LINKING TWO MINIMAL TRIANGULATIONS OF $\mathbb{C}P^2$ (*)

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Sommario. - *Si presenta un algoritmo per collegare due triangolazioni "minimali" del piano proiettivo complesso* $\mathbb{C}P^2$. *La prima è la triangolazione simpliciale a 9 vertici di Kühnel* [BK]*; la seconda è la triangolazione contratta di* $\mathbb{C}P^2$ *costruita dal terzo autore in* [G].

Summary. - *We present an explicit algorithm for linking two "minimal" triangulations of the complex projective plane* $\mathbb{C}P^2$. *The first one is the "9-vertex" simplicial triangulation found by Kühnel* [BK]*; the second one is the contracted triangulation of* $\mathbb{C}P^2$*, built by the third author in* [G].

1. Introduction.

In 1983 W. Kühnel built a simplicial triangulation of the complex projective plane $\mathbb{C}P^2$, denoted CP^2_9, with 9 vertices, 36 edges, 84 triangles, 90 tetrahedra and 36 4-simplexes (see [BK] and [K]). The list of its 4-simplexes, each represented by a 5-tuple of integers out of $\{1, 2, \ldots, 9\}$, is shown in Table 1.

Table 1

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12456	45789	78123
23564	56897	89231
31645	64978	97312
12459	45783	78126
23567	56891	89234
31648	64972	97315
23649	56973	89316
31457	64781	97124
12568	45892	78235
31569	64893	97236
12647	45971	78314
23458	56782	89125

The above triangulation is proved to be minimal with respect to the number of vertices; moreover, it is unique with this property [KL]. The nine links of its vertices are all isomorphic to one of the Brückner-Grünbaum 3-spheres with 8 vertices enumerated in [Br] and [GrS], and more precisely to the "non-polytopal" one (see also [Ba], [MY]).

The minimality (and uniqueness) of the Kühnel triangulation, with respect to the number of vertices, is meaningless in the larger context of *pseudocomplexes*. In fact, a result by Pezzana [P_1] assures that each closed connected n-manifold M^n is always homeomorphic to the space of a pseudocomplex $\mathbf{K}$, with exactly $n+1$ vertices (called a *contracted triangulation* of M^n). As recalled in the next section, this fact allows to represent M^n by an edge-coloured graph Γ (called a *crystallization* of M^n), which is in some sense "dual" of $\mathbf{K}$.

The "simplest" contracted triangulation of $\mathbb{C}P^2$ has 5 vertices, 10 edges, 20 triangles, 20 tetrahedra and 8 4-simplexes. It is proved to be minimal with respect to the number of 4-simplexes (as well as to the number of vertices) [G, Lemma 1 and Corollary 2]. The associated crystallization (which we shall denote by Π^8) is shown in Figure 1.

Figure 1

As it is easy to check, Π^8 regularly imbeds into the orientable surface of genus two, with respect to all cyclic permutations of the colour-set. Since $\mathbf{S}^n$ is the only closed n-manifold of regular genus zero [FG$_2$], and $\mathbf{S}^1 \times \mathbf{S}^3$ is the only closed 4-manifold of regular genus one [C], then Π^8 also realizes the least regular genus among all crystallizations of the complex projective plane.

The main purpose of this paper is to present an effective algorithm for linking together these two minimal representations of $\mathbb{C}P^2$.

The first step is to exhibit a coloured simplicial subdivision CP^2_{26} of CP^2_9, with 26 vertices, 167 edges, 438 triangles, 490 tetrahedra and 196 4-simplexes (hence much more economical than the first barycentric subdivision of CP^2_9, which has 4320 4-simplexes). The "dual" graph Π^{196} (of order 196 and regular genus 36), is then transformed into Π^8 by cancelling 94 dipoles.

The reduction algorithm is realized

by a Turbopascal Program. The same program, working in dimension n, allows to get the regular genus of a graph (in fact the genera of all its regular imbeddings) and a presentation of the fundamental group of the represented manifold (see [Ch]).

