# CONVERGENCE GROUPS: SEQUENTIAL COMPACTNESS AND GENERALIZATIONS (*)

by DIKRAN N. DIKRANJAN (in Udine)(**)

SOMMARIO. - *Si studia la compattezza sequenziale dei gruppi di convergenza, le sue generalizzazioni (completezza, coarseness, precompattezza sequenziale, compattezza categorica ecc.), le relazioni tra esse nonchè il loro impatto sulla struttura algebrica del gruppo.*

SUMMARY. - *Guided by the known facts in the case of topological groups, we study sequential compactness of convergence groups, its generalizations (completeness, coarseness, sequential precompactness, categorical compactness etc.), the relations between them and their impact on the algebraic structure of the underlying group.*

Throughout this paper a group is an abelian group, a *convergence group* is a group $G$ equipped with a compatible sequential convergence $\mathcal{L} \subseteq G^{\mathbf{N}} \times G$ satisfying the usual axioms - uniqueness of limits, convergence of the constant sequences, the subsequence axiom, the Urysohn axiom. A convergence group $G$ is *sequentially compact* if every sequence of $G$ has a converging subsequence. Sequential compactness should be considered as a natural counterpart of compactness in the case of convergence groups. A convergence group is *sequentially precompact* if each sequence has a Cauchy subsequence. As in the case of topological groups we have

"sequentially compact" $\Leftrightarrow$ "complete"&"sequentially precompact".

The next generalization of sequential compactness proved to be very fruitful. A convergence group $G$ is *coarse* if every sequentially continuous algebraic isomorphism $G \to H$ is an isomorphism [FZ1]. This is a

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Dedicated to Professor Mario Dolcher.

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(**) Indirizzo dell'Autore: Dipartimento di Matematica ed Informatica, Università di Udine, via Zanon 6, 33100 Udine (Italia).
E-mail: dikranja@dimi.uniud.it

natural counterpart of minimality in the case of convergence groups (a Hausdorff topological group $(G, \tau)$ is *minimal* if $G$ admits no Hausdorff group topologies coarser than $\tau$ [DPS]). It was believed for some time that coarse groups should be sequentially precompact since Prodanov and Stoyanov [PS] proved that minimal topological groups are precompact ([FZ1]). Simon and Zanolin [SZ] produced a counterexample under the assumption of the Continuum hypothesis.

The relation between coarseness and completeness has been studied to a greater extent. Frič, Zanolin and the author proved that coarse divisible abelian groups are complete ([FZ1, Theorem 5] for torsion-free groups, and [DFZ, Proposition 3.1] in the general case). To study further this relation and measure how far is a convergence group from being complete the author introduced an appropriate technique in [D2]. It led to the final description of the class of groups on which all coarse convergences are complete (see Theorem 3.3 below) and the construction of sequentially compact convergences on each algebraically compact group in [D4]. The present paper develops new tools (partially used already in [D4]), particularly suitable for the study of sequentially precompact convergence groups.

In Part 1 we recall all necessary definitions and facts from [D, D2, DFZ, FZ1, N] in order to make the paper practically selfcontained. Part 2 is divided in four sections. In the first and last ones we discuss sequential precompactness and introduce the *precompact radical* which measures the failure of precompactness and permits to consider two "approximations" of precompactness. Here we consider also other functorial subgroups, as well as the functorial topologies and convergences defined by means of them. We show that metrizable coarse groups are sequentially precompact. In the second and third section we define a natural construction of enlarging of a given sequential convergence $\mathcal{L}$ on a fixed group $G$ and discuss the properties of these enlargements. In case $\mathcal{L}$ is precompact, these enlargements describe all convergences on $G$ coarser than $\mathcal{L}$, and in particular the coarse and the sequentially compact ones among them.

In Part 3 we apply the technique developed in Part 2 to obtain a new coarseness criterion for sequentially precompact groups stronger than the known ones. We also obtain a very rigid algebraic restraint for a reduced group to admit complete coarse convergences. It extends Orsatti's theorem [O1] describing the groups which are compact in their natural topology to the case of convergence groups by replacing "compact" by "coarse and complete". Moreover, the rigidity of this condition is extended also to the convergence in question: only the functorial convergence defined by means of the natural topology of $G$ may have this property (this aspect of our result generalizes also the uniqueness established in [O2]). This result gives a restriction also for the (smaller) class $\mathcal{C}$ of groups admitting sequentially compact convergences, but we show that $\mathcal{C}$ contains all divisible groups. We offer also a strong criterion for non-coarseness of infinite products depending only on the algebraic structure of the groups in question.

In Part 4 we collect many open questions remaining in this area of convergence groups.

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## 1. Basic Definitions and Properties.

NOTATION. In what follows $\mathbf{N}$ denotes the set of (positive) natural numbers, $\mathbf{P}$ the set of prime numbers, $\mathbf{Z}$ the group of integer numbers, $\mathbf{Q}$ the group of rational numbers, $\mathbf{T}$ the unit circle group and $MON$ the set of strictly monotone maps $\mathbf{N} \to \mathbf{N}$. We fix $|X|$ for denoting the cardinality of a set $X$. The symbols $\omega$ and $\mathbf{c}$ stand for the first infinite cardinal and the cardinality of the continuum respectively. If $X$ is a subset of a group $G$, then $<X>$ is the subgroup of $G$ generated by $X$. We denote by $G^{(\tau)}$ the direct sum of $\tau$ copies of the group $G$, by $t(G)$ - the torsion subgroup of $G$, by $G[n]$ the subgroup of elements $x$ of $G$ with $nx = 0$ $(n \in \mathbf{N})$, by $s(G)$ - the socle of $G$ (i. e. $s(G) = \oplus_{p \in \mathbf{P}} s_p(G)$, where $s_p(G) = G[p]$), by $r(G)$ - the free-rank of $G$, by $r_p(G)$ - the $p$-rank of $G$ (i. e., $r_p(G) = \dim_{\mathbf{Z}/p\mathbf{Z}} s_p(G)$).

A sequence $S = \{x_n\}_{n=1}^{\infty}$ in a group $G$ will be understood as a map $S : \mathbf{N} \to G$ with $S(n) = x_n$ for $n \in \mathbf{N}$. A *subsequence* of $S$ is a composition $S \circ s$ with $s \in MON$. Let $\mathcal{P}$ be a property concerning sequences. We say that a sequence $S$ has *definitely* $\mathcal{P}$ if a subsequence $S \circ s$ has $\mathcal{P}$ for a shift $s \in MON$, i. e. $s(n) = n + c$ for some $c \in \mathbf{N}$. The family of all sequences in $G$ has a natural group structure inherited from $G^{\mathbf{N}}$.

A *(sequential) convergence group* is a couple $(G, \mathcal{L})$, where $G$ is a group and $\mathcal{L} \subseteq G^{\mathbf{N}} \times G$ is a *convergence*, i. e. this is the family of all pairs $(S, x)$ such that $S$ is a sequence *converging* to $x$, we denote this by $x = \lim_{\mathcal{L}} S$, or $S \xrightarrow{\mathcal{L}} x$, or simply $S \longrightarrow x$. The convergence $\mathcal{L}$ satisfies five axioms: uniqueness of limits (if $S \to x$ and $S \to y$, then $x = y$), convergence of the constant sequences, the subsequence axiom (if $S \to x$ then for every $s \in MON$ $S \circ s \to x$), the Urysohn axiom ($S \to x$ whenever for every $s \in MON$ there exists $s' \in MON$ such that $S \circ s \circ s' \to x$) and $\mathcal{L}$ is compatible with the group structure of $G$ (if $S \to x$ and $S' \to x'$ then $S - S' \to x - x'$). Equivalently, a convergence $\mathcal{L}$ is (the graph of) a homomorphism $\lim_{\mathcal{L}} : H \to G$, where $H$ is a subgroup of $G^{\mathbf{N}}$ (of *converging sequences*) containing the image $\Delta(G)$ of the diagonal homomorphism $G \to G^{\mathbf{N}}$, such that $\lim_{\mathcal{L}} \circ \Delta = id_G$ and the remaining subsequence axiom and Urysohn axiom are satisfied. We denote by $\mathcal{L}(0)^{-1}$ the family $\lim_{\mathcal{L}}^{-1} 0$ of all sequences converging to zero in $(G, \mathcal{L})$. This is in fact an isomorphic copy of the subgroup $\mathcal{L} \cap (G^{\mathbf{N}} \times \{0\})$ of the group $G^{\mathbf{N}} \times G$.

For a subset $M$ of a convergence group $G$ *the closure* of $M$ is the set of all limits of sequences contained in $M$. We say that $M$ is *closed* (resp. *dense*) if the closure of $M$ coincides with $M$ (resp. with $G$); $U \subseteq G$ is *open* if $G \setminus U$ is closed. (This defines a (sequential) topology $\tau$ on the group $G$, but in general $(G, \tau)$ is not a topological group [N].) On the other hand, every topological group $(G, \tau)$ gives rise to a convergence group by declaring $S \to x$ for every sequence converging in $\tau$, we denote by $\mathcal{L}_\tau$ this convergence. When $\tau$ is metrizable, we speak of *metric* convergence $\mathcal{L}_\tau$.

Quotients, products, completions. A homomorphism $f : G \to H$ between convergence groups is *sequentially continuous* if for each converging sequence $S \to x$ in $G$ the sequence $f(S)$ converges to $f(x)$ in $H$. Let $G$ be a convergence group, $N$ a closed subgroup of $G$ and $f : G \to G/N$ the quotient homomorphism. Unless otherwise stated, the group $G/N$ will be equipped with the convergence defined as follows: $f(S) \to f(x)$ in $G/N$ for a sequence $S$ in $G$ and $x \in G$ if for each $s \in MON$ there exist a sequence $S'$ in $N$ and $s' \in MON$ such that $S \circ s \circ s' + S' \to x$ in $G$. It is easy to see that this definition is correct, $f$ is sequentially continuous and has the universal property characterizing the quotient.

