \longrightarrow (\Phi, \Psi)$ such that $P_1\Xi = T_1$, $P_2\Xi = T_2$ and $\gamma\Xi = \tau$. There is an elementary description of (Φ, Ψ). Its objects are triples $(A, c : \Phi A \longrightarrow \Psi B, B)$ and a morphism $(A, c, B) \longrightarrow (A', c', B')$ is a pair (a, b) of morphisms $a : A \longrightarrow A'$ and $b : B \longrightarrow B'$ such that

$$\begin{array}{ccc} \Phi A & \xrightarrow{\Phi a} & \Phi A' \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\ \Psi B & \xrightarrow[\Psi b]{} & \Psi B' \ . \end{array}$$

Let $\mathbf{I}$ be a connected category and choose an object I_0 of $\mathbf{I}$ which will remain fixed throughout this section. We define the *universal covering category* $U\mathbf{I}$ of $\mathbf{I}$ by the comma category construction

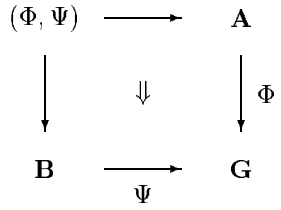
$$\begin{array}{ccc} U\mathbf{I} & \longrightarrow & \mathbf{1} \\ {\scriptstyle P}\downarrow & \Downarrow & \downarrow{\scriptstyle \ulcorner I_0 \urcorner} \\ \mathbf{I} & \xrightarrow[Q]{} & \pi(\mathbf{I}) \ . \end{array}$$

Thus an object of $U\mathbf{I}$ is a morphism $p : I_0 \longrightarrow I$ in $\pi_1(\mathbf{I})$; a morphism $i : p \longrightarrow q$ in $U\mathbf{I}$ is a morphism i of $\mathbf{I}$ such that



Our first objective is to show that $U\mathbf{I}$ is simply connected, and to this end we have the following result.

THEOREM 1. *π_1 preserves comma objects of the form*

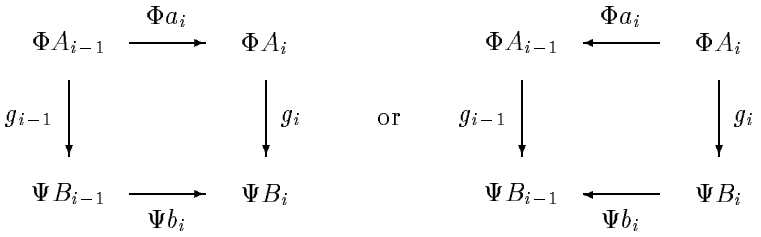


where $\mathbf{G}$ *is a groupoid.*

Proof. There is a canonical functor $\Theta : \pi_1(\Phi, \Psi) \longrightarrow (\pi_1\Phi, \pi_1\Psi)$ induced by the universal property of the comma object. Since $\mathbf{G}$ is a groupoid, $\mathbf{G} \xrightarrow{\cong} \pi_1\mathbf{G}$, and if we make the identification, the objects of $\pi_1(\Phi, \Psi)$ and $(\pi_1\Phi, \pi_1\Psi)$ are the same. Θ is the identity on objects. A morphism from (A, g, B) to (A', g', B') in $\pi_1(\Phi, \Psi)$ is an equivalence class of words

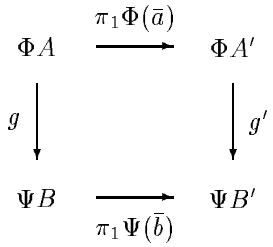
$$(a_n, b_n)^{\epsilon_n} \cdots (a_2, b_2)^{\epsilon_2} (a_1, b_1)^{\epsilon_1}$$

where



depending on whether $\epsilon_i = +1$ or -1. In either case, $g_i(\Phi a_i)^{\epsilon_i} = (\Psi b_i)^{\epsilon_i} g_{i-1}$. We also require that $(A, g, B) = (A_0, g_0, B_0)$ and $(A', g', B') = (A_n, g_n, B_n)$.