We wish to thank Josef Eschgfäller for his helpful suggestions during the realization of this program.

2. Main Definitions and Notations.

All spaces and maps considered in this work belong to the piece-wise-linear category, for which we refer to [Gl] or [RS]. The prefix "PL" will always be omitted. For graph theory, see [W].

By a *coloured n-complex* [BM] is meant a pseudocomplex $\mathbf{K}$ of dimension n [HW, p. 49], endowed with a map φ from the set $\mathbf{S}_0(\mathbf{K})$ of vertices of $\mathbf{K}$ to the "colour-set" $\mathbb{N}_{n+1} = \{1 \in \mathbb{Z} | 1 \leq i \leq n+1\}$, such that $\varphi|\mathbf{S}_0(\sigma)$ is injective for each simplex σ of $\mathbf{K}$. This is equivalent to say that φ is a *vertex-colouring* of $\mathbf{K}$, by means of $n+1$ colours.

A *contracted n-complex* is, by definition, an n-dimensional pseudocomplex $\mathbf{K}$ with Card $\mathbf{S}_0(\mathbf{K}) = n{+}1$. Such a $\mathbf{K}$ always admits a straightforward colouring φ, which is unique up to permutations of the colour-set $\mathbb{N}_{n+1}$.

Let now $\mathbf{K}$ be a coloured n-complex, triangulating a closed n-manifold M^n (i.e. whose space $|\mathbf{K}|$ is homeomorphic to M^n). Let further $\Gamma = \Gamma(\mathbf{K})$ be the dual 1-skeleton of $\mathbf{K}$. The vertex-colouring $\varphi : \mathbf{S}_0(\mathbf{K}) \to \mathbb{N}_{n+1}$ induces a "dual" edge-colouring $\gamma : \mathbf{E}(\Gamma) \to \mathbb{N}_{n+1}$ on Γ: if $\mathbf{e} \in \mathbf{E}(\Gamma)$ is the edge of Γ dual of the $(n-1)$ simplex σ^{n-1} of $\mathbf{K}$, and $\varphi(\mathbf{S}_0(\sigma^{n-1})) = \mathbb{N}_{n+1} - \{c\}$, then set $\gamma(\mathbf{e}) = c$.

By the definition itself, it turns out that Γ is an $(n+1)$-*coloured graph*, i.e. a multigraph (hence possibly with multiple edges, but no loops), regular of degree $n+1$, endowed with an edge-colouring $\gamma : \mathbf{E}(\gamma) \to \mathbb{N}_{n+1}$ (this simply means that $\gamma(\mathbf{e}_1) \neq \gamma(\mathbf{e}_2)$ for every pair $\mathbf{e}_1$, $\mathbf{e}_2$ of adjacent edges of Γ). It completely represents M^n: in fact, knowing Γ, it is possible to uniquely invert the described procedure, in order to reconstruct $\mathbf{K}$ (and therefore $M^n \cong |\mathbf{K}|$).

It is easy to see that every simplicial complex admits a coloured subdivision. This implies that all closed connected n-manifolds M^n can be represented by $(n+1)$-coloured graphs. The main result of $[P_1]$, $[P_2]$ assures that among such graphs there always exists at least one *crystallization* (i.e. a graph Γ, whose associated pseudocomplex $\mathbf{K}$ is contracted). This is equivalent to say that for each $c \in \mathbb{N}_{n+1}$, the subgraph $\Gamma_{\hat{c}}$, obtained by deleting all edges coloured c from Γ, is connected. An alternative proof of the above existence theorem is contained in [LM]; it makes use of the *moves* introduced in [FG$_1$], that we now recall, since they compose the single steps of the reduction algorithm, presented in the following section.

