FINITE QUOTIENTS OF HYPERBOLIC TETRAHEDRAL GROUPS (*)

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SOMMARIO. - *Viene data una classificazione dei gruppi finiti proiettivi lineari* $PSL(2,q)$ *e* $PGL(2,q)$ *che sono quozienti del gruppo iperbolico tetraedrale associato al tetraedro iperbolico di volume minimo. Questi gruppi finiti possono essere considerati analoghi 3-dimensionali dei gruppi di Hurwitz nella dimensione 2, che sono i quozienti finiti del gruppo iperbolico triangolare (2, 3, 7) associato al triangolo iperbolico dell'volume minimo oppure, in modo equivalente, i gruppi di automorfismi dell'ordine massimo* $84(g-1)$ *di superfici iperboliche (o di Riemann) di genere* g.

SUMMARY. - *We classify the finite projective linear groups* $PSL(2,q)$ *and* $PGL(2,q)$ *which are quotients of the hyperbolic tetrahedral group belonging to the hyperbolic tetrahedron of smallest volume. These finite quotients can be considered 3-dimensional analogues of the Hurwitz groups in dimension 2 which are the finite quotients of the hyperbolic (2, 3, 7)-triangle group belonging to the hyperbolic triangle of smallest volume or equivalently, the automorphism groups of maximal order* $84(g-1)$ *of closed hyperbolic (or Riemann) surfaces of genus* g.

1. Introduction.

There exists an extensive literature on the finite groups which are quotients of the $(2,3,7)$-triangle group, see [2]. The $(2,3,7)$-*triangle group* is the group of isometries of the hyperbolic plane $\mathbb{H}^2$ consisting of the orientation preserving elements in the group generated by the reflections in the sides of a hyperbolic triangle with angles $\pi/2$, $\pi/3$ and $\pi/7$. The finite quotients of the $(2,3,7)$-triangle group are called *Hurwitz groups* and are of interest both from an algebraic and geometric point of view. Algebraically, they are the finite groups with a generating system of "minimal

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type" (which are not too special): two generators of orders 2 and 3 whose product has order 7. Geometrically, they are exactly the finite groups of automorphisms (resp. isometries) of closed Riemann (resp. hyperbolic) surfaces of genus $g > 1$ of maximal possible order $84(g-1)$. This comes from the fact that the quotient $\mathbb{H}^2/(2,3,7)$ of the hyperbolic plane $\mathbb{H}^2$ by the triangle group $(2,3,7)$ is the orientable hyperbolic 2-orbifold of smallest possible volume (or equivalently, of maximal negative Euler characteristic). Particular emphasis has been on *simple* Hurwitz groups. The most important (frequent: see [8] for a formal approach to this intuitive notion) class of simple groups are the *projective special linear* or *linear fractional groups* $PSL(2,q)$ over the Galois field $\mathbb{F}_q$ of order $q = p^n$, p prime. The Hurwitz groups of type $PSL(2,q)$ have been classified by Macbeath in [9].

In dimension 3, that is for group actions of large orders on hyperbolic 3-manifolds, the situation is more complicated. The orientable hyperbolic 3-orbifold of minimal volume is not known (and Euler characteristics are 0 in dimension 3). The orientable hyperbolic 3-orbifolds of smallest known volumes are either *tetrahedral orbifolds* , that is quotients of hyperbolic 3-space $\mathbb{H}^3$ by tetrahedral groups, or admit a 2-fold covering by a tetrahedral orbifold (see [10]). A *tetrahedral group* is the group of orientation preserving elements in the group of isometries in $\mathbb{H}^3$ generated by the reflections in the faces of a hyperbolic tetrahedron all of whose dihedral angles are of the form π/n, $n \in \mathbf{Z}_+$. In contrast to the infinitely many hyperbolic triangle groups there exist only 9 bounded hyperbolic tetrahedra of this type, as found by Lanner (see [1] or [13]). The tetrahedron $\mathcal{T}$ of smallest volume among these 9 *Lanner tetrahedra* is shown in Figure 1, where a number n at an edge denotes a dihedral angle π/n.