On the other hand, a morphism in $(\pi_1\Phi, \pi_1\Psi)$ is a pair $(\bar{a}, \bar{b})$ of equivalence classes of words $\bar{a} = a_n^{\epsilon_n} \cdots a_2^{\epsilon_2} a_1^{\epsilon_1}$, $\bar{b} = b_m^{\eta_m} \cdots b_2^{\eta_2} b_1^{\eta_1}$ such that



i.e. $g'\Phi(a_n)^{\epsilon_n} \cdots \Phi(a_2)^{\epsilon_2}\Phi(a_1)^{\epsilon_1} = \Psi(b_m)^{\eta_m} \cdots \Psi(b_2)^{\eta_2}\Psi(b_1)^{\eta_1} g$.

The effect of Θ on morphisms is now clear:

$$\Theta((a_n, b_n)^{\epsilon_n} \cdots (a_2, b_2)^{\epsilon_2}(a_1, b_1)^{\epsilon_1}) = (a_n^{\epsilon_n} \cdots a_2^{\epsilon_2} a_1^{\epsilon_1}, b_n^{\epsilon_n} \cdots b_2^{\epsilon_2} b_1^{\epsilon_1}).$$

It is not necessary to check that Θ is well defined; it is by virtue of the universal property of comma objects.

Θ has an inverse Λ which is necessarily the identity on objects. On morphisms

$$\Lambda(\bar{a}, \bar{b}) = (a_n, 1_{B'})^{\epsilon_n} \cdots$$
$$\ldots (a_2, 1_{B'})^{\epsilon_2}(a_1, 1_{B'})^{\epsilon_1}(1_A, b_m)^{\eta_m} \cdots (1_A, b_2)^{\eta_2}(1_A, b_1)^{\eta_1}.$$

The domain of $(a_{i+1}, 1_{B'})^{\epsilon_{i+1}}$ which is the codomain of $(a_i, 1_{B'})^{\epsilon_i}$ is $g'_i = g'(\Phi a_n)^{\epsilon_n} \cdots (\Phi a_{i+1})^{\epsilon_{i+1}}$, $0 \leq i \leq n$. The domain of $(1_A, b_{j+1})$ which

is the codomain of $(1_A, b_j)$ is $g_j = (\Psi b_j)^{\eta_j} \cdots (\Psi b_1)^{\eta_1} g$, $0 \leq j \leq m$. The only compatibility which must be checked is that the domain of $(a_1, 1_{B'})^{\epsilon_1}$ is the same as the codomain of $(1_A, b_m)^{\eta_m}$, and this follows because $(\bar{a}, \bar{b})$ is a morphism $(A, g, B) \longrightarrow (A', g', B')$.

That Λ is well defined is easily checked by observing that it respects the basic instances (i)-(iv) of the congruence given in section 1. For example, if $(\bar{a}, \bar{b})$ and $(\bar{a}', \bar{b}')$ differ only in that $a_i^1 a_{i+1}^1$ in $\bar{a}$ is replaced by $(a_i a_{i+1})^1$ in $\bar{a}'$, then $\Lambda(\bar{a}, \bar{b})$ and $\Lambda(\bar{a}', \bar{b}')$ differ only in that the first has factors $(a_i, 1_{B'})^1 (a_{i+1}, 1_{B'})^1$ whereas the second has $(a_i a_{i+1}, 1_{B'})^1$ so that $\Lambda(\bar{a}, \bar{b}) \sim \Lambda(\bar{a}', \bar{b}')$. The other instances are equally straightforward.

Because $a^\epsilon 1^{\epsilon'} \sim a^\epsilon$ it follows that $\Theta\Lambda = 1_{(\pi_1\Phi, \pi_1\Psi)}$. To see that $\Lambda\Theta = 1_{\pi_1(\Phi,\Psi)}$, observe that

$$(1, b)(a, 1) = (a, b) = (a, 1)(1, b).$$

These equalities are not as innocent as they look. In order to factor (a, b) as $(1, b)(a, 1)$ one uses the fact that $\mathbf{G}$ is a groupoid. The other relations hold quite generally in any comma category.

This completes the proof: it is not necessary to check functoriality of Λ as it is inverse to the functor Θ. $\diamondsuit$

REMARK 2. We have given the proof in such detail because we consider this to be the main result of the paper. We were not able to get a proof based on the universal property of π_1. Rather, we consider the result as a basic extra property of π_1, to be used in conjunction with the universal property.

