ESH WHICH INDUCE βN (*)

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SOMMARIO. - *In questo lavoro si studiano compattificazioni di Hausdorff di spazi localmente compatti che possono ottenersi a partire da ESH (abbreviazione di Essential Semilattice Homomorphism, vedi* [1]*). In particolare, si studiano gli ESH che inducono $\beta\mathbf{N}$ e si prova che non esistono ESH definiti sulla famiglia di tutti i sottoinsiemi aperti di $\beta\mathbf{N}\setminus\mathbf{N}$ che inducono $\beta\mathbf{N}$. Usando un teorema di Van Douwen, il precedente risultato viene esteso al caso di spazi non pseudocompatti.*

SUMMARY. - *This is a paper on Hausdorff compactifications of locally compact spaces which can be obtained by mean of ESH (Essential Semilattice Homomorphism, see* [1]*). We study the ESH which induce $\beta\mathbf{N}$ and we prove that there is no ESH defined on the family of all the open subsets of $\beta\mathbf{N}\setminus\mathbf{N}$ which induces $\beta\mathbf{N}$. Using a theorem of Van Douwen, we extend the previous result to non-pseudocompact spaces.*

Introduction.

Let X be a locally compact Hausdorff space and K a compact Hausdorff space. A map $f : X \to K$ is said to be singular if the inverse images of all nonempty open subsets of K are not relatively compact in X. Singular maps may be used to obtain compactifications of locally compact Hausdorff spaces. In fact, if $f : X \to K$ is singular, we can construct a Hausdorff compactification of X by putting on $X \cup K$ the topology generated by the open subsets of X and the sets of the form $U \cup \big(f^{-1}(U) \setminus F\big)$, where U is open in K and F is a compact subspace of X.

Since every singular map is dense, the Stone-Čech compactification $\beta\mathbf{N}$ of the discrete space $\mathbf{N}$ of the natural numbers, is not singular, that is, it cannot be obtained in the manner just described. More generally, if X is not pseudocompact then βX is not singular.

The following generalization of the previous construction is presented in [1]. Let $\mathcal{B}$ be a basis for the open subsets of K, closed with respect to finite unions

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and let $\mathcal{N}_X$ be the set of all open non- relatively compact subsets of X together with the empty set. Then $\pi : \mathcal{B} \to \mathcal{N}_X$ with $\pi(U) \neq \emptyset$ for every $U \neq \emptyset$ is said to be an essential semilattice homomorphism (ESH for short) if the following conditions are satisfied:

$ESH1$) $X - \pi(K)$ is compact;
$ESH2$) the symmetric difference $\pi(U \cup V)\Delta(\pi(U) \cup \pi(V))$ is relatively compact, for every $U, V \in \mathcal{B}$;
$ESH3$) if $U, V \in \mathcal{B}$ and $\overline{U} \cap \overline{V} = \emptyset$, then $\pi(U) \cap \pi(V)$ is relatively compact.

Now, a compactification of X is attached to any essential semilattice homomorphism, considering on $X \cup K$ the topology generated by the open subsets of X and the sets of the form $U \cup (\pi(U) \setminus F)$ with $U \in \mathcal{B}$ and F a compact subset of X. Such a compactification, denoted by $X \cup_\pi K$, is called an ESH-compactification. Also, we say that π induces $X \cup_\pi K$.

In [1], it is proved that, if αX is a compactification of X and $\alpha X \setminus X$ is 0-dimensional, then αX is an ESH-compactification. The same conclusion holds if βX is the Stone-Cech compactification of a paracompact space X. Thus, $\beta\mathbf{N}$ is an ESH-compactification which is not singular. Following the proof of Thm. 5 (or Thm. 4) in [1], it is possible to obtain $\beta\mathbf{N}$ from an ESH defined on the family of all the clopen subsets of $\beta\mathbf{N} \setminus \mathbf{N}$.