Let Γ be an $(n+1)$-coloured graph, representing a manifold M^n. By a B-residue (B being a subset of $\mathbb{N}_{n+1}$) is meant a connected component of the subgraph $\Gamma_B = (\mathsf{V}(\Gamma), \gamma^{-1}(B))$. A subgraph Θ of Γ, formed by two vertices x, y joined by h edges with colours $c_1, \ldots, c_h$, is called an *h-dipole* $(1 \leq h \leq n)$ if x and y belong to different $(\mathbb{N}_{n+1} - \{c_1, \ldots, c_h\})$-residues. *Cancelling* Θ from Γ means to form the graph Γ', where $\mathsf{V}(\Gamma') = \mathsf{V}(\Gamma) - \{x, y\}$ and where $E(\Gamma')$ is obtained from $\mathbf{E}(\Gamma) - \mathbf{E}(\Theta)$ by "pasting together" the pairs of equally coloured edges coming to x and y from outside Θ. *Adding* Θ to Γ' means the inverse procedure. The colours $c_1, \ldots, c_h$ are said to be *involved* in the dipole Θ. The main result of [FG$_1$] assures that if two $(n+1)$-coloured graphs Γ and Γ' represents two closed connected n-manifolds M^n, M'^n, then M^n and M'^n are homeomorphic iff Γ and Γ' can be obtained from each other by cancelling and/or adding a finite number of dipoles (see also [F]). General surveys on the above arguments are contained in [FGG] and [V].

3. The Algorithm.

As hinted in §1, we show how to produce a rather economical coloured subdivision CP^2_{26} of CP^2_9. First of all, we relabel the nine vertices of CP^2_9, by setting:

$$\eta(i) = \begin{cases} i, & \text{for } 1 \leq i \leq 5; \\ i(\text{mod } 5), & \text{for } 6 \leq i \leq 9\ . \end{cases}$$

Let $\Sigma(CP^2_9) = \{\mathbf{e}^1_1, \ldots, \mathbf{e}^1_{d_1},\ \mathbf{e}^2_1, \ldots, \mathbf{e}^2_{d_2},\ \mathbf{e}^5_1, \ldots, \mathbf{e}^5_{d_5}\}$ be the set of all edges $\mathbf{e}^r_{jr}$ of CP^2_9 $(1 \leq r \leq 5,\ 1 \leq j_r \leq d_r\}$ whose endpoints are both labelled r by η. Let further $d_0 = 1$. For each pair (s, j_s), $0 \leq s \leq 5$, $1 \leq j_s \leq d_s$, we shall construct a subdivision $\mathbf{K}^s_{j_s}$ of CP^2_9 and a vertex-labelling $\eta^s_{j_s} : S_0(\mathbf{K}^s_{j_s}) \to \mathbb{N}_5$, such that:

1) $\quad \eta^s_{j_s}|\mathbf{S}_0(CP^2_9) = \eta$;

2) $\quad \Sigma(\mathbf{K}^s_{j_s}) = \{\mathbf{e}^h_{j_h} \in \Sigma(CP^2_9)|$ either $h = s,\ j_h > j_s$ or $h > s\}$.

For, let $\mathbf{K}^0_1 = CP^2_9$, $\eta^0_1 = \eta$ be the starting point of the subdivision algorithm. Suppose now given a subdivision $H = \mathbf{K}^t_{j_t}$ and a vertex-labelling $\psi = \eta^t_{j_t} : S_0(\mathbf{H}) \to \mathbb{N}_5$, satisfying conditions (1) and (2). If $t = 5$ and $j_t = d_5$, then the algorithm stops and $\mathbf{H}$ is the desired coloured subdivision of CP^2_9, since ψ is a vertex-colouring. If this is not the case, then we shall construct a subdivision $\tilde{\mathbf{H}} = \mathbf{K}^m_{j_m}$ and a vertex-labelling $\tilde{\psi} = \eta^m_{j_m}$: $\mathbf{S}_0(\tilde{\mathbf{H}}) \to \mathbb{N}_5$, again satisfying conditions (1) and (2), where either $m = t$, $j_m = j_t + 1$ (if $j_t < d_t$), or $m = t+1, j_m = 1$ (if $j_t = d_t$).