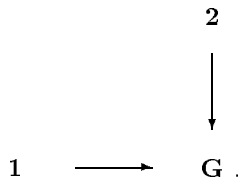
REMARK 3. The condition that $\mathbf{G}$ be a groupoid is necessary. Indeed if G and G' are two objects of $\mathbf{G}$, the comma category

$$\begin{array}{ccc} \mathbf{G}(G, G') & \longrightarrow & \mathbf{1} \\ \downarrow & \Downarrow & \downarrow{\scriptstyle G} \\ \mathbf{1} & \underset{G'}{\longrightarrow} & \mathbf{G} \end{array}$$

is the discrete category on the hom set $\mathbf{G}(G, G')$. Applying π_1 leaves this unchanged but the new comma category is the hom set $(\pi_1\mathbf{G})(G, G')$. Thus

if π_1 preserves all comma objects of functors into $\mathbf{G}$, $\mathbf{G}$ must be isomorphic to $\pi_1\mathbf{G}$, i.e. it must be a groupoid.

REMARK 4. Since we assume that $\mathbf{G}$ is a groupoid, any natural transformation between functors into it is a natural isomorphism. Thus the comma category is also the pseudo-pullback. The pseudo-pullback is similar to the comma category described at the beginning of the section, except that the natural transformations γ and τ are required to be natural isomorphisms. Its objects are triples (A, c, B) where c is an isomorphism. The rest is the same. But the result is not true for general pseudo-pullbacks either. It is again necessary that $\mathbf{G}$ be a groupoid. To see this, one must consider pseudo-pullbacks of diagrams of the form



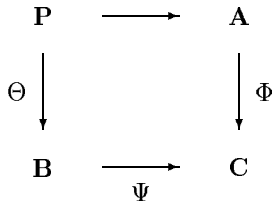
We leave details to the sceptical reader.

REMARK 5. Properties similar to this result have been considered in the literature for π_0. See [2] and references there.

REMARK 6. This is a nice example of how category theory combines the topological and the algebraic. It is not clear what condition should replace that of being a groupoid if one wished to prove a similar result for pullbacks or homotopy pullbacks of spaces.

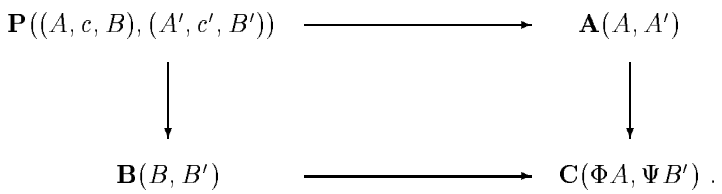
PROPOSITION 1. *A pseudo-pullback of an equivalence of categories is again an equivalence.*

Proof: Let



be a pseudo-pullback and Φ an equivalence functor.

For objects (A, c, B) and (A', c', B') of $\mathbf{P}$, the hom set can be described as a pullback in **Set**



The map on the right is a composite

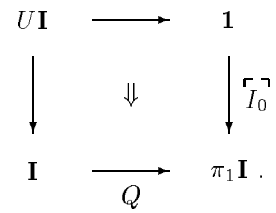
$$\mathbf{A}(A, A') \xrightarrow{\Phi} \mathbf{C}(\Phi A, \Phi A') \xrightarrow{\mathbf{C}(\Phi A, c')} \mathbf{C}(\Phi A, \Psi B'),$$

and the bottom map is similar. Since c' is an isomorphism, $\mathbf{C}(\Phi A, c')$ is a bijection, and since Φ is full and faithful, the first map (also called Φ) is a bijection. Thus the composite is a bijection, so its pullback is also one and therefore Θ is full and faithful.