The first section of the present paper is concerned with the essential semilattice homomorphisms which induce $\beta\mathbf{N}$ and, more generally, compactifications with 0-dimensional remainders. From these results, we shall deduce that there is no ESH defined on the family of all the open subsets of $\beta\mathbf{N} \setminus \mathbf{N}$ which induce $\beta\mathbf{N}$. In section two, we prove, using a theorem by Van Douwen ([4]), that the previous result extends to non-pseudocompact spaces.

1. All spaces are assumed to be Hausdorff and locally compact.

We denote by $\mathcal{N}_X$ the family of all open nonrelatively compact subsets of a space X together with the empty set. Moreover, $A\Delta B$ will denote the symmetric difference of the sets A and B.

Let αX be an ESH-compactification induced by π. Then, it is easy to show that a slight modification of π gives an ESH which also induces αX. The following proposition is a consequence of the fact that finite unions and intersections are compatible with the equivalence relation in $\mathcal{N}_X$ defined by $A \sim B$ if $A\Delta B$ is relatively compact.

PROPOSITION 1.1. *If $\pi : \mathcal{B} \to \mathcal{N}_X$ is an ESH which induces αX, then every map $\pi' : \mathcal{B} \to \mathcal{N}_X$ such that $\pi(U)\Delta\pi'(U)$ is relatively compact for every $U \in \mathcal{B}$ is an ESH which also induces αX.*

Now, let αX be a compactification of a space X with 0-dimensional remainder $K = \alpha X \setminus X$. In [1] it is shown that αX is an ESH-compactification. An ESH which induces αX can be constructed, for example, in the following way: let $\mathcal{B}^*$ be the family of the clopen subsets of K. If $U \in \mathcal{B}^*$, we choose an open set O containing U, whose complement is a neighborhood of $K \setminus U$. Then $\pi : \mathcal{B}^* \to \mathcal{N}_X$ defined $\pi(U) = O \cap X$ is an ESH which induces αX.

Now, we show that, if π and π' are ESH defined on $\mathcal{B}^*$ which induce αX, then they are "almost" the same ESH.

PROPOSITION 1.2. *Let $\alpha X \setminus X$ be 0-dimensional and let $\pi, \pi' : \mathcal{B}^* \to \mathcal{N}_X$ be two ESH which induce αX. Then $\pi(U)\Delta\pi'(U)$ is relatively compact for all $U \in \mathcal{B}^*$.*

Proof. Let $U \in \mathcal{B}^*$ and let $V = K \setminus U$. By $ESH3)$ we have that $F = \overline{\pi(U) \cap \pi(V)}$ and $F' = \overline{\pi'(U) \cap \pi'(V)}$ are compact subsets of X. Now, we claim that, if we put

$$S = (U \cup \pi(U)) \cap (U \cup \pi'(U))$$

and

$$T = (V \cup \pi(V) \setminus F) \cap (V \cup \pi'(V) \setminus F')$$

then $\pi(U)\Delta\pi'(U) \subset \alpha X \setminus (S \cup T) = L$. In fact, if $x \in \pi(U) \setminus \pi'(U)$, then $x \notin S$. Moreover $x \notin \pi(V) \setminus F$, since $\pi(U)$ and $\pi(V) \setminus F$ are disjoint. Therefore $x \notin T$. Hence $x \in L$. In similar way, we can see that $x \in \pi'(U) \setminus \pi(U)$ implies $x \in L$. Since L is a compact subset of X, the conclusion follows. $\diamondsuit$