Let $\{(G_i, \mathcal{L}_i) : i \in I\}$ be a family of convergence groups. Their Cartesian product $G$ will be always provided with the convergence $\mathcal{L}$ such that $S \to x$ in $(G, \mathcal{L})$ iff for each projection $p_i : G \to G_i$ the sequence $p_i(S)$ converges to $p_i(x)$ in $(G_i, \mathcal{L}_i)$.

A sequence $S$ in a convergence group is a *Cauchy sequence* if for every $s \in MON$ the sequence $S \circ s - S$ convergences to zero. Every convergent sequence is a Cauchy sequence. A Cauchy sequence with a constant subsequence is convergent. It is worth noting that these notions do not depend on the numeration of the sequence. In fact, if $\pi$ is a permutation of $\mathbf{N}$, then $S$ is convergent (Cauchy) iff $S \circ \pi$ has the same property. In this way one may speak of convergence (being Cauchy) of countably infinite sets instead of sequences. This permits us to form finite unions of sequences and discuss their convergence. If $\{(G_i, \mathcal{L}_i) : i \in I\}$ is a family of convergence groups and $G$ their Cartesian product with projections $p_i$ as above, then a sequence $S$ in $G$ is a Cauchy sequence iff each $p_i(S)$ is a Cauchy sequence. Therefore $G$ is complete iff each $G_i$ is complete.

A sequential convergence group $G$ is *complete* if every Cauchy sequence in $G$ is convergent. Let us recall that every convergence group $(G, \mathcal{L})$ has a standard completion called the *Novak completion* and denoted by $(\tilde{G}, \tilde{\mathcal{L}})$ (a non-trivial sequence $S$ in $\tilde{G}$ is convergent in $\tilde{\mathcal{L}}$ if and only if $S$ is contained in finitely many cosets $x_i + G$ and the corresponding subsequences $S_i - x_i$ are Cauchy sequences in $G$ [FZ3]. A convergence group $G$ may have other completions non-isomorphic to $\tilde{G}$ ([N]), however every complete convergence group containing $(G, \mathcal{L})$ as a dense convergence subgroup has as underlying group $\tilde{G}$. For other good categorical properties of $\tilde{G}$ see item (e) of Fact 1.1 below.

Coarse groups. The set $\mathbf{S}_G$ of all group convergences on a group $G$ will be always considered with the partial order defined by inclusion. Then the poset $\mathbf{S}_G$ has as bottom element the discrete convergence (where only definitely constant sequences converge). The coatoms of $\mathbf{S}_G$ are called *coarse convergences* [FZ1]. Here we recall some properties of coarse groups which will frequently be used in the paper (particularly the external criterion (c) and the internal one (d)).

Fact 1.1.

a) [FZ1] Let $(G, \mathcal{L})$ be a convergence group. Then there exists a coarse convergence $\overline{\mathcal{L}}$ on $G$ containing $\mathcal{L}$.

b) [FZ1] A closed subgroup of a coarse group is a coarse group.

c) [DFZ] Let $G$ be a dense subgroup of a convergence group $G'$. If $G$ is coarse then $G$ is *essential* in $G'$ (i. e. meets non-trivially each non-trivial subgroup of $G'$). If $G'$ is coarse and $G$ is essential in $G'$, then $G$ is coarse.

d) [FZ1] A convergence group $G$ is coarse iff every sequence $S$ of $G$ has one of the following two properties:

$(C_1)$ there exists a subsequence of $S$ converging to 0;

$(C_2)$ there exist $n \in \mathbf{N}$, $s_1, \ldots, s_n \in MON$, $k_1, \ldots, k_n \in \mathbf{Z}$ and $x \in G$, $x \neq 0$, such that $k_1 S_\circ s_1 + \ldots + k_n S_\circ s_n \to x$.

e) [FK] If $f : G \to K$ is a sequentially continuous homomorphism into a complete convergence group $K$ then there exists a sequentially continuous homomorphism $\tilde{f} : \tilde{G} \to K$ coinciding with $f$ on $G$. If $f$ is a embedding, then $\tilde{f}(\tilde{G})$ coincides with the closure of $f(G)$ in $K$ (so that if $f(G)$ is dense in $K$, then $\tilde{f}$ is an algebraic isomorphism). Moreover, if $\{x_n\}$ is a sequence in $G$ such that $f(x_n) \to y$ in $K$, then $x_n \to \tilde{f}^{-1}(x)$ in $\tilde{G}$.

Note that (a) is in strong contrast with the topological case (many groups, e. g. $\mathbf{Q}$, admit no minimal group topology at all). In (e) the inverse $\tilde{f}^{-1}$ need not be sequentially continuous on the closure of $f(G)$ in $K$.

The following cardinal invariants were introduced in [D2] in order to measure the non-completeness of a convergence group. For a convergence group $G$ and $p \in \mathbf{P}$ define $\rho(G) = r(\tilde{G}/G)$, $\rho_p(G) = r_p(\tilde{G}/G)$ and $\sigma_p(G) = r_p(\tilde{G}[p]/G[p])$. The group $G$ is said to be *t-complete* (resp. *b-complete, f-complete, s-complete, r-complete*) if $\rho(G) = 0$ (resp. $\tilde{G}/G$ is a bounded torsion group, $\tilde{G}/G$ is a finite group, $\sigma_p(G) = 0$ for every $p \in \mathbf{P}$, $\rho_p(G) = 0$ for every $p \in \mathbf{P}$). If $G$ is coarse, then $G$ is essential in $\tilde{G}$ by Fact 1.1 (c). This is equivalent to $\rho(G) = 0$ and $\sigma_p(G) = 0$ for each $p \in \mathbf{P}$. If $G$ is torsion-free and $\rho_p(G) = \sigma_p(G) = 0$, then $pG$ is closed in $G$ ([D2, Lemma 2.1]). Thus for a torsion-free coarse group $G$ $\rho_p(G) = 0$ implies $pG$ closed.

## 2. Precompactness, Enlargements and Functorial Subgroups

SEQUENTIAL PRECOMPACTNESS. Here we consider a natural generalization of Cauchy sequences. A sequence $S$ in a convergence group $(G, \mathcal{L})$ is said to be *$\mathcal{L}$-totally bounded* if each subsequence of $S$ has a an $\mathcal{L}$-Cauchy subsequence. Clearly a Cauchy sequence is totally bounded, but while the union of two Cauchy sequences need not be a Cauchy sequence, finite unions of totally bounded sequences (in particular, Cauchy sequences) is always totally bounded.

In analogy, a sequence $S$ in a convergence group $(G, \mathcal{L})$ is said to be *$\mathcal{L}$-totally unbounded* if it has no $\mathcal{L}$-Cauchy subsequences. A sequence $S$ is not $\mathcal{L}$-totally bounded iff $S$ has a subsequence which is $\mathcal{L}$-totally unbounded.

We are not going to use this notion in the sequel.

Denote by $\mathbf{B}_{(G,\mathcal{L})}$ or simply by $\mathbf{B}_{\mathcal{L}}$ the set of all $\mathcal{L}$-totally bounded sequences in $(G,\mathcal{L})$. Then $\mathbf{B}_{\mathcal{L}}$ is a subgroup of $G^N$. If $\Sigma$ is a set of sequences in $(G,\mathcal{L})$ then $(G,\mathcal{L})$ is said to be *$\Sigma$-sequentially precompact* if $\Sigma \subseteq \mathbf{B}_{\mathcal{L}}$. We abbreviate $G^N$-sequentially precompact to *sequentially precompact.* The convergence group $(G,\mathcal{L})$ is *strongly sequentially precompact* if there exists a sequentially compact group containing $(G,\mathcal{L})$ as a convergence subgroup. Note that each sequentially compact group is strongly sequentially precompact and each strongly sequentially precompact group is sequentially precompact, however it is not known if the last two properties coincide as in the case of topological groups (cf. [FZ1,SZ] and Question 4.4).

The next lemma shows that sequential precompactness is preserved under countable products and sequentially continuous homomorphic images. It is easy to see that this holds for sequential compactness and strong sequential precompactness as well (apply the lemma and the preservation of completeness under products).

LEMMA 2.1. a) *If the group $G$ is $\Sigma$-sequentially precompact and $f : G \to H$ is a surjective sequentially continuous homomorphism, then $H$ is $f(\Sigma)$-sequentially precompact.*

b) *Let for each $n \in \mathbf{N}$ $G_n$ be a convergence group and $\Sigma_n \subseteq G_n^{\mathbf{N}}$. Then the group $\prod G_n$ is $\prod \Sigma_n$-sequentially precompact iff $G_n$ is $\Sigma_n$-sequentially precompact for each $n \in \mathbf{N}$.*

*Proof.* a) follows from the fact that for a Cauchy sequence $S$ in $G$ the sequence $f(S)$ is a Cauchy sequence in $G/H$.

The necessity in b) follows from a). To prove the sufficiency assume that $G_n$ is $\Sigma_n$-sequentially precompact for each $n \in \mathbf{N}$ and take a sequence $S$ in $\prod \Sigma_n$. Define by induction $s_1, \ldots, s_n, \ldots \in MON$ such that for $S_n = S \circ s_1 \circ \ldots \circ s_n$ the sequence $p(S_n)$ is Cauchy. Now define the "diagonal subsequence" $D$ by setting $D(n) = S_n(n)$. Then for each $n \in \mathbf{N}$ $D$ coincides definitely with a subsequence of $S_n$, hence $p_n(D)$ is Cauchy. Then also $D$ is Cauchy. $\diamondsuit$

One can easily obtain from b) that sequential (pre)compactness is preserved also under $\Sigma$-products in the sense of Corson [C]. In general even products of $\mathbf{c}$ sequentially compact groups may fail to be sequentially precompact. In fact, Frič [F3] showed that the power $\mathbf{Z}(2)^{\mathfrak{c}}$ is not coarse. Since it is complete, this yields that $\mathbf{Z}(2)^{\mathfrak{c}}$ is not sequentially precompact.