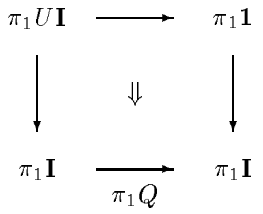
Since Φ is essentially surjective on objects, for any B in $\mathbf{B}$ there exist A in $\mathbf{A}$ and an isomorphism $c : \Phi A \longrightarrow \Psi B$ in $\mathbf{C}$. Then (A, c, B) is an object of $\mathbf{P}$ and $\Theta(A, c, B) = B$, i.e. Θ is surjective on objects. Thus Θ is an equivalence. ♢

COROLLARY 1. *The universal covering category, $U\mathbf{I}$, of a connected category $\mathbf{I}$ is simply connected.*

Proof. Recall that $U\mathbf{I}$ is defined as the comma category



By theorem 1,



is also a comma square, and since $\pi_1\mathbf{I}$ is a groupoid, it is a pseudo-pullback. $\pi_1 Q$ is an isomorphism and so an equivalence. Thus by proposition 1, $\pi_1 U\mathbf{I} \longrightarrow \pi_1\mathbf{1} = \mathbf{1}$ is an equivalence of categories, i.e. $U\mathbf{I}$ is simply connected. ♢

EXAMPLE 2. Let $\mathbf{G}$ be a group considered as a one-object category. Then $\pi_1\mathbf{G} \cong \mathbf{G}$ so the universal covering category is

$$\begin{array}{ccc} */\mathbf{G} & \longrightarrow & \mathbf{1} \\ {\scriptstyle P}\downarrow & \Downarrow & \downarrow{\scriptstyle *} \\ \mathbf{G} & \xrightarrow[1_{\mathbf{G}}]{} & \mathbf{G}\ . \end{array}$$

$*/\mathbf{G}$ is the chaotic category with the elements of $\mathbf{G}$ as objects. The functor P takes the unique morphism $g \longrightarrow h$ to $hg^{-1} : * \longrightarrow *$ in $\mathbf{G}$. So we see here very clearly that $U\mathbf{G}$ is simply connected and covers $\mathbf{G}$.

EXAMPLE 3. Let $\mathbf{D}$ be a connected graph and $F\mathbf{D}$ the free category generated by $\mathbf{D}$. The morphisms of $F\mathbf{D}$ are paths

$$D_0 \longrightarrow D_1 \longrightarrow D_2 \longrightarrow \cdots \longrightarrow D_n\ .$$

The morphisms of $\pi_1 F\mathbf{D}$ can be represented by reduced paths

$$D_0 \longleftrightarrow D_1 \longleftrightarrow D_2 \longleftrightarrow \cdots \longleftrightarrow D_n$$

where $\longleftrightarrow$ means an edge $\longrightarrow$ or $\longleftarrow$, where there are no occurrences of

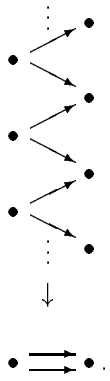
$$D \xrightarrow{x} D' \xleftarrow{x} D \quad \text{nor} \quad D' \xleftarrow{x} D \xrightarrow{x} D'.$$

Choose a base object D_0. Then $UF\mathbf{D}$ has as objects all such reduced paths $p : D_0 \longrightarrow D$. There is at most one morphism $p \longrightarrow q$ between any two paths and there is one if and only if $qp^{-1} \in F\mathbf{D}$, i.e. if and only if, except for a common initial part, q consists of forward arrows and p of backward arrows.

E.g. Let $\mathbf{D} = \bullet\circlearrowright$. Then $F\mathbf{D} = \mathbf{N}$ the monoid, $\pi_1 F\mathbf{D} = \mathbf{Z}$ the group, and $UF\mathbf{D} = \mathbf{Z}$ the poset.

The standard example of the circle covered by a helix also has an analogue here. Let $\mathbf{D} = \{\alpha, \beta : 0 \rightrightarrows 1\}$. Then $F\mathbf{D} = \mathbf{D}$ (with identities adjoined). In $\pi_1 F\mathbf{D}$ the reduced paths from 0 to itself are $\{(\beta^{-1}\alpha)^n | n \in \mathbf{Z}\}$ and from 0 to 1 they are $\{\alpha(\beta^{-1}\alpha)^n | n \in \mathbf{Z}\}$, and so on. In any case we see that $\pi_1 F\mathbf{D} \simeq \mathbf{Z}$ as mentioned before. If we choose 0 as base object, $UF\mathbf{D}$

has as objects $(\beta^{-1}\alpha)^n$ and $\alpha(\beta^{-1}\alpha)^n$, i.e. two copies of $\mathbf{Z}$. There is a morphism over α, $(\beta^{-1}\alpha)^n \longrightarrow \alpha(\beta^{-1}\alpha)^n$ and a morphism over β, $(\beta^{-1}\alpha)^{n+1} \longrightarrow \alpha(\beta^{-1}\alpha)^n$. Thus we have the picture of $UF\mathbf{D} \longrightarrow F\mathbf{D}$:



