MINIMAL SPACES, MAXIMAL PRE-ANTIS (*)

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SOMMARIO. - *Cosa dovremmo intendere dicendo che uno spazio topologico è "minimale" in una famiglia di spazi? Ovviamente la risposta dipende dalla natura della famiglia di spazi e, criticamente, dalla relazione d'ordine considerata tra gli spazi in oggetto che, nei casi significativi non è un ordinamento parziale ma solo un quasi ordinamento. Identificheremo due modi per definire la minimalità degli oggetti in questo contesto; il primo è adottato da Ginsburg e Sands mentre l'altro è (apparentemente) più debole.*
Sia P un invariante topologico. Uno spazio si dirà "anti-P" (in accordo con Bankston) quando i suoi soli P sottospazi sono quelli la cui cardinalità da sola garantisce che essi sono P. Quando anti-P è equivalente a Q diremo che P è un "pre-anti" per Q. Esamineremo le relazioni tra l'esistenza di vari pre-anti estremali per Q e l'esistenza, in particolari famiglie, di spazi che sono minimali in un senso opportuno.

SUMMARY. - *What should we mean by saying that a topological space is "minimal" among a family of spaces? Obviously it depends on the nature of the family and, critically, on the ordering relation being considered between the spaces in question which, in important cases, is not a partial order but only a quasi-order. We identify two ways to assign minimality to objects in such a context; one is that adopted by Ginsburg and Sands, the second is (apparently) weaker.*
Let P denote a topological invariant. A space is called "anti-P" (following Bankston) when its only P subspaces are those whose cardinalities alone guarantee that they must be P. When anti-P is equivalent to Q then P is called a "pre-anti" for Q. We shall explore relationships between the existence of various extremal pre-antis for Q and the occurrence, within particular families, of spaces that are minimal in one sense or the other.

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1. Introduction.

The mechanisms by which the topological invariants *connected, compact, perfect* give rise to diametrically opposed invariants *totally disconnected, pseudofinite* [7], *scattered* are in essence identical. This was realised by Paul Bankston who used these instances as guidelines for a general procedure [1], the "anti-" operation, which may be applied to any invariant in topology (and in other contexts: see [5]). To be explicit: given a property P, one first identifies the three classes of cardinals

$$\begin{aligned} \mathrm{spec}(P) &= \{\alpha : \text{every space on } \alpha \text{ points is } P\}, \\ \mathrm{proh}(P) &= \{\alpha : \text{no space on } \alpha \text{ points is } P\}, \\ \mathrm{ind}(P) &= \{\alpha : \text{some but not all spaces on } \alpha \text{ points are } P\}. \end{aligned}$$

Then the "total negation of P" is the property *anti-P* described thus: X is anti-P iff whenever Y is a subspace of X and Y is a P space, then the cardinality $|Y|$ of Y belongs to $\mathrm{spec}(P)$.

In other words, an anti-P space is one which forbids its subspaces to be P, necessarily excepting those whose cardinalities oblige them to be P.

The correspondence $P \to$ anti-P is not *onto* since, for example, a non-hereditary property can never be of the form anti-P. If, however, we restrict our attention to the class of hereditary properties then it is 'very nearly onto' (see [1] for details); such a restriction of the discussion may therefore be helpful in investigating certain aspects of the total negation operation. In contrast, no reasonable limitation on the generality of discourse will force it to be *one-to-one*: Matier and McMaster [3] have given a simple example of an invariant which, even in the context of *finite* topological spaces, coincides with anti-P for almost 1,500,000 distinct hereditary P! Consequently, a major objective in any attempt to achieve full understanding of Bankston's operation will need to be an investigation, for each P, of the structure of the class

$$\mathrm{anti}^{-1}(P) := \{Q : \text{anti-}Q = P\}$$

of so-called *pre-antis* for P; or, at least, of a significant portion of that class, for example the hereditary pre-antis.

Such an investigation was carried out in [3] for the special case where the least cardinal in $\mathrm{ind}(P)$ is a positive integer n, the main conclusions (for hereditary P) being:

i) P possesses hereditary pre-antis iff each non-P space has an n-element non-P subspace;

ii) anti-P is then the (implicatively) weakest hereditary pre-anti for P, while the strongest is composed of the n-element non-P spaces together with all spaces on fewer than n elements;

iii) if Q_1 and Q_2 are hereditary pre-antis for P then so is any hereditary property intermediate in logical strength between Q_1 and Q_2.