THEOREM 2.2. *Let $G$ be a convergence group and $K$ be a sequentially compact subgroup of $G$ such that the quotient $G/K$ is coarse (sequentially compact). Then $G$ is coarse (resp. sequentially compact).*

*Proof.* Let $\varphi: G \to G/K$ be the canonical homomorphism. Consider first the case when $G/K$ is coarse. Fix a sequence $S$ in $G$ and consider the sequence $\varphi(S)$ in $G/K$. Since $G/K$ is coarse there are two possibilities according to Fact 1.1 (d).

CASE 1. There exists a subsequence $\varphi(S{\circ}s) \to 0$. Then there exists $s' \in MON$ and a sequence $T$ in $K$ such that $S{\circ}s{\circ}s' + T \to 0$ in $G$. By the sequential compactness of $K$ there exists $s'' \in MON$ such that $T{\circ}s'' \to z$ in $K$. Hence $S{\circ}s{\circ}s'{\circ}s'' \to z$ in $G$. Therefore $S$ satisfies either $(C_1)$ or $(C_2)$ of Fact 1.1 (d) depending on whether $z = 0$ or not.

CASE 2. There exist $n \in \mathbf{N}$, $s_1, \ldots, s_n \in MON$, $k_1, \ldots, s_n \in \mathbf{Z}$ and $x \in G \setminus K$, such that $k_1\varphi(S{\circ}s_1) + \ldots + k_n\varphi(S{\circ}s_n) \to \varphi(x)$. Then there exists $s' \in MON$ and a sequence $T$ in $K$ such that $\Sigma_i k_i S{\circ}s_i{\circ}s' + T \to x$ in $G$. By the sequential compactness of $K$ there exists $s'' \in MON$ such that $T{\circ}s'' \to z$ in $K$. Then $\Sigma_i k_i S{\circ}s_i{\circ}s'{\circ}s'' \to x - z$ and obviously $x - z \notin K$. So for $k_i$, $s_i{\circ}s'{\circ}s''$ $(i = 1, \ldots, n)$ and $x - z \neq 0$ $(C_2)$ holds.

The case when $G/K$ is sequentially compact, being much easier, is dealt analogously. $\diamondsuit$

This theorem generalizes Proposition 2.3 of [DFZ] where the case of a product $G = (G/K) \times K$ was considered.

Now we prove that metric coarse convergences are sequentially precompact.

THEOREM 2.3. *Let $(G, \tau)$ be a metric topological group such that the metric convergence $\mathcal{L} = \mathcal{L}_\tau$ is coarse. Then:*

a) $\tau$ is minimal and $\mathcal{L}$ is sequentially precompact;

b) $G$ is essential in the (compact) completion $\hat{G}$ of $(G, \tau)$;

c) $\mathcal{L}$ is sequentially compact iff it is complete.

*Proof.* a) According to [DPS] a metrizable group $G$ is said to be *$M$-minimal* if every continuous isomorphism of $G$ into a metrizable topological group $H$ is open. It is proved there that abelian $M$-minimal groups are minimal, hence precompact (Theorem 2.9.2). Let us see that the group $(G, \tau)$ is $M$-minimal. Assume $\sigma \leq \tau$ is a metrizable group topology on $G$. Thus the coarseness of $\mathcal{L}$ yields $\mathcal{L} \subseteq \mathcal{L}_\tau \subseteq \mathcal{L}_\sigma \subseteq \mathcal{L}$. Hence $\mathcal{L}_\sigma = \mathcal{L}_\tau$. Since these topologies are metrizable, this yields $\sigma = \tau$. This proves that $\tau$ is $M$-minimal. Item b) follows from Fact 1.1 (c), item c) follows from a). $\diamondsuit$

ENLARGEMENTS. Now we present a coarsening construction for convergences isolated from a general kind of construction frequently used in topological algebra [P1]. In the case of a sequentially precompact convergence $\mathcal{L}$ it appeared in [D1]. Following Frič [F2] we call it *enlargement* and remove the sequential precompactness condition.

FACT 2.4. Let $(G, \mathcal{L})$ be a convergence group. Then every subgroup $F$ of $\tilde{G}$ satisfying $F \cap G = 0$ is closed.

*Proof.* Assume that $S$ is a convergent sequence definitely contained in $F$. By the definition of $\tilde{G}$ there is a subsequence $S'$ of $S$ which is contained in a coset of $G$. Then the difference $x_n - x_m$ for any couple of members of $S'$, with sufficiently large $m$ and $n$, is both in $F$ and $G$. By the hypothesis $F \cap G = 0$ this difference is zero. Thus $S$ is definitely a constant sequence, so that $\lim S \in F$. $\diamondsuit$

Throughout the rest of this Section $(G, \mathcal{L})$ will be a convergence group, $F$ will be a subgroup of $\tilde{G}$ with $F \cap G = 0$ and $\varphi$ the canonical homomorphism $\varphi : \tilde{G} \longrightarrow \tilde{G}/F$.

DEFINITION 2.5. Let $(G, \mathcal{L})$ be a convergence group, $F$ be a subgroup of $\tilde{G}$ with $F \cap G = 0$ and $\varphi : \tilde{G} \longrightarrow \tilde{G}/F$ be the canonical homomorphism. We denote by $\mathcal{L}_F$ (and call *F-enlargement* of $\mathcal{L}$) the convergence induced on $G$ by $\tilde{G}/F$ under the (monomorphic) restriction $\varphi|_G$.

In the next lemma we collect some easy to check properties of $F$-enlargements.

LEMMA 2.6. a) *With $G, \mathcal{L}$ and $F$ as above, a sequence $S$ in $G$ converges to 0 in $\mathcal{L}_F$ iff for every $s \in MON$ there exist $s' \in MON$ and an element $f \in F$ such that $S \circ s \circ s'$ converges to $f$ in $\tilde{G}$.*

b) *The assignment $F \mapsto \mathcal{L}_F$ is a one-to-one order preserving correspondence between subgroups of $\tilde{G}$ with $G \cap F = 0$ and group convergences on $G$ coarser than $\mathcal{L}$; in particular, $\mathcal{L}_{(0)} = \mathcal{L}$.*

REMARK 2.7. a) Note that $\mathcal{L}_F$ can be defined also without the condition $G \cap F = 0$, but then one has various inconveniences and in particular uniqueness of limits fails (actually the closure of 0 in $\mathcal{L}_F$ is $G \cap F$).

b) If the reader is still not satisfied by this apparently external characterization of the $F$-enlargement he can make it intrinsic by replacing the subgroup $F$ of $\tilde{G}$ by a non-empty family $\mathcal{F}$ of Cauchy sequences in $(G, \mathcal{L})$ closed under the usual equivalence relation ($S \sim S'$ if $S - S' \to 0$) and subtraction. So in these terms the restraint $F \cap G = 0$ is given by asking $\mathcal{F}$ to have no convergent sequences beyond those in $\mathcal{L}^{-1}(0)$. Now $S \in \mathcal{L}_F^{-1}(0)$ iff for every $s \in MON$ there exists $s' \in MON$ such that $S \circ s \circ s' \in \mathcal{F}$.

According to the above lemma the assignment $F \mapsto \mathcal{L}_F$ defines a poset isomorphism between the poset of all subgroups $F$ of $\tilde{G}$ with $F \cap G = 0$ and a subset of $\mathbf{S}_G$ consisting of convergences $\mathcal{L}'$ containing $\mathcal{L}$. To describe the image of this isomorphism observe that $\mathcal{L}_F(0)^{-1} \subseteq \mathbf{B}_{\mathcal{L}}$ holds always. In the next theorem we show that this condition is also sufficient for a convergence to be an $F$-enlargement of $\mathcal{L}$ for some subgroup $F$ of $\tilde{G}$. In other words, the $F$-enlargements of a given convergence $\mathcal{L}$ on $G$ coincide precisely with the convergences $\mathcal{L}'$ on $G$ containing $\mathcal{L}$ and such that each $\mathcal{L}'$-zero sequence is $\mathcal{L}$-totally bounded.

Generalizing [F2, Sec. 2], where $G = \mathbf{Q}$ and $\mathcal{L}$ is the metric convergence we suggest the following

Definition 2.8. Let $(G, \mathcal{L})$ be a convergence group and $\mathcal{M}$ be a convergence on $G$ containing $\mathcal{L}$. The convergence $\mathcal{M}_{\mathcal{L}}$ on $G$ having as zero sequences $\mathcal{M}(0)^{-1} \cap \mathbf{B}_{\mathcal{L}}$ will be called *$\mathcal{L}$-bounded part* of $\mathcal{M}$.

Theorem 2.9. *Let $(G, \mathcal{L})$ be a convergence group and let $\mathcal{M}$ be a convergence on $G$ containing $\mathcal{L}$. Then there exists a subgroup $F$ of $\tilde{G}$ satisfying*

$$F \cap G = 0 \text{ and } \mathcal{L}_F \subseteq \mathcal{M}, \tag{1}$$

*and $F$ is the greatest subgroup of $\tilde{G}$ satisfying (1). Moreover, $\mathcal{L}_F = \mathcal{M}_{\mathcal{L}}$, consequently $\mathcal{M} = \mathcal{L}_F$ whenever $(G, \mathcal{L})$ is $\mathcal{M}(0)^{-1}$-sequentially precompact. The following conditions are equivalent:*

a) *each $\mathcal{M}$-Cauchy sequence in $G$ is $\mathcal{L}$-totally bounded;*

b) $\mathbf{B}_{\mathcal{M}} = \mathbf{B}_{\mathcal{L}}$;

c) *$\mathcal{M} = \mathcal{L}_F$ and $\tilde{G}/F$ coincides with the Novak completion of $(G, \mathcal{M})$.*

*Proof.* Denote by $G'$ the Novak completion of $(G, \mathcal{M})$. The identity $\iota : (G, \mathcal{L}) \to (G, \mathcal{M})$ extends to a sequentially continuous homomorphism $\tilde{\iota} : \tilde{G} \to G'$ by Fact 1.1 (e). Set $F = \ker \tilde{\iota}$, then $F \cap G = 0$ holds obviously. To verify the rest of (1) note that by the categorical properties of the quotient there exists a sequentially continuous monomorphism $g : \tilde{G}/F \longrightarrow G'$ such that $g\varphi = \tilde{\imath}$. Now the inclusion in (1) follows from the definition of $\mathcal{L}_F$.