3. Connected Limits.

Let $\mathbf{I}$ be a connected category and I_0 a fixed object of $\mathbf{I}$. Let G be the group of endomorphisms of I_0 in $\pi_1\mathbf{I}$, $G = (\pi_1\mathbf{I})(I_0, I_0)$. Let $U\mathbf{I}$ be the universal covering category of $\mathbf{I}$, based at I_0. There is a right action of G on $U\mathbf{I}$. For each $g \in G$ we have a functor

$$U\mathbf{I} \xrightarrow{R_g} U\mathbf{I}$$

$R_g(p) = pg$ on objects, and for $i : p \longrightarrow q$ in $U\mathbf{I}$, $R_g(i) = i : pg \longrightarrow qg$.

PROPOSITION 2. *The quotient of $U\mathbf{I}$ by this action (i.e. the colimit in* **Cat***) is $P : U\mathbf{I} \longrightarrow \mathbf{I}$.*

Proof. Since P selects the codomain of a path, we see that $PR_g = P$ for all g.

Let $F : U\mathbf{I} \longrightarrow \mathbf{A}$ be such that $FR_g = F$ for all $g \in G$. We must show that there is a unique $H : \mathbf{I} \longrightarrow \mathbf{A}$ such that $HP = F$. Since $\mathbf{I}$ is connected, for every I there is a morphism $p : I_0 \longrightarrow I$ in $\pi_1\mathbf{I}$, i.e. an object p in $U\mathbf{I}$ such that $P(p) = I$. Thus P is onto on objects. P is also onto on morphisms (but not full!) since for $i : I \longrightarrow I'$ in $\mathbf{I}$, we also have $i : p \longrightarrow ip$ in $U\mathbf{I}$ and $P(i : p \longrightarrow ip) = i$. Thus if H exists it is

necessarily unique. $H(I)$ must be $F(p)$ and $H(i)$ must be $F(i : p \longrightarrow ip)$. If $q : I_0 \longrightarrow I$ is another morphism in $\pi_1\mathbf{I}$, then $p^{-1}q : I_0 \longrightarrow I_0$ is an element of G and $R_{p^{-1}q}(p) = q$ so $F(p) = F(q)$ and $F(i : p \longrightarrow ip) = F(i : q \longrightarrow iq)$. Thus H is well defined, and is now easily seen to be functorial. $\diamondsuit$

Such colimit formulas in **Cat** correspond to general constructions of limits and colimits in category theory. To see how this comes about, let $\mathbf{A}$ be a category and let $\mathbf{Cat}/\mathbf{A}$ be the category of categories over $\mathbf{A}$. There is a functor $\Delta : \mathbf{A} \longrightarrow \mathbf{Cat}/\mathbf{A}$ which takes A to the slice category

$$\Delta(A) = (\mathbf{A}/A \xrightarrow{\partial_0} \mathbf{A}).$$

For a diagram $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}$, a morphism

$$\begin{array}{ccc} \mathbf{I} & \longrightarrow & \mathbf{A}/A \\ & {\scriptstyle\Gamma}\searrow \quad \swarrow{\scriptstyle\partial_0} & \\ & \mathbf{A} & \end{array}$$

is exactly the same as a cocone

$$\gamma : \Gamma \longrightarrow A.$$

Thus the partial left adjoint to Δ exists at Γ if and only if Γ has a colimit, and the value of the partial left adjoint is that colimit.