Figure 1

We denote the corresponding tetrahedral group by T; similarly, we have the hyperbolic tetrahedron $\mathcal{T}'$ with associated tetrahedral group T'. The main result of the present work is the classification of finite quotients of type $PSL(2,q)$ and $PGL(2,q)$ of the tetrahedral groups T and T', by torsion-free subgroups. These subgroups are then the universal covering groups of closed hyperbolic 3-manifolds with large isometry groups (containing subgroups $PSL(2,q)$ or $PGL(2,q)$), the best known one among them being the Seifert-Weber hyperbolic dodecahedral space. Similar methods apply to the remaining 7 tetrahedral groups, with some appropriate modifications of the numerical computations.

2. Preliminaries and Statement of Results.

The quotient $\mathbb{H}^3/T$ of hyperbolic 3-space by the tetrahedral group T is a closed orientable hyperbolic 3-orbifold whose universal covering group (in the sense of orbifolds, isomorphic to the orbifold fundamental group) is the tetrahedral group T; its underlying topological space is the 3-sphere S^3 and its singular set is the 1-skeleton of the tetrahedron $\mathcal{T}$ (the singular set is the projection of the fixed point sets of the nontrivial elements in T, see [11] for basic definitions about orbifolds). The singular points of an orientable 3-orbifold can be of one of the following types (the stabilizer of a point in the universal covering projecting to the given point): cyclic $\mathbf{Z}_n$, dihedral $\mathbb{D}_n = (2,2,n)$, tetrahedral $\mathbf{A}_4 = (2,3,3)$, octahedral $\mathbf{S}_4 = (2,3,4)$ or dodecahedral $\mathbf{A}_5 = (2,3,5)$, where $(2,3,n)$ denotes a triangle group, spherical for $n \leq 5$. Of course, these are just the finite subgroups of the orthogonal group $SO(3)$. For example, the vertices of $\mathcal{T}$ are of types $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbf{S}_4$ and $\mathbf{A}_5$, see Figure 1; in particular, these groups are subgroups of T. Similar remarks apply to the tetrahedral group T' associated to the tetrahedron $\mathcal{T}'$. A presentation of T resp. T' is as follows:

$$T \quad \text{resp.} \quad T' \;=\; \mathbb{D}_5 \underset{Z_5}{*} \mathbf{A}_5/\langle (a\,x)^k \rangle,$$

where $k = 4$ resp. 5 in case of T resp. T', $\mathbf{A}_5 = \langle x, y \mid x^2 = y^3 = (xy)^5 = 1\rangle$, $\mathbb{D}_5 = \langle a, b \mid a^2 = b^2 = (a\,b)^5 = 1\rangle$ and the amalgam $\mathbf{Z}_5$ is generated by $a\,b = (x\,y)^{-1}$. These presentations can be obtained in one of the following ways: either by applying Poincaré's theorem on fundamental polyhedra to the group generated by the reflections in the faces of the tetrahedron and then applying the Reidemeister-Schreier subgroup method to find a presentation of the subgroup of index 2 of orientation preserving elements, see [1] or [12]; or by computing the orbifold fundamental group $\pi_1(\mathbb{H}^3/T) \cong T$ of the quotient orbifold by either the orbifold version of Van Kampen's theorem ([7]) or by applying the Wirtinger-method for knots to the complement of the 1-skeleton of the tetrahedron, see [6, Prop.1].

Now, for a given surjection

$$\phi : T \to PSL(2,q)$$

with torsionfree kernel, we have the restriction

$$\varphi = \phi|_{\mathbf{A}_5} : \mathbf{A}_5 \to PSL(2,q)$$

which is injective. Therefore, in order to find the finite quotients of T of type $PSL(2,q)$, we shall start with an inclusion

$$\varphi : \mathbf{A}_5 \to PSL(2,q)$$

and try to extend it to a surjection

$$\phi : T \to PSL(2,q),$$

with torsionfree kernel.

LEMMA 1. *The following are equivalent*

i) $\mathbf{A}_5 \subset PSL(2,q)$;

ii) $\mathbf{A}_5 \subset PGL(2,q)$;

iii) $q \equiv \pm 1 \ (mod\, 10)$ *or* $q = 5^n$;

iv) $\sqrt{5} \in \mathbb{F}_q$ *that is* 5 *has a square root in* $\mathbb{F}_q$.