The edge $\mathbf{e}^{(1)} = \mathbf{e}^m_{j_m} \in \Sigma(\mathbf{H})$ (with m as before) has both endpoints labelled m by ψ. Let m_1 be the first element of $\mathbb{N}_5$-$\{m\}$ (in the natural order) such that Card $\{\psi^{-1}(m_1) \cap \mathbf{S}_0(Lk(\mathbf{e}^{(1)}; \mathbf{H}))\}$ is minimal in $\mathbb{N}_5$-$\{m\}$. Roughly speaking, m_1 is one of the colours different from m (in fact the first you meet) which are less used for labelling the vertices of the link of $\mathbf{e}^{(1)}$ in $\mathbf{H}$. Let $\mathbf{H}^{(1)}$ be the stellar subdivision of H on $\mathbf{e}^{(1)}$ [Gl]: it has only one more vertex $\mathbf{v}^{(1)}$ (i.e. the barycenter of $\mathbf{e}^{(1)}$), whose link is $Lk(\mathbf{v}^{(1)}; \mathbf{H}^{(1)}) = \partial\mathbf{e}^{(1)} * Lk(\mathbf{e}^{(1)}; \mathbf{H})$. Note that $\partial\mathbf{e}^{(1)}$ is a 0-sphere, with both vertices labelled m. Now, define $\psi^{(1)} : \mathbf{S}_0(H^{(1)}) \to \mathbb{N}_5$ as follows:

$$\psi^{(1)}(\mathbf{w}) = \begin{cases} \psi(\mathbf{w}) & \text{if } \mathbf{w} \neq \mathbf{v}^{(1)}; \\ m_1 & \text{if } \mathbf{w} = \mathbf{v}^{(1)}. \end{cases}$$

If no vertex of $Lk(\mathbf{v}^{(1)}; \mathbf{H}^{(1)})$ is labelled m_1 by $\psi^{(1)}$, then set $\mathbf{H}^{(1)} = \tilde{\mathbf{H}}$, $\psi^{(1)} = \tilde{\psi}$. Conversely, apply the same procedure on each edge $\mathbf{e}^{(2)}_k = \mathbf{v}^{(1)} * \mathbf{w}_k$ of $\mathbf{H}^{(1)}$ with $\psi^{(1)}(\mathbf{w}_k) = m_1$, choosing the labels of the new vertices (the barycenters of the edges $\mathbf{e}^{(2)}$'s) in $\mathbb{N}_5 - \{m, m_1\}$. The subdivision $\tilde{\mathbf{H}}$ of $\mathbf{H}$ and the vertex-labelling $\tilde{\psi}$ we are looking for, are obtained by at most four such steps.

The final product of the described algorithm is a simplicial coloured subdivision CP^2_{26} of CP^2_9; it strongly depends on the order fixed on $\Sigma(CP^2_9)$ and on the colours chosen at each step. Its "dual" 5-coloured graph $\Gamma(CP^2_{26})$ will be called Π^{196}, since it has 196 vertices (each represented by an integer out of $\mathbb{N}_{196}$), corresponding to the 196 4-simplexes of CP^2_{26}. It is memorized on the computer as a list of 498 lines (records),shown in Table 2. The first two elements of each line belong to $\mathbb{N}_{196}$ and represent two different vertices i, j of Π^{196} $(i < j)$; the remaining elements of each line form a subset $\mathcal{B} = \mathcal{B}(i,j)$ of $\mathbb{N}_5$ and represent the colour-set of the edges joining i and j in Π^{196}. Note that this set always reduces to a single colour, since Π^{196} admits no multiple edges.