Partial left adjoints are part of the categorical folklore. A functor $U : \mathbf{B} \longrightarrow \mathbf{A}$ has a left adjoint at A if the functor $\mathbf{A}(A, U-) : \mathbf{B} \longrightarrow \mathbf{Set}$ is representable. If we choose a representing object FA for each such A, F extends uniquely to a functor on a *full* subcategory $\mathbf{A}_0$ of $\mathbf{A}$. F preserves colimits in the following sense: if $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}_0$ has a colimit A in $\mathbf{A}$, then FA is defined if and only if $F\Gamma$ has a colimit in $\mathbf{B}$ and then

$$\varinjlim F\Gamma \cong F \varinjlim \Gamma.$$

Thus if we have a colimit in **Cat**, say $\varinjlim_J \Phi(J) \cong \mathbf{I}$ for $\Phi : \mathbf{J} \longrightarrow \mathbf{Cat}$, then for any $\mathbf{A}$ and any diagram $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}$ we get a diagram of diagrams $\Gamma_J : \Phi(J) \longrightarrow \mathbf{A}$ by composition with the injections, and Γ is the colimit of the Γ_J in $\mathbf{Cat}/\mathbf{A}$. Since $\varinjlim$ is a partial left adjoint to $\Delta : \mathbf{A} \longrightarrow \mathbf{Cat}/\mathbf{A}$, we see that

$$\varinjlim_I \Gamma(I) \cong \varinjlim_J \varinjlim_K \Gamma_J(K)$$

and provided each $\varinjlim_K \Gamma_J(K)$ exists, the left side exists if and only if the right side does.

Of course this discussion can be dualized. Colimit formulas in **Cat** also give rise to general constructions of limits. So if $\varinjlim_J \Phi(J) \cong \mathbf{I}$ then

$$\varprojlim_I \Gamma(I) \cong \varprojlim_J \varprojlim_K \Gamma_J(K).$$

Perhaps the simplest example of such a formula is

$$A \times B \times C \cong (A \times B) \times C$$

which comes from the formula

$$\mathbf{3} = \mathbf{2} + \mathbf{1}$$

in **Cat**, where **3**, **2**, **1** represent discrete categories with 3, 2, 1 objects respectively.

As the whole paper is stated in terms of limits it is this last version which we use.

THEOREM 2. *A connected limit can be computed as a simply connected limit followed by the fixed object of a group action (the limit of a diagram indexed by a group).*

Proof. Apply the above discussion to the result of Proposition 2. ♢

COROLLARY 2. *Connected limits can be computed using fibered products and fixed objects of group actions.* ♢

PROPOSITION 3. *If a category* **I** *has a weak initial object, then* **I** *limits can be computed using equalizers and intersections.*

Proof. Let W be a weak initial object for **I**, i.e. $\mathbf{I}(W, I) \neq \emptyset$ for all $I \in \mathbf{I}$. Let $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}$ where **A** has equalizers and intersections. For every pair of arrows $\alpha, \beta : W \rightrightarrows I$ in **I**, let

$$A_{\alpha,\beta} \rightarrowtail \Gamma(W) \rightrightarrows \Gamma(I)$$

be an equalizer, and let $A = \bigcap A_{\alpha,\beta}$. Then A is the limit of Γ. For any I there is at least one morphism $\alpha : W \longrightarrow I$ so we have γI :

$A \rightarrowtail \Gamma(W) \longrightarrow \Gamma(I)$. γI is independent of the choice of α as $A \subseteq A_{\alpha,\beta}$ for all $\alpha, \beta : W \longrightarrow I$. It follows that γ is a cone $A \longrightarrow \Gamma$. For any other cone $\kappa : B \longrightarrow \Gamma$, $\Gamma(\alpha)\kappa(W) = \kappa(I) = \Gamma(\beta)\kappa(W)$ for all α, β as above, so $\kappa(W)$ factors through each $A_{\alpha,\beta}$ and thus through A, as $\kappa(W) = \gamma(W)a$. Then

$$\kappa(I) = \Gamma(\alpha)\kappa(W) = \Gamma(\alpha)\gamma(W)a = \gamma(I)a.$$

Finally a is unique as $\gamma(W)$ is monic. ♢

COROLLARY 3. *Connected limits can be computed using fibered products and equalizers.*

Proof. This follows immediately from Corollary 1 and Proposition 3 as a group has a weak initial object, and an intersection is a fibered product. ♢

The fact that we chose a base object in order to construct the universal covering category, on which everything else depends, is somewhat unsatisfactory. To be sure, any other choice would give an isomorphic category, but the isomorphism is not unique. The methodology of category theory suggests that we search for a choice-free construction. This is what led to the fundamental *groupoid* rather than fundamental group. The solution lies in Kan extensions and the fact that, in some sense, $Q : \mathbf{I} \longrightarrow \pi_1\mathbf{I}$ is the family of all universal covering categories with the action of $\pi_1\mathbf{I}$ built in. We end this section with a theorem which gives a precise formulation of such a base free theory.