Now consider a subgroup $F'$ of $\tilde{G}$ satisfying (1) and pick an element $x \in F'$. Then there exists a sequence $S \to x$ in $\tilde{G}$. Thus $S \to 0$ in $\mathcal{L}_{F'}$. By (1) this gives $S \to 0$ in $\mathcal{M}$. On the other hand, $S = \tilde{\imath}(S) \to \tilde{\imath}(x)$ in $G'$. Since $G'$ has unique limits $\tilde{\imath}(x) = 0$. This proves that $x \in F$, so that each subgroup of $\tilde{G}$ satisfying (1) is contained in $F$.

To prove that $\mathcal{L}_F$ contains the $\mathcal{L}$-bounded part of $\mathcal{M}$ take an $\mathcal{L}$-totally bounded sequence $S \to 0$ in $(G, \mathcal{M})$. We have to show that $S \to 0$ in $\mathcal{L}_F$. Since $\mathcal{L}_F$ satisfies Urysohn's axiom it is enough to see that $S$ has a subsequence converging to 0 in $(G, \mathcal{L}_F)$. By assumption there exists an $\mathcal{L}$-Cauchy subsequence $S'$ of $S$. Denote by $x$ the $\mathcal{L}$-limit of $S'$ in $\tilde{G}$. Applying $\tilde{\imath}$ gives $\tilde{\imath}(x) = \lim_{G'} \tilde{\imath}(S') = \lim_{\mathcal{M}} S' = 0$. Thus $x \in F$. By the definition of $\mathcal{L}_F$ this means that $S'$ $\mathcal{L}_F$-converges to 0 in $G$. This proves that the $\mathcal{L}$-bounded part of $\mathcal{M}$ is contained in $\mathcal{L}_F$. To prove that $\mathcal{L}_F$ coincides with the $\mathcal{L}$-bounded part of $\mathcal{M}$ it suffices to recall that $\mathcal{L}_F$ coincides with its $\mathcal{L}$-bounded part.

Since $\mathcal{M} \supseteq \mathcal{L}$, clearly $\mathbf{B}_{\mathcal{L}} \subseteq \mathbf{B}_{\mathcal{M}}$, so the equivalence of a) and b) is straightforward. By the first part of the proof b) implies that $\mathcal{M} = \mathcal{L}_F$ since $\mathcal{M}(0)^{-1} \subseteq \mathbf{B}_{\mathcal{M}}$. To prove the rest of the implication b) $\Rightarrow$ c) observe that by means of the monomorphism $g$ $\tilde{G}/F$ can be identified with a convergence subgroup of $G'$. We have to prove that $g$ is surjective. Let now $S$ be an $\mathcal{M}$-Cauchy sequence in $G$. By a) each subsequence $S'$ of $S$ has a subsequence $S''$ which is $\mathcal{L}$-Cauchy. If $S'' \to x$ in $\tilde{G}$, then $S'' = \tilde{\imath}(S'') \to \tilde{\imath}(x)$ in the subgroup $g(\tilde{G}/F)$ of $G'$, so that $S''$ is convergent in $g(\tilde{G}/F)$. Since $S$ is $\mathcal{M}$-Cauchy, this yields that $S$ is convergent in $g(\tilde{G}/F)$. Thus every $\mathcal{M}$-Cauchy sequence has a limit in $g(\tilde{G}/F)$. Thus $g(\tilde{G}/F)$ coincides with the Novak completion of $(G, \mathcal{M})$ as abstract group. To show they coincide also as convergence groups it suffices to apply the definition of quotient convergence to check that the quotient convergence of $\tilde{G}/F$ has the same property as the convergence of the Novak completion. Namely, a sequence $S$ in $\tilde{G}/F$ converges iff each subsequence $S'$ of $S$ has a subsequence $S''$ contained into a coset $x + G$ and $S'' - x$ is an $\mathcal{M}$-Cauchy sequence.

Assume that c) holds. Then $\mathcal{M} = \mathcal{L}_F$ and $\tilde{G}/F$ is complete. To check a) take an $\mathcal{M}$-Cauchy sequence $S$ in $G$. By the completeness of $\tilde{G}/F$ there exists $x = \lim_{\tilde{\mathcal{M}}} S$ in $\tilde{G}/F$. Let $x = g(y)$ for some $y \in \tilde{G}$. Then there exists an $\mathcal{L}$-Cauchy sequence $T$ in $G$ such that $y = \lim_{\tilde{\mathcal{L}}} T$ in $\tilde{G}$. Then $T = \varphi(T) \xrightarrow{\tilde{\mathcal{M}}} x = g(y)$. Hence $\lim_{\mathcal{M}}(S - T) = 0$. By

Lemma 2.6 for each $s \in MON$ there exist $s' \in MON$ and $f \in F$ such that $(S-T)_\circ s_\circ s' \to f$ in $\tilde{G}$. Since $T_\circ s_\circ s' \to y$ in $\tilde{G}$ we get $S_\circ s_\circ s' \to f+y$ in $\tilde{G}$. Hence $S_\circ s_\circ s'$ is $\mathcal{L}$-Cauchy. This proves that every $\mathcal{M}$-Cauchy sequence is $\mathcal{L}$-totally bounded, so a) holds. ♢

PROPERTIES OF THE ENLARGEMENT. In the next theorem we give again, to ease the reference, the main properties of the enlargement established in Theorem 2.9.

THEOREM 2.10. *Let $(G,\mathcal{L})$ be a convergence group. Then the assignment $F \mapsto \mathcal{L}_F$ is an order preserving bijection between the set of all subgroups $F$ of $\tilde{G}$ with $F \cap G = 0$ and the set of all $\mathcal{L}$-bounded convergences on $G$ containing $\mathcal{L}$. The Novak completion of $(G,\mathcal{L}_F)$ coincides with $\tilde{G}/F$ iff $\mathbf{B}_{\mathcal{L}_F} = \mathbf{B}_{\mathcal{L}}$. In such a case $(G,\mathcal{L}_F)$ is complete iff $\tilde{G} = G+F$, i. e. $F$ splits off.*

According to the above corollary $\tilde{G}/F$ is complete iff $\mathbf{B}_{\mathcal{L}_F} = \mathbf{B}_{\mathcal{L}}$. It will be important to find a more convenient form of this condition.

LEMMA 2.11. *Let $(G,\mathcal{L})$ be a convergence group, $F$ a subgroup of $\tilde{G}$ with $F \cap G = 0$ and $p \in \mathbf{P}$. Consider the embedding $G \hookrightarrow \tilde{G}/F$. Then $s_p(\tilde{G}/F) \subseteq G$ iff $s_p(\tilde{G}) \subseteq G+F$ and*

$$p\tilde{G} \cap F \subseteq pG + pF. \tag{2}$$

*Proof.* Let $s_p(\tilde{G}/F) \subseteq G$. Then clearly $s_p(\tilde{G}) \subseteq G+F$ holds. To prove (2) take an element $x \in \tilde{G}$ such that $px \in F$. Then the coset $x+F$ belongs to $s_p(\tilde{G}/F)$, hence by our hypothesis $x = g+f$ for some $g \in G$, $f \in F$. Obviously $px = pg + pf \in pG + pF$.

Conversely, let $s_p(\tilde{G}) \subseteq G+F$ and (2) hold. If the coset $x+F$ in $\tilde{G}/F$ belongs to $s_p(\tilde{G}/F)$ then $px \in F$. Thus by (2) there exist $g \in G$ and $f \in F$ such that $px = pg + pf$. Set $t = x-g-f$, then $pt = 0$, so $t \in s_p(\tilde{G}) \subseteq G+F$. Consequently $t = g_1 + f_1$ with $g_1 \in G$ and $f_1 \in F$. This gives $x = (g+g_1)+(f+f_1) \in G+F$, so $x+F \subseteq G+F$.

♢

COROLLARY 2.12. *Let $(G,\mathcal{L})$ be a convergence group and $F$ be a sub-*

*group of* $\tilde{G}$ *with* $F \cap G = 0$ *such that* $\tilde{G}/F$ *is complete. Then for every* $p \in \mathbf{P}$ $\rho_p(G, \mathcal{L}_F) = r_p(\tilde{G}/(G+F))$ *and* $\rho(G, \mathcal{L}_F) = r(\tilde{G}/(G+F))$. *Moreover:*

a) $(G, \mathcal{L}_F)$ *is b-complete iff* $n\tilde{G} \subseteq G + F$ *for some* $n \in \mathbf{N}$*;*

b) $(G, \mathcal{L}_F)$ *is* $f$*-complete iff* $G + F$ *has finite index in* $\tilde{G}$*;*

c) $(G, \mathcal{L}_F)$ *is* $t$*-complete iff* $\tilde{G}/(G+F)$ *is torsion;*

d) $(G, \mathcal{L}_F)$ *is* $s$*-complete iff for each* $p \in \mathbf{P}$ $s_p(\tilde{G}) \subseteq G + F$ *and (2) holds.*

*Proof.* Since the Novak completion of $(G, \mathcal{L}_F)$ coincides with $\tilde{G}/F$ by Theorem 2.10, a)-c) follow from the definitions. Item d) follows from Lemma 2.11. $\diamondsuit$

The following seems the most important application of Theorem 2.9.