Taken together, these findings furnish a complete description of the hereditary pre-antis of P, as forming an interval $[h_{\min}, h_{\max}]$ in the implicative lattice of hereditary topological invariants. Notice the importance in (i) and (ii) of the *minimal* character of the n-element non-P spaces: minimal not merely in the cardinality sense, but also in the sense of being embeddable into every non-P space.

A parallel analysis of the case where the least member of ind(P) is $\aleph_o$ is facilitated by the elegant and surprising result on minimal countably infinite spaces obtained by Ginsburg and Sands [2]. They listed five simple topological spaces on $\aleph_o$ points, each of which is 'minimal' in the sense of being homeomorphic to all of its equicardinal subspaces, and they established that *every* infinite space contains an embedded copy of at least one of these five. Now if P has $\aleph_o$ as its 'least indecisive cardinal' and possesses any hereditary pre-antis, it has been shown [4] that they again constitute an interval $[h_{\min}, h_{\max}]$ of hereditary invariants, where $h_{\min}$ is anti-P and $h_{\max}$ comprises those of the five Ginsburg and Sands spaces that are not P, together with all finite spaces. Again the *minimality with respect to embeddability* of certain non-P spaces is the key idea of the demonstration.

Turning now to the case where the least member of ind(P) exceeds $\aleph_o$, we find that the quest for further such results is hampered not only by the absence of a 'Ginsburg and Sands' analogue for uncountable spaces but also by the difficulty of interpreting the term 'minimal'. Since the ordering relation between topological spaces that is implicit in the "anti-" operation is the one *sub* described by

$$X \text{ sub } Y \text{ iff } X \text{ is homeomorphic to a subspace of } Y$$

the root of this difficulty lies in the fact that *sub* is only a quasi-order rather than a partial order: that is to say, it is reflexive and transitive but not generally anti-symmetric - for instance, $(0,1)$ sub $[0,1]$ and $[0,1]$ sub $(0,1)$ yet $(0,1)$ and $[0,1]$ are substantially different spaces. (Of course *sub* does act as a partial order on the (homeomorphism classes of) *finite* topological spaces, which is why the first case discussed above is so relatively straightforward.) We have addressed the question of minimality in a quasi-ordered setting in a previous article [6] from which the following definitions and observations are taken.

2. Minimality and the 'anti'-operation.

DEFINITIONS. Let $\mathcal{F}$ denote a family of topological spaces, X an element of $\mathcal{F}$ and *sub* as defined above. We call X
strongly quasi-minimal in $\mathcal{F}$ if Y sub $X, Y \in \mathcal{F}$ imply Y homeomorphic to X,
weakly quasi-minimal in $\mathcal{F}$ if Y sub $X, Y \in \mathcal{F}$ imply X sub Y.

The abbreviations sqm, wqm will be employed. Further, if $\mathcal{G}$ is a subfamily of $\mathcal{F}$, we say that

$\mathcal{G}$ *supports* $\mathcal{F}$ if for each $X \in \mathcal{F}$ there is $Y \in \mathcal{G}$ such that Y sub X.

NOTES.

i) sqm in $\mathcal{F}$ implies wqm in $\mathcal{F}$.

ii) The converse to (i) is not generally valid.

iii) In any $\mathcal{F}$ which is partially-ordered by sub the sqm, wqm and minimal elements coincide.

iv) [2] Take $\mathcal{T}$ as the family of all infinite topological spaces, $\mathcal{GS}$ the class of five spaces singled out by Ginsburg and Sands. Then the sqm/wqm elements of $\mathcal{T}$ are precisely the members of $\mathcal{GS}$. Also $\mathcal{GS}$ supports $\mathcal{T}$.

Now let P be a given hereditary invariant. Our objective here is to indicate how the above ideas can be used to explore the questions:

1) Does P possess pre-antis of a particular kind?

2) When does a class of pre-antis for P possess maximum/maximal elements?

3) How do such 'extremal' pre-antis relate to the other elements of the class?

Answers to these questions naturally depend very much on the type of property P given and on the type of pre-anti sought. In the present note we shall restrict attention to an important special case by making the additional assumptions

$$\mathrm{ind}(P) \neq \phi,$$

$$\mathrm{proh}(P) = \phi.$$

(As an illustration, when P is hereditary compactness, we have $\mathrm{spec}(P) = [1, \aleph_o)$, $\mathrm{ind}(P) = [\aleph_o, \ )$, $\mathrm{proh}(P) = \phi$.)