THEOREM 3. *Let* $\mathbf{I}$ *be any category and* $\mathbf{A}$ *a category with fibered products. Then* $\mathbf{A}$ *has right Kan extensions along* $Q : \mathbf{I} \longrightarrow \pi_1\mathbf{I}$, *i.e. the functor*

$$\mathbf{A}^{\pi_1\mathbf{I}} \xrightarrow{Q^*} \mathbf{A}^{\mathbf{I}}$$

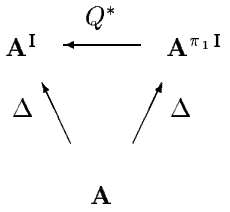
obtained by composition with Q, *has a right adjoint,* Ran_Q.

Proof. For $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}$, $Ran_Q\Gamma(I_0)$ is given by the limit of the diagram

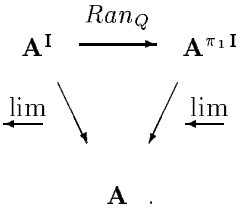
$$(Q, I_0) \xrightarrow{P} \mathbf{I} \xrightarrow{\Gamma} \mathbf{A}$$

if these limits exist. These limits *do* exist as (Q, I_0) is simply connected; it is the universal covering category of the component of I_0 in $\mathbf{I}$. ♢

The relation to limits is the following. We have a commutative diagram



which gives, on passing to right adjoints

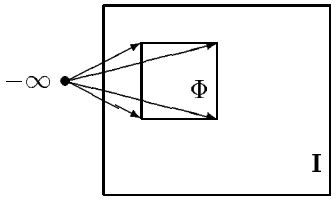


If $\mathbf{I}$ is connected, $\pi_1 \mathbf{I}$ has a weak initial object and $\varprojlim_{\pi_1 \mathbf{I}}$ can be computed using equalizers and fibered products (Proposition 3).

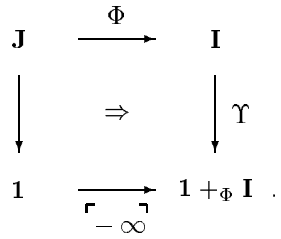
4. Direct Constructions.

Lest someone think that we have obscured matters with our "slick" conceptual proof, we give a direct construction of connected limits from equalizers and fibered products. The direct approach has the added advantage of working also for finite limits, which our method does not.

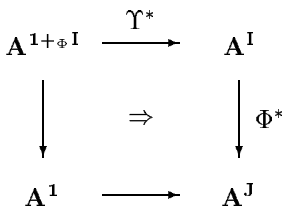
Let $\Phi : \mathbf{J} \longrightarrow \mathbf{I}$ be a functor. We construct a new category $\mathbf{1} +_\Phi \mathbf{I}$ by adding to $\mathbf{I}$ a cone on Φ:



We add a new object $-\infty$, and for every J in $\mathbf{J}$ we add a new morphism $\iota_J : -\infty \longrightarrow \Phi J$ and then we impose the equations $(\Phi j)\iota_J = \iota_{J'}$ for each $j : J \longrightarrow J'$ in $\mathbf{J}$. What we have described is a *cocomma category* construction



It has a universal property dual to that of comma category. This means that for any category $\mathbf{A}$,



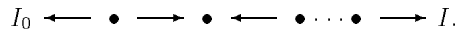
is a comma square. In concrete terms, a diagram $\mathbf{1} +_\Phi \mathbf{I} \longrightarrow \mathbf{A}$ is the same as a diagram $\Gamma : \mathbf{I} \longrightarrow \mathbf{A}$ together with a cone $\alpha : A \longrightarrow \Gamma\Phi$.