Moreover, if this is the case, there are exactly 2 conjugacy classes of subgroups $\mathbf{A}_5$ *in* $PSL(2,q)$ *but only one in* $PGL(2,q)$*, for each* $q = p^n$, $p \neq 5$*, or* $q = 5^{2n}$*. There is only one conjugacy class of subgroups* $\mathbf{A}_5$ *in* $PSL(2, 5^{2n+1})$.

Proof. For $PSL(2,q)$, this is proved in [3] and [6]; for $PGL(2,q)$ it follows then from the fact that $\mathbf{A}_5$ is generated by two elements of order 3 and 5 which necessarily lie in $PSL(2,q)$. The last statement follows from the classification of subgroups $\mathbf{A}_5$ in projective linear groups given in [3].

A matrix $A \in GL(2,q)$ projects to an element in $PSL(2,q) \subset PGL(2,q)$ if and only if its determinant $det\, A$ is a square in $\mathbb{F}_q$. Multiplying with $(\sqrt{det\, A})^{-1} \in \mathbb{F}_q$ we may assume then that $det\, A = 1$ which we shall always do in the following. Then the trace of an element $A \in PSL(2,q)$ is well defined up to sign. It will be convenient in the following to consider $PGL(2,q)$ as a subgroup of $PSL(2,q^2)$. Given $A \in GL(2,q)$ its determinant is always a square in the quadratic extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$. Then $(\sqrt{det\, A})^{-1} A$ has determinant 1 and represents an element in $PSL(2,q^2)$ so its trace is again well-defined up to sign; in the following, by the trace of an element of $PGL(2,q)$, we shall understand this trace. Then we have:

LEMMA 2. *The trace of an element in* $PGL(2,q)$ *belongs to* $\mathbb{F}_q \subset \mathbb{F}_{q^2}$ *if and only if the element lies in* $PSL(2,q) \subset PGL(2,q)$.

An element in $PSL(2,q)$ is called *parabolic* if its trace is equal to ± 2. A proof of the following Lemma can be found in [3] or [4].

LEMMA 3.

a) *Two non-parabolic elements in* $PSL(2,q)$ *(or* $PGL(2,q) \subset PSL(2,q^2)$*) are conjugate if and only if they have the same trace (up to sign).*

b) *A non-parabolic element in* $PSL(2,q)$ *(or* $PGL(2,q) \subset PSL(2,q^2)$*) has order* $2,3,4$ *resp.* 5 *if and only if its trace is equal to* $0, \pm 1, \pm\sqrt{2}$ *resp.* $\pm(1 \pm \sqrt{5})/2$ *(where the last stays for four numbers).*

For the triangle group

$$(2,3,m) = \langle x, y \mid x^2 = y^3 = (x\,y)^m = 1\rangle$$

and the dihedral group

$$\mathbb{D}_m = (2,2,m) = \langle a, b \mid a^2 = b^2 = (a,b)^m = 1\rangle,$$

let $G_{m,k}$ be the group

$$G_{m,k} := \mathbb{D}_m \underset{Z_m}{*} (2,3,m)/\langle (a\,x)^k \rangle,$$

where $\mathbf{Z}_m$ is generated by $a\,b = (x\,y)^{-1}$.

All groups $G_{m,k}$ are spherical, euclidean or hyperbolic tetrahedral groups, where in the hyperbolic case the tetrahedron may be bounded, cusped (some vertices on the sphere at infinity) or of infinite volume (some vertex beyond the sphere at infinity of $\mathbb{H}^3$), see [11, Ch. 13] for a classification of tetrahedra. In particular, the groups $\mathbb{D}_m$ and $(2,3,m)$ can be considered as subgroups of $G_{m,k}$.

The following result is an elaboration and generalization of results proved in [5] and [6], where we were interested in the case $m = 7$, $k \in \{2,3,4,5\}$.

PROPOSITION. *Let $\varphi : (2,3,m) \to PSL(2,q)$ be a homomorphism with torsionfree kernel and let $\gamma \in \mathbb{F}_q$ be the trace of the element $\varphi(x\,y)$ of order m in $PSL(2,q)$ which we assume non-parabolic. Let $\tau \in \mathbb{F}_q$ resp. $\mathbb{F}_{q^2} - \mathbb{F}_q$ be the trace of an element of order k in $PSL(2,q)$ resp. $PGL(2,q) - PSL(2,q)$, and let*

$$C(\tau,\gamma) := \tau^2\,\gamma^2 - 4\,(\tau^2 + \gamma^2) + 12,$$

$C(\tau,\gamma) \in \mathbb{F}_q$.