Table 2

1	2	colours: 2	11	25	colours: 4	22	51	colours: 2
1	3	colours: 5	11	26	colours: 3	23	37	colours: 5
1	4	colours: 1	12	27	colours: 1	23	52	colours: 3
1	5	colours: 3	12	28	colours: 4	23	39	colours: 2
1	6	colours: 4	12	111	colours: 2	24	53	colours: 2
2	7	colours: 5	13	18	colours: 2	24	38	colours: 5
2	8	colours: 1	13	29	colours: 3	24	52	colours: 1
2	9	colours: 3	13	25	colours: 1	24	50	colours: 4
2	10	colours: 4	14	30	colours: 5	25	37	colours: 2
3	11	colours: 1	14	31	colours: 4	25	54	colours: 3
3	7	colours: 2	14	32	colours: 2	25	39	colours: 5
3	19	colours: 3	14	122	colours: 1	26	55	colours: 2
3	13	colours: 4	15	33	colours: 5	26	56	colours: 5
4	11	colours: 5	15	34	colours: 2	26	57	colours: 4
4	8	colours: 2	15	35	colours: 3	27	58	colours: 2
4	14	colours: 3	15	36	colours: 1	27	59	colours: 4
4	39	colours: 4	16	21	colours: 2	28	59	colours: 1
5	9	colours: 2	16	46	colours: 1	28	60	colours: 3
5	20	colours: 5	16	32	colours: 3	28	61	colours: 2
5	15	colours: 4	16	47	colours: 4	29	53	colours: 5
5	45	colours: 1	16	48	colours: 5	29	38	colours: 2
6	10	colours: 2	17	27	colours: 3	29	54	colours: 1
6	13	colours: 5	17	37	colours: 4	29	62	colours: 4
6	39	colours: 1	18	38	colours: 3	30	63	colours: 4
6	53	colours: 3	18	37	colours: 1	30	65	colours: 1
7	17	colours: 1	19	40	colours: 2	30	66	colours: 2
7	12	colours: 3	19	41	colours: 5	30	120	colours: 3
7	18	colours: 4	19	26	colours: 1	31	63	colours: 5
8	17	colours: 5	19	92	colours: 4	31	67	colours: 1
8	21	colours: 3	20	42	colours: 4	31	68	colours: 2
8	23	colours: 4	20	43	colours: 2	31	69	colours: 3
9	12	colours: 5	20	44	colours: 1	32	66	colours: 5
9	21	colours: 1	20	64	colours: 3	32	68	colours: 4
9	22	colours: 4	21	27	colours: 5	32	70	colours: 1
10	24	colours: 3	21	49	colours: 4	33	71	colours: 2
10	18	colours: 5	22	28	colours: 5	33	72	colours: 3
10	23	colours: 1	22	49	colours: 1	33	73	colours: 1
11	17	colours: 2	22	50	colours: 3	33	74	colours: 4