Proposition 1. [1, page 242] *The property non-P is a pre-anti for P.*

Proposition 2. [3, prop. 3] *Call a space 'critical' when its cardinality is the least member of ind(P) and put*

$\mathcal{C} = \{Y : Y$ *is critical and every critical subspace of* Y *is non-*$P\}$.

Then P has hereditary pre-antis if and only if $\mathcal{C}$ supports non-P.

Lemma 3. *Let $\mathcal{W}$ denote the class of wqm members of non-P. If P has a maximal pre-anti then $\mathcal{W}$ supports non-P.*

Proof. Suppose that $\mathcal{W}$ does not support non-P, that is, there is a non-P space X such that whenever Y sub X and Y is non-P, there is a non-P space Z_Y sub Y where no subspace of Z_Y is homeomorphic to Y. We shall show that the assertion 'Q is a maximal pre-anti for P' leads to a contradiction.

Notice first that $\mathrm{spec}(Q) = \phi$: for otherwise, those Q spaces whose cardinalities belong to $\mathrm{ind}(Q)$ would form a pre-anti for P strictly stronger than Q. It follows that any space possessing a Q subspace is non-P. Now as X is not P, i.e. not anti-Q, we can choose Y sub X with $Y \in Q$ and $|Y| \in \mathrm{ind}(Q)$. The space Y is non-P since it contains itself as a Q subspace, so a corresponding Z_Y can be found. Define a property Q^* thus:

$$Q^* = Q \backslash \{Y\}.$$

We now verify that Q^* (which again is strictly stronger than Q) is another pre-anti for P. On the one hand, a P space cannot contain a Q^* subspace, since such a subspace would of course be Q. On the other, for any non-P space W, W is not anti-Q so there exists U sub W where U is Q; if $U \neq Y$ then U is Q^*, while if $U = Y$ then we have Z_Y sub W where Z_Y is Q^*: thus in both cases we find a Q^* subspace of W whose cardinality cannot belong to $\mathrm{spec}(Q^*) = \mathrm{spec}(Q) = \phi$, which shows that W is not anti-Q^*. Hence $P = \text{anti-}Q^*$ as claimed.

In order to access a converse to Lemma 3 it is convenient to employ the following equivalence relation $\equiv$ on $\mathcal{W}$:

$$X \equiv Y \text{ iff } X \text{ sub } Y \qquad (X, Y \in \mathcal{W}).$$

LEMMA 4. *Suppose that a subclass $\mathcal{W}'$ of $\mathcal{W}$ supports non-P and that Q is a given pre-anti for P. Each member Δ of the partition Π of $\mathcal{W}'$ induced by $\equiv$ includes a Q space. However the space $U_\Delta \in Q \cap \Delta$ is chosen, the property*

$$Q_\Pi = \{U_\Delta : \Delta \in \Pi\}$$

is a maximal pre-anti for P.

Proof. For each $\Delta \in \Pi$ select any space V_Δ in Δ. Since $P = \text{anti-}Q$, V_Δ is not anti-Q so there exists W_Δ sub V_Δ with $W_\Delta \in Q$ and $|W_\Delta| \in \mathrm{ind}(Q)$. Since W_Δ cannot be P, it follows that $W_\Delta \in \mathcal{W}$: for if Z sub W_Δ where Z is non-P then Z sub $V_\Delta \in \mathcal{W}$ so V_Δ sub Z also, yielding W_Δ sub Z. Thus $W_\Delta \in \Delta \cap Q$.

No P space can contain a Q_Π subspace since P is hereditary and $Q_\Pi \Rightarrow$ non-P, so P implies anti-Q_Π. If X is non-P we can choose W sub X where $W \in \mathcal{W}'$, identify the member Δ of Π which includes W, and observe that U_Δ sub W sub X gives U_Δ sub X. Now since U_Δ is non-P and $\mathrm{proh}(P) = \mathrm{spec}(\text{non-}P) = \phi$ we see that $|U_\Delta| \in \mathrm{ind}(P)$, so some spaces of cardinality $|U_\Delta|$ are P (and consequently cannot be Q_Π); hence $|U_\Delta| \in \mathrm{ind}(Q_\Pi)$ and X is not anti-Q_Π. This establishes Q_Π as a pre-anti for P.