Corollary 2.13. *Let* $(G, \mathcal{L})$ *be a sequentially precompact convergence group. Then the assignment* $F \mapsto \mathcal{L}_F$ *is an order preserving bijection between the set of all subgroups* $F$ *of* $\tilde{G}$ *with* $F \cap G = 0$ *and the set of all convergences on* $G$ *containing* $\mathcal{L}$*. The Novak completion of* $(G, \mathcal{L}_F)$ *coincides with* $\tilde{G}/F$*. Moreover the following are equivalent:*

a) $(G, \mathcal{L}_F)$ *is sequentially compact;*
b) $(G, \mathcal{L}_F)$ *is complete;*
c) $\tilde{G} = G + F$.

*Proof.* Follows from Theorem 2.10 since now every convergence $\mathcal{M}$ containing $\mathcal{L}$ is $\mathcal{L}$-bounded and satisfies $\mathbf{B}_{\mathcal{M}} = \mathbf{B}_{\mathcal{L}} = G^N$. $\diamondsuit$

Now we give a criterion for coarseness of the $F$-enlargement.

Theorem 2.14. *Let* $(G, \mathcal{L})$ *be a sequentially precompact convergence group. Then for a subgroup* $F$ *of* $\tilde{G}$ *with* $F \cap G = 0$ *the following are equivalent:*

a) $(G, \mathcal{L}_F)$ *is a coarse convergence group;*
b) $F$ *is maximal with the property with* $F \cap G = 0$*;*
c) $F + G$ *is essential in* $\tilde{G}$ *and for each* $p \in \mathbf{P}$ *(2) holds.*

*Proof.* The equivalence of a) and b) follows from Corollary 2.13.

a) $\Rightarrow$ c) By the coarseness criterion a) implies that $G$ is essential in $\tilde{G}/F$. By Corollary 2.13 $\tilde{G}/F$ is complete and so we can apply Corollary 2.12. The essentiality of $G$ in $\tilde{G}/F$ yields that $(G, J\mathcal{L}_F)$ is $t$-complete and $s$-complete. This implies c).

c) $\Rightarrow$ a) The same argument as above yields that $G$ is essential in its completion $\tilde{G}/F$. Now we are going to apply the coarseness criterion Fact 1.1 (e). Let $S$ be a sequence in $G$. By the sequential precompactness of $(G, \mathcal{L})$ it will have a subsequence $S'$ converging to an element $x$ of $\tilde{G}/F$. If $x \neq 0$, then $S$ satisfies $(C_1)$ from Fact 1.1 (e). Assume that $x \neq 0$, then by the essentiality of $G$ in $\tilde{G}/F$ there exists $n \in \mathbf{N}$ such that $nx \in G\backslash\{0\}$. Now the sequence $nS'$ converges to $nx$ in $(G, \mathcal{L}_F)$. Thus the sequence S satisfies in this case the condition $(C_2)$ in Fact 1.1 (e). This proves that $(G, \mathcal{L}_F)$ is coarse. $\diamondsuit$

Note that if in the above situation $G$ is torsion-free then $F$ contains $t(\tilde{G})$ and $F$ is a pure subgroup of $\tilde{G}$ whenever $\mathcal{L}_F$ is coarse.

The $F$-enlargements prove to be very useful not only in the case of sequentially precompact convergences $\mathcal{L}$. Nice examples can be obtained if one takes a metric convergence $\mathcal{L}$ such that the metric completion of $G$ is locally compact. For $G = \mathbf{Q}$ and $\mathcal{L}$ the usual Euclidean metric on $\mathbf{Q}$ the reader may see [F2], where also unbounded convergences are considered.

Functorial subgroups and convergences. Functorial subgroups in the category of topological groups were studied in [D3]. They can be introduced also in the category **ConGr** of convergence groups and sequentially continuous homomorphisms by assigning to each $G \in$ **ConGr** a subgroup $\mathbf{r}(G)$ such that for every morphism $f: G \to H$ in **ConGr** $f(\mathbf{r}(G)) \subseteq \mathbf{r}(H)$. We begin with the simplest examples determined by the abstract-group structure.

A subset $M$ of a group $G$ is *c-closed* if $M$ is closed in $G$ equipped with any sequential convergence. Since every convergence is contained in a coarse one (Fact 1.1 (a)) a subset $M$ of $G$ is $c$-closed iff it is closed with respect to every coarse convergence on $G$. Obviously a finite union of $c$-closed sets is $c$-closed. For every $n \in \mathbf{N}$ the subgroup $G[n]$ is $c$-closed, so by virtue of [D, Theorem 2.2], these functorial subgroups will play an important role in our exposition. Here we list some of their properties without proof:

a) $G[0] = G$, $G[1] = 0$ and $G[m] \subseteq G[n]$ whenever $m$ divides $n$;

b) $(\oplus_\alpha G_\alpha)[n] = \oplus_\alpha G_\alpha[n]$;

c) if $d$ is GCD of the (possibly infinite) family of integers $\{n_\alpha\}$, then $G[d] = \bigcap_\alpha G[n_\alpha]$;

d) if $n$ is the LCM of $n_1, ..., n_s$ then $G[k] = G[n_1] + ... + G[n_s]$;

e) if $\{g_\alpha\}$ is a family of elements of $G$ and $\{n_\alpha\}$ are integers, then for $M = \{x \in G : (\forall\alpha) n_\alpha x = g_\alpha\}$ either $M = \emptyset$ or $M = m + G[d]$ holds, where $m \in M$ is arbitrary and $d$ is the GCD of $\{n_\alpha\}$.

The set $M$ described in e) is $c$-closed. In Theorem 3.1 below we characterize the groups $G$ such that $c$-closedness of the subgroups $pG$ is equivalent to the stronger condition given in e).

For a group $G$ the functorial subgroups $nG$, $n \in \mathbf{N}$, are used to define a topology $\nu_G$, called *natural topology* of $G$ ([O1,O3]). It has as a base at 0 the filterbase $\{nG\}_{n\in\mathbf{N}}$. Following [F] and [O3] set $G_\omega = \cap_{n=1}^{\infty} nG$ and $p^\omega G = \cap_{n=1}^{\infty} p^n G$ for $p \in \mathbf{P}$. Then $\nu_G$ is Hausdorff iff $G_\omega = 0$. In case $G$ is torsion-free this is equivalent to reducedness of $G$ (i. e. $G$ has no divisible subgroups beyond 0). Finally, $\nu_G$ is precompact iff $G/nG$ is finite for every $n \in \mathbf{N}$ and $G_\omega = 0$.

For a prime number $p$ the functorial subgroups $p^n G$ of $G$ define the $p$-adic topology $\tau_p$ of $G$ having as local base at 0 the filterbase $\{p^n G\}_{n\in\mathbf{N}}$. We note that the $p$-adic topology $\tau_p$ of $G$ is Hausdorff iff $p^\omega G = 0$. Further, $\tau_p$ is precompact iff $p^\omega G = 0$ and $G/p^n G$ is finite for each $n \in \mathbf{N}$.

These topologies are functorial: every homomorphism $(G, \nu_G) \to (H, \nu_H)$ is continuous, analogously for the $p$-adic topology. The convergences $\mathcal{L}_{\nu_G}$ and $\mathcal{L}_{\tau_p}$ (briefly $\mathcal{L}_p$) related to these topologies will be important for us. Obviously, they are functorial in the same sense. Usually we impose conditions ensuring that the topologies $\nu_G$ and are $\tau_p$ Hausdorff in order to let the respective convergences have unique limits.

Denote by **Prec** the full subcategory of all sequentially precompact groups in **ConGr**. Then a typical functorial subgroup in the category **ConGr** is defined by setting for each $G \in$ **ConGr** $\pi(G)$ to be the intersection of all sequentially continuous homomorphisms $G \to H$ with $H \in$ **Prec**. Obviously $\pi(G/\pi(G)) = 0$ always holds (so that $\pi$ is a *radical* [D3]). Let us call $\pi(G)$ *precompact radical* of $G$. The full subcategory **Pr** of convergence groups $G$ with $\pi(G) = 0$ contains **Prec** and is closed under taking subgroups and products. One can also consider the intermediate full subcategory **Pre** having as objects the groups $G \in \mathbf{ConGr}$ admitting a coarser sequentially precompact convergence. Define a new functorial subgroup by setting for $G \in \mathbf{ConGr}$ $\pi_{\mathbf{T}}(G)$ to be the intersection of all sequentially continuous homomorphism $f\colon G \to \mathbf{T}$. Then $\pi(G) \subseteq \pi_{\mathbf{T}}(G)$.

PROPOSITION 2.16. a) *Each countable group in* **Pr** *belongs to* **Pre** *and each coarse group in* **Pre** *belongs to* **Prec**. *Hence every countable coarse group in* **Pr** *is sequentially precompact.*

b) $\pi_{\mathbf{T}}(G) = G$ *for every sequentially compact divisible group* $G$ *with* $|G| < \mathbf{c}$.

*Proof.* a) follows easily from Lemma 2.1.

b) Let $f\colon G \to \mathbf{T}$ be a sequentially continuous homomorphism. Then the subgroup $f(G)$ of $\mathbf{T}$ must be divisible and sequentially compact, hence closed. The only closed subgroups of $\mathbf{T}$ are the finite ones and $\mathbf{T}$ itself. Since divisible finite groups are trivial and $f$ cannot be surjective, we conclude that $f$ is the zero homomorphism. $\diamondsuit$

Compact topological groups have many non-trivial convergent sequences even when they are not metrizable, nevertheless we have the following consequence of the above proposition.

COROLLARY 2.17. *Let* $G$ *be a sequentially compact divisible convergence group with* $|G| < \mathbf{c}$. *Then every sequentially continuous homomorphism of* $G$ *into a compact topological group is trivial.*

In Corollary 3.10 below we show that such groups exist in profusion (every divisible with $|G| < \mathbf{c}$ admits a sequentially compact convergence).