34	71	colours: 5	47	68	colours: 3	62	99	colours: 1
34	51	colours: 4	48	91	colours: 1	63	118	colours: 3
34	75	colours: 3	48	94	colours: 4	63	115	colours: 1
34	76	colours: 1	48	95	colours: 2	63	116	colours: 2
35	72	colours: 5	48	124	colours: 3	64	103	colours: 5
35	75	colours: 2	49	59	colours: 5	64	114	colours: 4
35	77	colours: 1	49	96	colours: 3	64	117	colours: 2
35	53	colours: 4	50	60	colours: 5	64	120	colours: 1
36	77	colours: 3	50	96	colours: 1	65	121	colours: 3
36	73	colours: 5	50	97	colours: 2	65	115	colours: 4
36	76	colours: 2	51	98	colours: 5	65	122	colours: 5
36	45	colours: 4	51	97	colours: 3	65	123	colours: 2
37	78	colours: 3	52	78	colours: 5	66	116	colours: 4
38	78	colours: 1	52	79	colours: 2	66	125	colours: 3
38	60	colours: 4	52	96	colours: 4	66	123	colours: 1
39	79	colours: 3	53	79	colours: 1	67	126	colours: 3
40	80	colours: 5	54	78	colours: 2	67	115	colours: 5
40	55	colours: 1	54	79	colours: 5	67	122	colours: 4
40	81	colours: 3	54	99	colours: 4	67	127	colours: 2
40	82	colours: 4	55	100	colours: 5	68	116	colours: 5
41	80	colours: 2	55	101	colours: 3	68	127	colours: 1
41	56	colours: 1	55	102	colours: 4	69	79	colours: 4
41	83	colours: 4	56	100	colours: 2	69	96	colours: 2
41	103	colours: 3	56	119	colours: 3	69	126	colours: 1
42	84	colours: 2	56	104	colours: 4	69	128	colours: 5
42	85	colours: 1	57	104	colours: 5	70	129	colours: 3
42	74	colours: 5	57	92	colours: 1	70	127	colours: 4
42	114	colours: 3	57	99	colours: 3	70	122	colours: 2
43	84	colours: 4	57	102	colours: 2	70	123	colours: 5
43	86	colours: 1	58	95	colours: 5	71	130	colours: 3
43	87	colours: 3	58	105	colours: 1	71	131	colours: 1
43	111	colours: 5	58	106	colours: 3	71	98	colours: 4
44	85	colours: 4	58	107	colours: 4	72	130	colours: 2
44	88	colours: 3	59	108	colours: 3	72	112	colours: 4
44	45	colours: 5	59	107	colours: 2	72	132	colours: 1
44	86	colours: 2	60	108	colours: 1	73	132	colours: 3
45	89	colours: 3	60	109	colours: 2	73	131	colours: 2
45	46	colours: 2	61	110	colours: 1	73	133	colours: 4
46	90	colours: 3	61	109	colours: 3	74	98	colours: 2
46	76	colours: 4	61	84	colours: 5	74	134	colours: 1
46	91	colours: 5	61	111	colours: 4	74	135	colours: 3
47	93	colours: 5	62	92	colours: 3	75	130	colours: 5
47	51	colours: 1	62	112	colours: 5	75	136	colours: 1
47	49	colours: 2	62	113	colours: 2	75	97	colours: 4

76	136	colours: 3	91	188	colours: 4	110	167	colours: 3
76	131	colours: 5	93	134	colours: 2	110	165	colours: 2
77	132	colours: 5	93	150	colours: 3	110	141	colours: 5
77	136	colours: 2	93	98	colours: 1	110	146	colours: 4
77	89	colours: 4	93	151	colours: 4	111	146	colours: 1
78	108	colours: 4	94	152	colours: 2	111	139	colours: 3
80	100	colours: 1	94	151	colours: 5	112	168	colours: 2
80	137	colours: 3	94	188	colours: 1	112	160	colours: 1
80	138	colours: 4	94	154	colours: 3	113	161	colours: 1
81	101	colours: 1	95	155	colours: 3	113	168	colours: 5
81	113	colours: 4	95	149	colours: 1	114	135	colours: 5
81	139	colours: 2	95	156	colours: 4	114	169	colours: 2
81	140	colours: 5	96	108	colours: 5	114	118	colours: 1
82	92	colours: 2	97	157	colours: 1	115	170	colours: 3
82	138	colours: 5	97	158	colours: 5	115	171	colours: 2
82	113	colours: 3	98	159	colours: 3	116	172	colours: 3
82	102	colours: 1	99	160	colours: 5	116	171	colours: 1
83	92	colours: 5	99	161	colours: 2	117	137	colours: 5
83	112	colours: 3	100	162	colours: 4	117	169	colours: 4
83	138	colours: 2	100	163	colours: 3	117	140	colours: 3
83	104	colours: 1	101	161	colours: 4	117	125	colours: 1
84	141	colours: 1	101	106	colours: 2	118	120	colours: 4
84	142	colours: 3	101	124	colours: 5	118	173	colours: 5
85	143	colours: 3	102	161	colours: 3	118	172	colours: 2
85	144	colours: 5	102	162	colours: 5	119	120	colours: 5
85	141	colours: 2	103	135	colours: 4	119	173	colours: 4
86	141	colours: 4	103	137	colours: 2	119	163	colours: 2
86	145	colours: 3	103	119	colours: 1	120	125	colours: 2
86	146	colours: 5	104	160	colours: 3	121	170	colours: 4
87	145	colours: 1	104	162	colours: 2	121	147	colours: 5
87	142	colours: 4	105	149	colours: 5	121	174	colours: 2
87	140	colours: 2	105	164	colours: 3	122	147	colours: 3
87	139	colours: 5	105	146	colours: 2	123	174	colours: 3
88	143	colours: 4	105	165	colours: 4	123	171	colours: 4
88	145	colours: 2	106	155	colours: 5	124	148	colours: 1
88	121	colours: 1	106	164	colours: 1	124	154	colours: 4
88	89	colours: 5	106	166	colours: 4	124	155	colours: 2
89	90	colours: 2	107	156	colours: 5	125	172	colours: 4
89	147	colours: 1	107	165	colours: 1	125	163	colours: 5
90	136	colours: 4	107	166	colours: 3	126	170	colours: 5
90	129	colours: 1	108	166	colours: 2	126	147	colours: 4
90	148	colours: 5	109	167	colours: 1	126	157	colours: 2
91	148	colours: 3	109	142	colours: 5	127	157	colours: 3
91	149	colours: 2	109	139	colours: 4	127	171	colours: 5