If it were not maximal, there would necessarily exist a proper subclass Π' of Π such that

$$Q_{\Pi'} = \{U_\Delta : \Delta \in \Pi'\}$$

satisfies anti-$Q_{\Pi'} = P$. Select $\Delta_o \in \Pi \backslash \Pi'$; the space U_{Δ_o} is not P so it must have a subspace U_{Δ_1} which is $Q_{\Pi'}$. Then, however, $U_{\Delta_1} \equiv U_{\Delta_0}$ which forces U_{Δ_1} and U_{Δ_0} to belong to the *same* member of the partition Π, contradictory to the choice of Δ_0.

PROPOSITION 5. *The following assertions are equivalent*:
(a) $\mathcal{W}$ *supports non-*P,
(b) P *has a maximal pre-anti*,
(c) *for each pre-anti* Q *of* P *there is a maximal pre-anti stronger than* Q.

Proof. Lemma 3 shows that (b) $\Rightarrow$ (a) while Proposition 1 makes it obvious that (c) $\Rightarrow$ (b). Let us now assume that (a) holds. By applying Lemma 4 to the case $\mathcal{W}' = \mathcal{W}$ we construct the maximal pre-anti Q_Π and observe that Q_Π implies Q : hence (c).

LEMMA 6. *Let* $W \in \mathcal{W}$ *and denote by* $[W]$ *the* $\equiv$*-class which includes* W. *Then* W *is sqm in non-*P *iff* $[W] = \{W\}$.

Proof. Almost immediate from the definitions.

PROPOSITION 7. P *has a maximum pre-anti if and only if non-*P *is supported by*

$$\mathcal{S} = \{S : S \textit{ is sqm in non-}P\}.$$

Proof. Let P have a maximum pre-anti. By Lemma 3 non-P is supported by $\mathcal{W}$. If some $\equiv$-class were to include two distinct elements, Lemma 4 (with $\mathcal{W}' = \mathcal{W}$ and $Q =$ non-P) shows how to form two distinct maximal pre-antis for P, producing a contradiction. So each $\equiv$-class is a singleton and $\mathcal{S} = \mathcal{W}$ supports non-P.

Conversely if $\mathcal{S}$ supports non-P then Lemma 4 (with $\mathcal{W}' = \mathcal{S}$) and Lemma 6 jointly show that $\mathcal{S}$ itself is a pre-anti stronger than any of the pre-antis for P, that is, $\mathcal{S}$ is a maximum pre-anti.

In summary so far:

THEOREM 1. *Let* P *be a hereditary property where* $ind(P) \neq \phi$ *and* $proh(P) = \phi$. *Then* P *has*

i) *a maximal pre-anti if and only if non-*P *is supported by its wqm members*,

ii) *a maximum pre-anti if and only if non-P is supported by its sqm members.*

PROBLEM. For which properties P are wqm and sqm distinct within non-P?

As a final illustration of the ideas here employed, we quote from [6] the following extension of Proposition 2:

THEOREM 2. *Let P be a property which has hereditary pre-antis and satisfies ind$(P) \neq \phi$. Then it possesses a maximum hereditary pre-anti if and only if the class of critical non-P spaces is supported by its wqm members.*

REFERENCES

[1] BANKSTON P., *The total negation of a topological property*, Illinois J. Math. **23** (1979), 241-252.

[2] GINSBURG J. and SANDS B., *Minimal infinite topological spaces*, Amer. Math. Monthly **86** (1979), 574-576.

[3] MATIER J. and MCMASTER T.B.M., *Total negations of hereditary properties*, Suppl. Rend. Circ. Mat. Palermo **29** (1992), 563-573.

[4] MATIER J. and MCMASTER T.B.M., *Minimal countable spaces and anti-properties*, Suppl. Rend. Circ. Mat. Palermo **29** (1992), 553-561.

[5] MATIER J. and MCMASTER T.B.M., *Iteration of the "anti" operation in ordered topological spaces and in other categorical contexts,* Boll. U.M.I. **7** (1993), 369-378.

[6] MATTHEWS P.T. and MCMASTER T.B.M., *A viewpoint on minimality in topology*, to appear in Irish Math. Soc. Bull.

[7] WILANSKY A., *Between T_1 and T_2*, Amer. Math. Monthly **74** (1967), 261-266.