## 3. Applications.

COARSENESS CRITERION. Applying Theorem 2.14 with $F = 0$ we obtain the following coarseness criterion for sequentially precompact groups.

COROLLARY 3.1. *A sequentially precompact group* $G$ *is coarse iff* $G$ *is* *essential in* $\tilde{G}$*.*

Sequential precompactness cannot be removed here, this is witnessed by any complete non-coarse group. This result is stronger than Criterion 2 in [DFZ] (Fact 1.1 (d)) since $\tilde{G}$ is not assumed to be *coarse*. Actually $\tilde{G}$ is rarely coarse (it was conjectured in [D2] that $\tilde{G}$ is coarse iff $\tilde{G}/G$ is finite).

It is well known that each abelian group admits a precompact group topology. It was proved in [D4] that similar result holds for sequential convergence groups, actually each group admits a strongly sequentially precompact convergence.

THEOREM 3.2. *Every group admits a sequentially precompact coarse convergence.*

*Proof.* By [D3, Theorem 2.2]) there exists a sequentially precompact convergence $\mathcal{L}$ on $G$. By Zorn's lemma there exists a subgroup $F$ of its Novak completion $\tilde{G}$ such that $F \cap G = 0$ and $F$ is maximal with this property. Now apply Theorem 2.14 b) to conclude that the enlargement $\mathcal{L}_F$ is coarse. ♢

This result gives an easy proof of the fact that infinite coarse groups are not discrete. It cannot be extended to non abelian groups as shown in [FZ1] (Ol'shanskiĭ's group admits only the discrete convergence).

WHEN ALL COARSE CONVERGENCES ARE COMPLETE. Here we describe the class $\mathcal{A}$ of groups on which all coarse convergences are complete.

A sequential convergence group $G$ is said to satisfy (CL) if for every $n \in \mathbf{N}$ the subgroup $nG$ is closed (D2]). Every sequentially compact groups satisfies (CL). It was proved in [D2] that for coarse groups the condition (CL) implies completeness, in particular (CL) is equivalent to completeness for torsion-free coarse groups ([D2, Theorem 2.2]). Now we see that a similar condition characterizes the class $\mathcal{A}$.

THEOREM 3.3. *For a group $G$ each of the following conditions are equivalent:*

a) *$G$ is either divisible or bounded torsion such that for every $p \in \mathbf{P}$ the $p$-torsion part of $G$ has the form* $\mathbf{Z}(p^k)^{(\alpha_o)} \oplus \mathbf{Z}(p^{k+1})^{(\alpha_1)} \oplus$

*$\ldots \oplus \mathbf{Z}(p^{k+s})^{(\alpha_s)}$ for some $k \geq 1$, $\alpha_o > 0$ and finite cardinals $\alpha_i$ for $i > 0$;*

b) *for every $p \in \mathbf{P}$ there exists a finite subgroup $F$ of $G$ and $n \in \mathbf{N}$ such that $pG = F + G[n]$;*

c) *for every $p \in \mathbf{P}$ the subgroup $pG$ is c-closed;*

d) *every coarse convergence on $G$ is complete.*

The proof of this theorem is given in [D4]. It makes use of the technique of enlargements and functorial subgroups developed in Part 2.

Groups with complete coarse convergences. Next we study the class $\mathcal{C}_c$ of groups admitting complete coarse convergences. Clearly $\mathcal{C}_c$ contains all finite groups and all groups admitting sequentially compact convergences. Moreover, $\mathcal{C}_c$ is closed under finite products, since coarseness is preserved by finite products ([F1]).

We begin the description of the smaller class of groups having a coarse convergence with (CL). As already mentioned, for torsion-free groups completeness is equivalent to the condition (CL), so this will completely describe the torsion-free groups in the class $\mathcal{C}_c$.

It was proved by Orsatti [O1] that a group $G$ such that $\nu_G$ is compact satisfies

$$G \cong \prod (\mathbf{Z}_p^{n_p} \times F_p), \tag{3}$$

where $\mathbf{Z}_p$ is the group of $p$-adic integers, $F_p$ is a finite $p$-group and $n_p \in \mathbf{N} \cup \{0\}$ for each prime $p$. Moreover, a group with (3) has a unique compact group topology, namely $\nu_G$ (see for example [O2, DPS]). Our next result shows that these results can be extended to the much larger class of coarse convergences satisfying (CL).

Theorem 3.4. *Let $\mathcal{L}$ be a coarse convergence on a group $G$ with $\nu_G$ Hausdorff. Then the following are equivalent:*

a) *$\mathcal{L} = \mathcal{L}_{\nu_G}$;*
b) *$\nu_G$ is precompact and $\mathcal{L}$ satisfies (CL);*
c) *$\mathcal{L}$ is sequentially compact and $\mathcal{L} = \mathcal{L}_{\nu_G}$.*

In case these conditions hold $G$ satisfies (3).

*Proof.* a) $\Rightarrow$ b) Assume that $\mathcal{L} = \mathcal{L}_{\nu_G}$. Then the coarseness of $\mathcal{L}$ implies in virtue of Theorem 2.3 that $\nu_G$ is precompact. It suffices to note now that $\mathcal{L}_{\nu_G}$ always satisfies (CL).

b) $\Rightarrow$ a). Assume $\mathcal{L}$ is a coarse convergence with (CL) on $G$ and $\nu_G$ is precompact. Then the subgroup $nG$ of $G$ is closed for each $n \in \mathbf{N}$. By the precompactness of $\nu_G$ the quotient $G/nG$ is finite, hence the subgroup $nG$ is also open for each $n \in \mathbf{N}$. Assume that $S \to 0$ in $\mathcal{L}$. Fix $k \in \mathbf{N}$ and assume that infinitely many members of the sequence are out of $kG$. Then the complement $G \setminus nG$ is closed and contains a subsequence $S'$ of $S$. This contradicts the fact that $S'$ converges to $0 \in kG$. This proves the inclusion $\mathcal{L} \subseteq \mathcal{L}_{\nu_G}$. Since $\mathcal{L}$ is coarse and $\nu_G$ is Hausdorff we conclude that $\mathcal{L} = \mathcal{L}_{\nu_G}$.

b) $\Rightarrow$ c) since (CL) implies completeness. Finally, c) $\Rightarrow$ a) is trivial.

The isomorphism (3), in case a)-c) hold, follows from Orsatti's theorem [O1] since now $\nu_G$ is compact. ♢

Corollary 3.5. *For a group $G$ satisfying (3) for finite $p$-groups $F_p$ and non-negative integers $n_p$ $\mathcal{L}_{\nu_G}$ is the unique coarse convergence satisfying (CL) on $G$.*

As we have seen above, the precompactness of $\nu_G$, required in the next corollary, is a natural condition.

Corollary 3.6. *If for the group $G$ the topology $\nu_G$ is precompact, then the following are equivalent for $G$:*

a) *$G$ admits a coarse convergence with (CL);*

b) *$G$ admits a sequentially compact convergence;*

c) *$G$ admits a compact metrizable group topology;*

d) *$\nu_G$ is compact;*

e) *for each prime $p$ there exist a finite $p$-group $F_p$ and $n_p$ such that (3) holds.*

*Proof.* Obviously e) $\Rightarrow$ d) $\Rightarrow$ c) $\Rightarrow$ b) $\Rightarrow$ a). The implication a) $\Rightarrow$ e) follows from Theorem 4.4. ♢

Corollary 3.7. *Let $G$ be a group such that $\nu_G$ is precompact, but not compact. Then $G$ admits no coarse convergence satisfying (CL), so in particular no sequentially compact convergences. If $G$ is torsion-free then*

$G \notin \mathcal{C}_c$.

The groups $G$ satisfying the hypothesis of the above corollary are the groups $G$ with $\nu_G$ precompact and which fail to satisfy the isomorphism (3). In particular, any group $G$ with $r(G) < \mathbf{c}$ and $\nu_G$ precompact works. For example every infinite finitely generated group has this property.

Now we give a local version of the above results which characterizes the $p$-adic convergence $\mathcal{L}_p$ of a group $G$. Note that the factors $\mathbf{Z}_p^{n_p} \times F_p$ in (3) are uniquely determined by $G$ since $\mathbf{Z}_p^{n_p} \times F_p \cong \bigcap\{mG : m \in N, (m,p) = 1\}$.

Corollary 3.8. *Let $p$ be a prime number and $G$ be a torsion-free group with $p^\omega G = 0$ and $G/pG$ finite. If $\mathcal{L}$ is a coarse complete convergence on $G$, then $\mathcal{L} = \mathcal{L}_p$ and $G \cong \mathbf{Z}_p^k$ for $k = r_p(G/pG)$.*

Groups with sequentially compact convergences. Next we study the class $\mathcal{C}$ of groups which admit sequentially compact convergences. Obviously, $\mathcal{C} \subseteq \mathcal{C}_c$ and the class $\mathcal{C}$ is closed under countable products, since sequential compactness is preserved under countable products. It was shown in [D4] that $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{C}$ contains all algebraically compact groups, while the torsion-free finite-rank groups in $\mathcal{C}$ are precisely the divisible ones.

According to the above results, for a torsion-free group $G$, such that $\nu_G$ is precompact, $G \in \mathcal{C}_c$ can be witnessed only by the convergence $\mathcal{L}_{\nu_G}$. Now (3) looks simpler.

Theorem 3.9. *Let $G$ be a torsion-free reduced group such that $G/pG$ is finite for each prime $p$. Then the following are equivalent for $G$:*

a) $G \in \mathcal{C}_c$.

b) $G \in \mathcal{C}$.

c) $G \cong \prod_{p \in \mathbf{P}} \mathbf{Z}_p^{k_p}$ *for some* $k_p \in \mathbf{N}$.