PROPOSITION 4. *If* $\mathbf{A}$ *has* $\mathbf{J}$ *limits, then* $\mathbf{A}$ *has right Kan extensions along* Υ, *i.e.* $\Upsilon^* : \mathbf{A}^{\mathbf{1}+_\Phi \mathbf{I}} \longrightarrow \mathbf{A}^{\mathbf{I}}$ *has a right adjoint,* Ran_Υ.

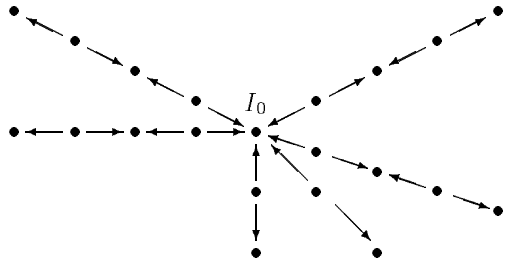
Proof. If we view $\mathbf{A}^{\mathbf{1}+_\Phi \mathbf{I}}$ as the comma category, then its objects are pairs (Γ, α) and $\Upsilon^*(\Gamma, \alpha) = \Gamma$. It is an easy calculation to see that $Ran_\Upsilon(\Gamma)$ is given by $(\Gamma, \gamma : \varprojlim \Gamma\Phi \longrightarrow \Gamma\Phi)$. ♢

In view of the discussion at the end of section 3, we can compute $\mathbf{I}$ limits, by first taking Ran_Υ, and then a $\mathbf{1} +_\Phi \mathbf{I}$ limit (assuming Ran_Υ exists).

Let $\mathbf{I}$ be connected. Choose some base object I_0 and for every other object I choose a path



Construct a graph by joining disjoint copies of each of these paths at the common node I_0



and let $\mathbf{J}$ be the free category generated by the graph (as there are no composable pairs, it suffices to add the identities). There is an obvious functor $\Phi : \mathbf{J} \longrightarrow \mathbf{I}$ which is onto on objects. Thus the category $\mathbf{1} +_\Phi \mathbf{I}$ has a weak initial object. Furthermore $\mathbf{J}$ is easily seen to be simply connected; any path of arrows starting at I_0 must retrace its steps in order to return to I_0.

Let $\mathbf{A}$ be a category with fibered products and equalizers. Then, by proposition 4, $\mathbf{A}$ has right Kan extensions along $\Upsilon : \mathbf{1} +_\Phi \mathbf{I} \longrightarrow \mathbf{I}$ as simply connected limits exist in $\mathbf{A}$. So $\mathbf{I}$ limits can be constructed as $\mathbf{1} +_\Phi \mathbf{I}$ limits of a Ran_Υ. Since $\mathbf{1} +_\Phi \mathbf{I}$ has a weak initial object, limits of that sort can be constructed in $\mathbf{A}$ using equalizers and fibered products. Each of these steps can be effectively carried out and in the case where $\mathbf{I}$ is finite or finitely generated all limits are finite.

Putting all this together we can give an explicit construction for the limit of a (finite) connected diagram.

1) Choose a vertex of the diagram, and for each other vertex choose a path of edges joining it to the first one.

2) Using (pullbacks, or in the infinite case) fibered products, take step by step the limit of the star-shaped diagram so obtained. This gives something like a cone for the original diagram except that it may have several arrows into some vertices and some of the triangles which should commute, don't.

3) Take the equalizer of each pair of maps corresponding to multiple edges or defective triangles.

4) Take the intersection of all these equalizers.

References

[1] Carboni A. and Johnstone P., *Connected limits, familially representable functors and Artin glueing*, Proceedings of the CTCS-5 meeting, Amsterdam (1993), Appl. Math. Structures in Comp. Sci. (to appear).
[2] Carboni A. and Paré R., *The Monotone-Light Factorization*, submitted.
[3] Cockett R., *List-Arithmetic Distributive Categories: Locoi*, J. Pure and Appl. Alg. **66** (1990) pp. 1-29.
[4] Lawvere F. W., *Functorial Semantics of Algebraic Theories*, Ph.D. Dissertation, 1963, Columbia University.
[5] Mac Lane S., *Categories for the Working Mathematician*, Springer-Verlag, New York 1971.
[6] Paré R., *Simply Connected Limits*, Canad. J. of Math. Vol. XLII, No 4, 1990, pp. 731-746.