Then φ extends to a homomorphism with torsionfree kernel $\phi : G_{m,k} \to PSL(2,q)$ resp. $PGL(2,q)$ such that $\phi(b\,y)$ has trace τ if and only if $C(\tau,\gamma)$ is a square resp. a non-square in $\mathbb{F}_q$. Moreover a given φ has at most 2 such extensions; if φ extends then any extension $\phi : (2,3,m) \to PSL(2,q^r)$ has image in $PSL(2,q)$ resp. $PGL(2,q)$.

Proof. By conjugation one may assume that $\varphi(x\,y)$ is in diagonal form (if this is not possible in $PSL(2,q)$ it will be possible in $PSU(2,q^2) \cong PSL(2,q)$). The elements $\varphi(x)$ resp. $\varphi(y)$ have orders 2 resp. 3 which gives some conditions for the coefficients of these matrices, using Lemma

3. Then one extends $\varphi : (2,3,m) \to PSL(2,q)$ to a homomorphism $\varphi' : \mathbb{D}_m \underset{Z_m}{*} (2,3,m) \to PGL(2,q)$ looking for all possible elements $\varphi'(a)$ of order 2 (trace 0) such that also $\varphi'(b) = \varphi'(a^{-1}\,y^{-1}\,x^{-1})$ has order 2. Now φ extends to $\phi : G_{m,k} \to PGL(2,q)$ if and only if, for one of the possible choices of $\varphi'(a)$, the element $\varphi'(a\,x)$ has order k or, more exactly, trace τ. For this one has to solve a quadratic equation in $\mathbb{F}_q$ whose discriminant is $C(\tau,\gamma)\cdot\alpha$, where $\alpha \in \mathbb{F}_q$ is a square if $\varphi'(a) \in PSL(2,q)$ and a non-square if $\varphi'(a) \in PGL(2,q) - PSL(2,q)$. This gives the number theoretical condition, see [5], [6] for the details of the computations. Moreover, if there exists an extension then there are at most 2 such extensions corresponding to the 1 or 2 solutions of the quadratic equation, with image in $PSL(2,q)$ resp. $PGL(2,q)$.

Our main results are as follows:

THEOREM 1. *There exists a surjection with torsionfree kernel*

$$\phi : T \to PSL(2,p^n) \quad \textit{resp.} \quad PGL(2,p^n)$$

exactly in the following cases:

I) $p \equiv \pm 1 \ (mod\,10)$

i) $p \equiv -1 \ (mod\,8)$: $PSL(2,p)$ *and* $PSL(2,p^2)$

ii) $p \equiv 1 \ (mod\,8)$:
$$\begin{cases} PSL(2,p) & \textit{if } 1+\sqrt{5} \textit{ is a square in } \mathbb{F}_p \\ PSL(2,p^2) & \textit{if } 1+\sqrt{5} \textit{ is a non-square in } \mathbb{F}_p \end{cases}$$

iii) $p \equiv -3 \ (mod\,8)$:
$$\begin{cases} PGL(2,p) & \textit{if } 1+\sqrt{5} \textit{ is a non-square in } \mathbb{F}_p \\ PSL(2,p^2) & \textit{if } 1+\sqrt{5} \textit{ is a square in } \mathbb{F}_p \end{cases}$$

iv) $p \equiv 3 \ (mod\,8)$: $PGL(2,p)$ *and* $PSL(2,p^2)$.

II) $p \equiv \pm 3 \ (mod\,10)$:
$$\begin{cases} PSL(2,p^2) & \textit{if } 1+\sqrt{5} \textit{ is a square in } \mathbb{F}_{p^2} \\ PSL(2,p^4) & \textit{if } 1+\sqrt{5} \textit{ is a non-square in } \mathbb{F}_{p^2} \end{cases}$$

III) $p = 5$: $PSL(2,5^2)$.