*Proof.* Let $p \in \mathbf{P}$. Since $G$ is torsion-free, the finiteness of $G/pG$ implies that all quotients $G/p^kG$ $(k \in \mathbf{N})$ are finite. For coprime natural numbers $(n,m) = 1$ one has $mnG = mG \cap nG$. Hence every quotient $G/kG$ $(k \in \mathbf{N})$ is finite. Thus $\nu_G$ is precompact by the reducedness of $G$. To prove the implication a) $\Rightarrow$ c) assume $G \in \mathcal{C}_c$ and fix a complete coarse convergence $\mathcal{L}$ on $G$. Then $\mathcal{L}$ satisfies also (CL) since $G$ is torsion-free. Now Corollary 3.6 applies. The implications c) $\Rightarrow$ b) and b) $\Rightarrow$ a) are trivial. $\diamondsuit$

As a direct consequence of Theorem 2.14 we obtain

COROLLARY 3.10. *Let $(G, \mathcal{L})$ be a sequentially precompact group. Then $G$ admits a sequentially compact convergence containing $\mathcal{L}$ iff $G$ splits off in its Novak completion $\tilde{G}$. All such convergences correspond to the various complements of $G$ in $\tilde{G}$.*

Now we give a large supply of groups in $\mathcal{C}$.

COROLLARY 3.11. *Let $G$ be a divisible group and let $\mathcal{L}$ be a sequentially precompact convergence on $G$. Then every coarse convergence on $G$ containing $\mathcal{L}$ is sequentially compact.*

*Proof.* According to the above corollary it suffices to note that $G$ splits in $\tilde{G}$. $\diamondsuit$

This corollary provides examples of groups satisfying the hypothesis of Corollary 2.17.

SEQUENTIALLY COMPACT CONVERGENCES ON $\mathbf{R}$. Compact metric group topologies on the reals were constructed for the first time by Halmos [H]. Such topologies are not comparable with the euclidean topology of $\mathbf{R}$. It was proved by Prodanov [P2] that the reals do not admit a minimal group topology coarser than the usual topology. To see that the counterpart for convergences strongly fails fix a sequentially precompact convergence $\mathcal{L}$ on $\mathbf{R}$ containing the metric convergence. To obtain such a convergence it suffices to take any continuous monomorphism of $\mathbf{R}$ into a compact metrizable group (see for example [HR] or [DPS, Chapter 3]). By Corollary 3.11 every coarse convergence on $G$ containing $\mathcal{L}$ is sequentially compact. Various interesting convergences on $\mathbf{R}$ were constructed by Frič [F2].

COARSENESS OF PRODUCTS. Now we give a negative result on coarseness of infinite products. Since a closed subgroup of a coarse group is coarse (Fact 1.1 (b)), it is enough to consider only products of coarse groups.

The following condition for a group $G$ is satisfied by free groups

$$\cap_{n=1}^{\infty} p_n G = 0 \quad \text{holds for every infinite sequence of distinct primes } \{p_n\}_{n=1}^{\infty}. \tag{$\Pi$}$$

It is easy to see that a group satisfying $(\Pi)$ is torsion-free. Among the subgroups $G$ of $\mathbf{Q}$ those satisfying ($\Pi$) are precisely the groups of finite type (i. e. those for which the set of primes appearing in the denominator of elements of $G$ is finite). An arbitrary torsion-free group $G$ satisfies ($\Pi$) iff all rank-one subgroups of $G$ satisfy ($\Pi$) (or equivalently, there exists a free subgroup $F$ of $G$ such that $G/F$ is torsion with only finitely many non-trivial primary components).

PROPOSITION 3.12. *Let $G$ be an infinite group satisfying ($\Pi$) with $G/pG$ finite for each prime $p$. Then for every coarse convergence $\mathcal{L}$ on $G$ only finitely many of the subgroups $pG$ ($p \in \mathbf{P}$) are closed, so that $\rho_p(G) \neq 0$ for all but finitely many primes $p$. Consequently $(G, \mathcal{L})$ is not $b$-complete.*

*Proof.* Let $\mathcal{L}$ be a coarse convergence such that $p_n G$ is closed for infinitely many primes $p_n$. Then ($\Pi$) implies that $\cap_{n=1}^{\infty} p_n G = 0$. This yields that the metric topology $\tau$ generated by the family $\{p_n G\}$ is Hausdorff. Since this topology is also precompact, a similar argument as that given in the proof of Theorem 3.4 shows that $\mathcal{L} = \mathcal{L}_\tau$. Now Theorem 2.3 yields that $\tau$ is a minimal group topology. This is false since omitting the first prime and using again ($\Pi$) we get a strictly coarser Hausdorff group topology on $G$. According to [D2, Lemma 2.1], $\rho_p(G) = 0$ implies $pG$ closed, since $G$ is torsion-free. Therefore $\rho_p(G) \neq 0$ for all but finitely many primes $p$. Hence for the Novak's completion $\tilde{G}$ of $(G, \mathcal{L})$ the quotient $\tilde{G}/G$ contains non-trivial $p$-torsion elements for all but finite number of primes $p$. Consequently $\tilde{G}/G$ is not bounded torsion, so that $(G, \mathcal{L})$ is not $b$-complete. $\diamondsuit$

Finitely generated free groups satisfy the hypothesis of the next theorem.

THEOREM 3.13. *Let $\{G_n\}_{n=1}^{\infty}$ be a family of coarse non-trivial groups satisfying ($\Pi$) and such that for each $n \in \mathbf{N}$ $G_n/pG_n$ is finite for each $p \in \mathbf{P}$. Then the product $\prod_{n=1}^{\infty} G_n$ is not coarse.*

*Proof.* According to [D, Theorem 1.2] the coarseness of $\prod_{n=1}^{\infty} G_n$ yields that all but a finite number of the groups $G_n$ are $b$-complete. By Proposition 3.12 the groups $G_n$ are not $b$-complete. $\diamondsuit$

Obviously, this theorem can be proved also for products $\prod_{n=1}^{\infty} G_n$ where for infinitely many $n \in \mathbf{N}$ $G_n/pG_n$ is finite for each $p \in \mathbf{P}$ and $G_n$ satisfies ($\Pi$).

In the following corollary we consider products of non-trivial groups.

COROLLARY 3.14. a) *Let $\mathcal{G} = \{G_i : i \in I\}$ be a family of convergence groups with coarse product. Then the number of finitely generated free groups in $\mathcal{G}$ is finite.*

b) *Let $\mathcal{G} = \{G_i : i \in I\}$ be a family of coarse finitely generated free groups. Then $\prod_i G_i$ is coarse iff $\mathcal{G}$ is finite.*

This corollary provides a new easy proof of the first example disproving the preservation of coarseness under countable products given in [DFZ, Example 2.2]. In fact, the groups $G_n$ in that example are all isomorphic to $\mathbf{Z}$, so satisfy the hypothesis of the Corollary 3.14. To understand better the force of b) one can compare with the topological case: if $G_p$ is the group $\mathbf{Z}$ provided with the $p$-adic topology, then $\prod_{p\in\mathbf{P}} G_p$ is totally minimal [DPS].

COARSE GROUP WITH SMALL NON-COARSE NOVAK COMPLETION. Here we give an example of a coarse convergence group $G$ such that $\tilde{G}$ is not coarse. An example with these properties was given already in [DFZ, Example 3.5, Proposition 3.5.1-3.5.3] by means of the free convergence group technique. The underlying group of that example is $G = \mathbf{Z}^{(\omega)}$, i. e. the free group of countable rank, and $\rho_2(G)$ is infinite. This shows that the group is far from being $r$-complete, in particular it is far from being complete. This property is rather undesirable from the point of view of the following conjecture made in [D2]: the Novak completion of a coarse group is coarse iff $G$ is $f$-complete. We are still unable to prove or disprove this conjecture, however, we provide here an example in the above direction with the following two advantages: a) its underlying group is much simpler than the group in [DFZ], b) $G$ is very close to being $f$-complete, in fact $\rho(G) = \rho_q(G) = \sigma_p(G) = \sigma_q(G) = 0$ for each $q \in \mathbf{P}$, $q \neq p$ and $\rho_p(G) = 1$. Moreover, our example has the following extremal property: being non $f$-complete, the quotient $\tilde{G}/G$ is (necessarily) infinite, but all its proper subgroups are finite.

EXAMPLE 3.15. Let $p \in \mathbf{P}$ and $G = \mathbf{Z}_{(p)}$ be the localization of $\mathbf{Z}$ at $p$, i. e. the subgroup of $\mathbf{Q}$ consisting of all rationals with no entries of $p$ in the denominator. Consider the $p$-adic convergence $\mathcal{L} = \mathcal{L}_p$ on $G$. It is precompact and its metric completion $\hat{G}$ is the group $\mathbf{Z}_p$ of $p$-adic integers. By Fact 1.1 (e) it coincides with the Novak completion of $G$ as abstract groups. We show now that any coarse convergence $\mathcal{M}$ on $G$ containing $\mathcal{L}$ has the properties mentioned above.