128	175	colours: 1	144	184	colours: 2	164	186	colours: 5
128	176	colours: 3	144	180	colours: 4	164	182	colours: 2
128	177	colours: 2	145	183	colours: 4	164	193	colours: 4
128	189	colours: 4	145	182	colours: 5	165	184	colours: 5
129	157	colours: 4	146	182	colours: 3	165	193	colours: 3
129	174	colours: 5	148	185	colours: 4	166	177	colours: 5
129	147	colours: 2	148	186	colours: 2	166	193	colours: 1
130	168	colours: 4	149	186	colours: 3	167	193	colours: 2
130	178	colours: 1	149	184	colours: 4	167	183	colours: 5
131	178	colours: 3	150	172	colours: 5	167	182	colours: 4
131	153	colours: 4	150	163	colours: 4	168	191	colours: 1
132	178	colours: 2	150	173	colours: 2	169	172	colours: 1
132	179	colours: 4	150	159	colours: 1	170	190	colours: 2
133	180	colours: 5	151	181	colours: 2	171	190	colours: 3
133	181	colours: 1	151	187	colours: 3	174	190	colours: 4
133	179	colours: 3	151	153	colours: 1	175	194	colours: 2
133	153	colours: 2	152	181	colours: 5	175	195	colours: 4
134	181	colours: 4	152	180	colours: 1	177	194	colours: 1
134	176	colours: 5	152	189	colours: 3	178	196	colours: 4
134	173	colours: 3	152	176	colours: 4	179	195	colours: 5
135	159	colours: 2	153	188	colours: 5	179	192	colours: 1
135	173	colours: 1	153	196	colours: 3	179	196	colours: 2
136	178	colours: 5	154	185	colours: 1	180	195	colours: 3
137	159	colours: 4	154	187	colours: 5	180	188	colours: 2
137	163	colours: 1	154	189	colours: 2	181	192	colours: 3
138	168	colours: 3	155	186	colours: 1	184	194	colours: 3
138	162	colours: 1	155	177	colours: 4	185	196	colours: 5
139	182	colours: 1	156	184	colours: 1	185	188	colours: 3
140	158	colours: 4	156	177	colours: 3	185	195	colours: 2
140	174	colours: 1	156	176	colours: 2	186	194	colours: 4
141	183	colours: 3	157	190	colours: 5	187	196	colours: 1
142	183	colours: 1	158	190	colours: 1	187	192	colours: 2
142	158	colours: 2	158	169	colours: 3	187	191	colours: 4
143	183	colours: 2	159	169	colours: 5	189	192	colours: 5
143	170	colours: 1	160	191	colours: 2	189	195	colours: 1
143	175	colours: 5	160	192	colours: 4	193	194	colours: 5
144	176	colours: 1	161	191	colours: 5			
144	175	colours: 3	162	191	colours: 3			

In the second part of the algorithm, we modify the already obtained graph Π^{196} by cancelling a finite sequence of dipoles from it.