THEOREM 2. *There exists a surjection with torsionfree kernel*

$$\phi : T' \to PSL(2,p^n) \quad \textit{resp.} \quad PGL(2,p^n)$$

exactly in the following cases (note that the case of $PGL(2,p^n)$ really does not occur):

I) $p \equiv \pm 1 \ (mod\, 10)$

i) *p non-square in $\mathbb{F}_{19}$: $PSL(2,p)$ and $PSL(2,p^2)$*

ii) *p square in $\mathbb{F}_{19}$:*

$$\begin{cases} PSL(2,p) & \text{if } (7+5\sqrt{5})/2 \text{ is a square in } \mathbb{F}_p \\ PSL(2,p^2) & \text{if } (7+5\sqrt{5})/2 \text{ is a non-square in } \mathbb{F}_p \end{cases}$$

II) $p \equiv \pm 3 \ (mod\, 10)$:

$$\begin{cases} PSL(2,p^2) & \text{if } (7+5\sqrt{5})/2 \text{ is a square in } \mathbb{F}_{p^2} \\ PSL(2,p^4) & \text{if } (7+5\sqrt{5})/2 \text{ is a non-square in } \mathbb{F}_{p^2} \end{cases}$$

III) $p = 5$: $PSL(2,5) \cong \mathbf{A}_5$.

3. Proof of the Theorems.

Suppose $p \neq 5$ first.

By Lemma 1 there exists a subgroup $\mathbf{A}_5$ (unique up to conjugation) in $PGL(2,q)$, $q = p^n$, $p \neq 5$, if and only if $q \equiv \pm 1 \ (mod\, 10)$. This leaves the possibilities $p \equiv \pm 1 \ (mod\, 10)$ and $p \equiv \pm 3 \ (mod\, 10)$. Any inclusion $\varphi : \mathbf{A}_5 \to PGL(2,q)$ is then conjugate to an inclusion $\varphi : \mathbf{A}_5 \to PSL(2,p)$ if $p \equiv \pm 1 \ (mod\, 10)$ or to an inclusion $\varphi : \mathbf{A}_5 \to PSL(2,p^2)$ if $p \equiv \pm 3 \ (mod\, 10)$. Starting with such an inclusion, we want to extend it to the tetrahedral group $T = G_{5,4}$.

Let be $\gamma \in \mathbb{F}_q$ be one of the 2 possible traces $(1 \pm \sqrt{5})/2$ (up to sign) of an element of order 5 in $PSL(2,q)$, $q = p$ or p^2. It is easy to see that an element of order 5 and its square have different traces therefore both values of γ occur as the trace of the element $\varphi(x\,y)$, for some inclusion φ.

We want to find an extension ϕ of φ such that $\phi(a\,x)$ has order 4; let $\tau := \pm\sqrt{2} \in \mathbb{F}_p$ or $\mathbb{F}_{p^2}$ be the trace of an element of order 4. Recall that 2 is a square in $\mathbb{F}_p$, that is $\pm\sqrt{2} \in \mathbb{F}_p$, if and only if $p \equiv \pm 1 \ (mod\, 8)$.

Then $C(\tau,\gamma) = 1 \pm \sqrt{5}$, and $(1+\sqrt{5})\,(1-\sqrt{5}) = -4$. Now -1 is a square in $\mathbb{F}_q$ if and only if $q \equiv 1 \ (mod\, 4)$. Therefore, if $q \equiv 1 \ (mod\, 4)$, both $1+\sqrt{5}$ and $1-\sqrt{5}$ are squares or non-squares in $\mathbb{F}_q$, whereas if $q \equiv 3 \ (mod\, 4)$ one is square and the other a non-square.

Suppose $p \equiv \pm 1 \ (mod\, 10)$ and $p \equiv -1 \ (mod\, 8)$. Then $p \equiv 3 \ (mod\, 4)$ so exactly one of the 2 values $C(\tau,\gamma)$ is a square in $\mathbb{F}_p$. By the Proposition, we get an extension to $PSL(2,p)$ and another one to $PSL(2,p^2)$

(but not to $PSL(2,p)$ in the second case where $C(\tau,\gamma)$ is a non-square in $\mathbb{F}_p$ but a square in $\mathbb{F}_{p^2}$). The first extension is surjective because $\mathbf{A}_5$ is a maximal subgroup of $PSL(2,p)$ and $\mathbf{A}_5$ has no elements of order 4 (see [3], [4] for the classification of maximal subgroups of projective linear groups). The second extension is also surjective because the image cannot be $PGL(2,q) \subset PSL(2,p^2)$ which would be the only other possibility, again by the classification of maximal subgroups of projective linear groups; this follows from the fact $\sqrt{2} \in \mathbb{F}_p$ which implies that every element of order 4 in $PGL(2,p)$ is already in $PSL(2,p)$, by Lemma 2.