In fact, by Theorem 2.14 $\mathcal{M} = \mathcal{L}_F$ for some subgroup $F$ of $\hat{G}$ with $F \cap G = 0$ and maximal with this property. Thus $F$ is a pure subgroup of $\hat{G}$, so that $F \not\subseteq p\hat{G}$. Since $p\hat{G}$ is a maximal proper subgroup of $\hat{G}$ it follows that $p\hat{G} + F = \hat{G}$, thus the quotient $\hat{G}/F$ is $p$-divisible. On the other hand, $\hat{G}$ is $q$-divisible for each $q \in \mathbf{P}$, $q \neq p$, so the quotient has the same property. Thus the quotient $\hat{G}/F$ is divisible. By the coarseness criterion $G$ is essential in $\hat{G}$ so $\hat{G}/F$ is torsion-free and has the same free-rank. Hence $\hat{G}/F \cong \mathbf{Q}$ as abstract groups. By Theorem 2.14 $\hat{G}/F$ coincides with the Novak completion $\tilde{G}$ of $(G, \mathcal{L}_F)$. Thus $\tilde{G}/G = \mathbf{Z}(p^\infty)$ - the Prüfer group. The well known properties of $\mathbf{Z}(p^\infty)$ imply all properties of the non-completeness measure of $(G, \mathcal{L}_F)$ mentioned above. It remains to see that $\tilde{G}$ is not coarse. This can be done as in [D1], providing an explicit convergence containing properly the convergence of $\tilde{G}$. We prefer here to apply the coarseness criterion from Fact 1.1 (d). Identifying $\tilde{G}$ with $\mathbf{Q}$ consider the sequence $S$ in $\tilde{G}$ defined by $S(n) = \frac{1}{p^n}$. Then no subsequence of $S$ convergences to 0 in $\tilde{G}$. Assume that $(C_2)$ holds from Fact 1.1 (d). Then there exist a non-zero $r \in \tilde{G}$, non-zero integers $t_1, t_2, ..., t_k$ and $k$ subsequences $S_1, ..., S_k$ of $S$ such that $S' = \Sigma_{i=1}^k t_i S_i$ convergences to $r$ in $\tilde{G}$. According to the definition of the convergence in $\tilde{G}$ this will imply that the differences of distinct members of $S'$ with sufficiently big numbers belong to $G$ - contradiction. $\diamondsuit$

## 4. Open Questions.

PRECOMPACT GROUPS. Every group admits a finest precompact group topology, namely that generated by all characters. We do not know if convergences have analogous property ([D4]).

QUESTION 4.1. *Does every group $G$ admit a finest sequentially precompact convergence?*

Positive answer to this question will yield that every $(G, \mathcal{L}) \in$ **Pre** admits a finest sequentially precompact topology coarser than $\mathcal{L}$ (so that **Prec** will be a bireflective subcategory of **Pre**). On the other hand, we do not know if **Pre** is closed under products, or equivalently

QUESTION 4.2. *Does a product of sequentially precompact groups admit a coarser sequentially precompact convergence?*

We do not know even if $\mathbf{Z}(2)^{\mathfrak{c}} \in$ **Pre** (obviously $\mathbf{Z}(2)^{\mathfrak{c}} \in$ **Pr**). However, it is clear that the conjunction of positive answers to both 4.1 and 4.2 is false. In fact, "Yes" to 4.2 yields that **Pr** = **Pre**, in particular $G/\pi(G) \in$ **Pre** for each convergence group $G$. With 4.1 true the subcategory **Prec** of sequentially precompact groups would be epireflective in **ConGr** hence closed under products - a contradiction.

As shown in Corollary 2.17, $\pi_{\mathbf{T}}(G) \neq G$ may happen with a sequentially compact group $G$. This phenomenon suggests the study of an appropriate subclass containing the metric compact topological groups.

PROBLEM 4.3. Describe the sequentially compact convergence groups $G$ having sufficiently many sequentially continuous homomorphism $f\colon G \to \mathbf{T}$. Are they necessarily metric?

No examples are known at present to distinguish sequential precompactness from strong sequential precompactness. This leaves open the following

QUESTION 4.4. *Is every sequentially precompact group $G$ strongly sequentially precompact? What if $G$ is also coarse?*

STABILITY UNDER QUOTIENTS. Some of our properties generalizing sequential compactness are stable under quotients, as sequential compactness itself and (strong) sequential precompactness. This fails in the case of coarseness and completeness. Following the case of topological groups ([DPS]) we call a convergence group $G$ *totally coarse* if every quotient of $G$ is coarse ([Z]). The following example shows that coarseness may be destroyed even by quotients w.r.t. simple finite subgroups. Fix a splitting $\mathbf{T} = \mathbf{Q}^{(\mathbf{c})} \oplus (\mathbf{Q}/\mathbf{Z})$ of the circle group $\mathbf{T}$ and let $G$ be the subgroup $\mathbf{Q}^{(\mathbf{c})} \oplus s(\mathbf{Q}/\mathbf{Z})$. Then $G$ is coarse, but for each prime $p$ the quotient $G/C_p$ w.r.t. the (unique) cyclic subgroup $C_p$ of $\mathbf{T}$ of order $p$ is not coarse (apply Corollary 3.1). This group is not complete, but it is still possible that complete coarse groups are totally coarse.

Uspenskiĭ [U] has recently shown that every topological group is a quotient of a Weil-complete group (this is not given explicitly in his paper [U], but it follows immediately from one of the main results). Corollary 2.13 says that for a sequentially precompact convergence group $G$ the quotients $\tilde{G}/F$ remain complete. This leads to

QUESTION 4.5. *When is completeness preserved under quotients or sequentially continuous isomorphisms? Is every convergence group a quotient of a complete group?*

Following the analogy with the topological case ([DPS, Chapter 7]) we call a group $G$ *totally complete* if all quotients of $G$ are complete, and *strongly complete* if every sequentially continuous isomorphic image of $G$ is complete. Obviously sequentially compact groups possess both properties, while coarse complete groups are strongly complete. Hence for the class $\mathcal{T}_c$ of groups admitting a totally complete and totally coarse convergence we have $\mathcal{C} \subseteq \mathcal{T}_c \subseteq \mathcal{C}_c$. Note that a torsion-free group $G$ belongs to $\mathcal{T}_c$ iff $G/G_\omega \in \mathcal{T}_c$. Hence the characterization of the torsion-free groups in $\mathcal{T}_c$ is bounded to that of reduced ones.

Precompactness of the coarse groups. The question of sequential precompactness of coarse groups is far from being resolved. We do not know if $CH$ can be removed from the only known example of a coarse group which is not sequentially precompact ([SZ]). It is not known whether that example is totally coarse; if "Yes" then it would be also totally complete by [D2, Theorem 3.2].

Question 4.6. *Is it consistent with $ZFC$ that every (totally) coarse group is sequentially precompact?*

The algebraic structure of a group $G$ may have an impact on the precompactness of the coarse convergences on $G$. It is not known whether all coarse convergences on the groups $\mathbf{Z}$ or $\mathbf{Q}$ are sequentially precompact. Theorem 3.2 suggests the following

Question 4.7. *Does every infinite group admit a non-precompact coarse convergence?*

Categorically compact groups. A convergence group $G$ is *c-compact* if for every convergence group $H$ the canonical projection $p: G \times H \to H$ sends closed subgroups of $G \times H$ to closed subgroup of $H$. It can be proved that $G$ is $c$-compact iff it is totally complete and strongly complete (i. e. every sequentially continuous homomorphic image of $G$ is complete, [D5]). In particular, sequential compactness implies $c$-compactness. It was proved recently by Uspenskiĭ and the author [DU] that categorically compact abelian topological groups (defined in analogous way) are compact. The proof is essentially based on the precompactness of the minimal abelian topological groups. On the other hand, the categorically compact linearly topologized modules are the linearly compact ones [DT]. Both facts suggest the following

Question 4.8. *Is every c-compact group sequentially compact (or at least coarse)?*

Question 4.9. *Is every (totally) coarse c-compact group sequentially compact?*

PRODUCTS. In all known examples of infinite products which fail to be coarse the groups are not complete.

QUESTION 4.10. *Are countable products of (totally) coarse (totally) complete groups coarse?*

If "Yes" then the class $\mathcal{C}_c$ (resp. $\mathcal{T}_c$) is closed under countable products. Productivity for totally complete, totally minimal topological groups was established in [ED].

It is easy to see that finite products of $c$-compact groups are again $c$-compact, we do not know if this can be extended to the case of (countably) infinite products.

The generalization of preservation of some property under finite products in the sense of Theorem 2.2 is known also as "three-space-problem" (see [EDS] where this problem is studied for (total) minimality of topological groups). We propose the following "multiple" question.

QUESTION 4.11. *Let $K$ be a closed subgroup of a convergence group $G$. If both $K$ and $G/K$ are (totally) coarse (or/and sequentially precompact) does $G$ have the same property? What if $K$ is complete?*

ALGEBRAIC VS SEQUENTIAL COMPACTNESS. By a result of Loś [L] (see also [F, Theorem 42.3]), algebraically compact groups are characterized as *groups admitting $\omega$-limits* (groups $G$ with a homomorphism $\omega$-lim: $G^{\mathbf{N}} \to G$ satisfying only the constant sequence axiom, so that the Urysohn axiom and the subsequence axiom are missing but now *all* sequences are convergent [F, §42]). The key to the proof is the fact that the quotient $G^{\mathbf{N}}/G^{(\mathbf{N})}$ is algebraically compact. In the case of groups admitting $\omega$-limits $G^N$ is the group of all converging sequences. Consider now a convergence group $(G,\mathcal{L})$. Then the set $p(\mathcal{L})$ of all converging sequences in $(G,\mathcal{L})$ is a subgroup of $G^{\mathbf{N}}$ with $G^{(N)} \subseteq \mathcal{L}^{-1}(0) \subseteq p(\mathcal{L})$ and $p(\mathcal{L})/G^{(\mathbf{N})} \cong G \times \mathcal{L}^{-1}(0)/G^{(\mathbf{N})}$. This suggests the following:

QUESTION 4.12. *Let $(G,\mathcal{L})$ be a sequentially compact group. Is then $p(\mathcal{L})/G^{(N)}$ algebraically compact?*

If "Yes" then also $G$ will be algebraically compact, hence the class $\mathcal{C}$

coincides with the class of algebraically compact groups in view of the other inclusion proved in [D4]. In the case of negative answer remains the possibility to use the inclusion $\mathcal{C} \subseteq \mathcal{C}_c$, so that it would be helpful to describe $\mathcal{C}_c$. At this stage we do not know even if $\mathbf{Z}^{(\omega)} \in \mathcal{C}_c$ (or $\mathbf{Z}^{(\omega)} \in \mathcal{C}$, note that this group is not algebraically compact).

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