The reduction algorithm is realized by a Turbopascal Program, which has one procedure to search and one to delete dipoles from the graph. The *searching procedure* acts on each line $(i, j; \mathcal{B})$ of the list representing the graph, checking if i and j belong to different $(\mathbb{N}_5 - \mathcal{B})$-residues.

Once a dipole $\Theta = (i, j; \mathcal{B})$ has been recognized, the *cancelling procedure* works in the following way:

1) it deletes the lines, which contain either i or j from Γ (Θ included) and puts them into a new graph $\Xi(\Theta)$;

2) for each colour $c \in \mathbb{N}_5 - \mathcal{B}$, it searches the two lines $(i, h; \mathcal{B}')$ (or $(h, i; \mathcal{B}')$), $(j, k; \mathcal{B}'')$ (or $k, j : \mathcal{B}''$)) of $\Xi(\Theta)$ containing i and j respectively, such that $c \in \mathcal{B}' \cap \mathcal{B}''$; then supposing $h < k$) either it adds a new line $(h, k; \{c\})$ to Γ (if no previous line of Γ contains both h and k), or it substitutes the line $(h, k; \mathcal{A})$ (already contained in Γ), by the $(h, k; \mathcal{A} \cup \{c\})$.

Note that each step reduces the order of the graph by two. The whole reduction procedure stops either when the resulting graph is devoid of dipoles, or when it reduces to a single line (which would imply that it represents a sphere). In both cases, the resulting graph is a crystallization: thus the described algorithm shows, among other, how to get a crystallization of a closed manifold out of a coloured complex triangulating it (see also [LM]).

The sequence of 94 dipoles (each represented only by the pair of its vertices) found and cancelled, starting from Π^{196} is shown in Table 3.

Table 3

1	2	3	11	4	8
5	9	6	10	7	17
12	27	13	18	14	30
15	33	16	21	19	41
20	42	22	50	23	52
24	53	25	37	26	56
28	59	29	38	31	63
32	66	34	71	35	72
36	73	39	79	40	80
43	84	44	85	45	46
47	49	48	91	51	98
54	78	55	100	57	104
58	95	60	108	61	111
62	92	64	114	65	12
67	115	68	116	69	96
74	135	75	130	76	131
77	132	81	113	82	139
83	112	86	141	87	142
88	112	89	90	93	151
94	188	97	159	99	160
101	161	102	162	103	119
105	149	106	155	107	156
109	166	110	146	123	174
124	148	70	129	127	157
133	153	136	178	137	163
138	168	143	170	145	183
154	185	158	169	117	140
164	186	165	184	167	193
171	190	179	196	182	191
126	147	150	172	175	194
125	177	187	195	189	192
120	144				

The output of the program is the graph Π^8 presented in Table 4. It has 8 vertices (belonging to the set $\{118, 128, 134, 152, 173, 176, 180, 181\}$, and 20 edges.

Table 4

118 173 colours: 2 5	134 181 colours: 2 4
118 180 colours: 3 4	134 176 colours: 5
118 176 colours: 1	134 173 colours: 1 3
128 176 colours: 2 3	152 181 colours: 3 5
128 180 colours: 5	152 180 colours: 1 2
128 181 colours: 1	152 176 colours: 4
128 173 colours: 4	

It turns out that the graphs Π^8 (built in [G] and shown in Figure 1) and Π^8 are isomorphic by the colour-preserving isomorphism Ψ induced by the following bijection ψ between the respective vertex-sets: $\psi(1) = 118$; $\psi(2) = 173$, $\psi(3) = 134$; $\psi(4) = 181$; $\psi(5) = 152$; $\psi(6) = 180$; $\psi(7) = 176$; $\psi(8) = 128$.

It would be interesting to apply the same algorithm to the 15-vertex triangulations of the 8-manifold presented in [BrK] (which is probably the quaternionic projective plane $\mathbb{H}P^2$), in order to get a (minimal?) crystallization of it.

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