The other cases under I and II are treated in a similar way.

Now suppose $p = 5$. Then every element of order 5 in $PSL(2,5^n)$ is parabolic. Because in the Proposition we assumed that $\varphi(x\,y)$ is non-parabolic, we exchange the roles of the element $\varphi(x\,y)$ and $\varphi'(a\,x)$ of orders 5 and 4, that is we start with an inclusion

$$\varphi : \mathbf{S}_4 = (2,3,4) \to PSL(2,5^{2n})$$

(in fact the non-squares in $\mathbb{F}_5$ are non-squares resp. squares in $\mathbb{F}_{5^n}$, n odd resp. even). Any such inclusion is conjugate to an inclusion

$$\varphi : \mathbf{S}_4 = (2,3,4) \to PSL(2,25),$$

by the classification of subgroups $\mathbf{S}_4$ in projective linear groups ([3]).
Now $\gamma = \pm\sqrt{2}$, $\tau = \pm 2$, so $C(\tau,\gamma) = -4 \equiv 1 \ (mod\, 5)$ which is a square. Therefore φ extends to a surjection

$$\phi : G_{4,5} \to PSL(2,25),$$

and this is the only possibility.

This finishes the *Proof of Theorem 1.*

The *Proof of Theorem 2* is analogous, with the following modifications.

Suppose $p \neq 5$. Now let $\gamma := \pm(1+\varepsilon\sqrt{5})/2$ and $\tau := \pm(1+\delta\sqrt{5})/2$ where ε, $\delta \in \{+1,-1\}$. Then

$$C(\tau,\gamma) = (9 + 5\,\varepsilon\,\delta - 5\sqrt{5}(\varepsilon+\delta))/4.$$

If $\varepsilon = -\delta$ then $C(\tau,\gamma) = 1$ is a square and an extension ϕ of φ always exists. However the image of ϕ is $\mathbf{A}_5$ in all cases. In fact, by an easy computation using generators and relations for $T' = G_{5,5}$ and permutations in $\mathbf{A}_5$ one sees that an inclusion

$$\varphi : \mathbf{A}_5 \to \mathbf{A}_5 \subset PSL(2,q)$$

extends in exactly 2 ways to a map

$$\phi : G_{5,5} \to \mathbf{A}_5 \subset PSL(2,q)$$

with torsionfree kernel, and that for both extensions $\phi(x\,y)$ and $\phi(a\,x)$ are not conjugate in $\mathbf{A}_5$, therefore one is conjugate to the square of the other and consequentely they have different traces in $PSL(2,q)$. By the Proposition, any inclusion

$$\varphi : \mathbf{A}_5 \to PSL(2,q)$$

has at most 2 different extensions

$$\phi : G_{5,5} \to PSL(2,q)$$

for the fixed $\tau (\neq \gamma)$ which are therefore already realized by the extensions

$$\phi : G_{5,5} \to \mathbf{A}_5 \subset PSL(2,q).$$

If $\varepsilon = \delta$ then $C(\tau,\gamma) = (7 \pm 5\sqrt{5})/2$ and

$$(7 + 5\sqrt{5})\,(7 - 5\sqrt{5})/4 = -19.$$

Now the traces of $\phi(x\,y)$ and $\phi(a\,x)$ for a possible extension ϕ are equal therefore by the above argument the image of ϕ cannot be $\mathbf{A}_5$. Now the proof is analogous to the proof of Theorem 1 noting that -19 is a square in $\mathbb{F}_p$ if and only if p is a square in $\mathbb{F}_{19}$, by the quadratic reciprocity law.

Finally, suppose $p = 5$. By a direct calculation with parabolic elements in $PSL(2,5)$, in analogy to the proof of the Proposition, one finds that up to conjugation the only possible surjective image is $PSL(2,5) \subset PSL(2,5^n)$.

REMARK. Up to conjugation in the symmetric group $\mathbf{S}_5$, there is exactly one surjection

$$\phi : G_{5,5} \to \mathbf{A}_5$$

with torsionfree kernel; the kernel is the universal covering group of the Seifert-Weber hyperbolic dodecahedral space (see [1]).

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