If $\mathbf{N}$ is the discrete space of natural numbers and A is a subset of $\mathbf{N}$, we put $A^* = (Cl_{\beta\mathbf{N}}A) \setminus \mathbf{N}$, where $\beta\mathbf{N}$ is the Stone-Cech compactification of $\mathbf{N}$. It is known that the family $\{A^* : A \subset \mathbf{N}\}$ is exactly the clopen subsets of $\mathbf{N}^* = \beta\mathbf{N}\setminus\mathbf{N}$. These sets form a basis for the topology of $\mathbf{N}^*$ which is closed with respect to finite unions and intersections. In the following we denote again by $\mathcal{B}^*$ the family of the clopen subsets of $\mathbf{N}^*$ and by $\mathcal{N}$ the set of all non- relatively compact (infinite) subsets of $\mathbf{N}$ together with the empty set. Moreover, $A^* \in \mathcal{B}^*$ will mean $A^* = (Cl_{\beta\mathbf{N}}A) \setminus \mathbf{N}$ with $A \subset \mathbf{N}$. If $A \subset \mathbf{N}$, we denote by $[A]$ the equivalence class of A respect to the relation $\sim$ defined before. In this case $A \sim B$ iff $A\Delta B$ is finite, that is $A^* = B^*$.

Now, the previous results imply the following one.

Proposition 1.3. *A map $\pi : \mathcal{B}^* \to \mathcal{N}$ is an ESH which induces $\beta\mathbf{N}$ if and only if the map $\epsilon : 2^{\mathbf{N}}/\sim\, \to 2^{\mathbf{N}}$ defined by $\epsilon([A]) = \pi(A^*)$ is a choice function on the family of equivalence classes of $2^{\mathbf{N}}$ satisfying $\epsilon([\emptyset]) = \emptyset$.* $\diamondsuit$

We remark that, if αX is an ESH-compactification induced by $\pi : \mathcal{B} \to \mathcal{N}_X$, then $\mathcal{B}$ contains all the clopen subsets of the remainder $K = \alpha X \setminus X$. In fact, such sets are open and compact and $\mathcal{B}$ is a basis closed with respect to finite unions.

Moreover if $\mathcal{B}' \subset \mathcal{B}$ is a basis closed with respect to finite unions then $\pi' = \pi|_{\mathcal{B}'} : \mathcal{B}' \to \mathcal{N}_X$ is again an ESH such that $X \cup_\pi K = X \cup_{\pi'} K$.

So if $\pi : \mathcal{B} \to \mathcal{N}$ is an ESH which induces $\beta\mathbf{N}$, one has $\mathcal{B}^* \subset \mathcal{B}$ and, by Proposition 1.3, it follows that $\pi(A^*)\Delta A$ is finite for all $A^* \in \mathcal{B}^*$.

Now we will extend the ESH described in Proposition 1.3, to a larger basis so that the extension is still an ESH (which induces again $\beta\mathbf{N}$). Similar extensions can be made when $\alpha X \setminus X$ is 0-dimensional.

Let $\mathcal{B}^\sim = \{B \subset \mathbf{N}^* : B$ is open and $\overline{B}$ is clopen$\}$. We note that $\mathcal{B}^\sim$, which contains $\mathcal{B}^*$, is a basis closed with respect to finite unions.

Proposition 1.4. *Let $\mathcal{B}$ be a basis for $\mathbf{N}^*$, closed with respect to finite unions such that $\mathcal{B}^* \subset \mathcal{B} \subset \mathcal{B}^\sim$ and let $\pi_0 : \mathcal{B}^* \to \mathcal{N}$ be an ESH which induces $\beta\mathbf{N}$. Then every map $\pi : \mathcal{B} \to \mathcal{N}$ such that $\pi(B)\Delta\pi_0(\overline{B})$ is finite for all $B \in \mathcal{B}$ is an ESH which induces $\beta\mathbf{N}$.*

Proof. The proof that π is an ESH is straightforward. Moreover, by Prop. 1.1 , both $\pi|_{\mathcal{B}^*}$ and π_0 induce $\beta\mathbf{N}$, hence, also π induces $\beta\mathbf{N}$. $\diamondsuit$

Proposition 1.5. *Let $\pi : \mathcal{B} \to \mathcal{N}_X$ be an ESH which induces a compactification αX and let $B_1, B_2 \in \mathcal{B}$. Then $\pi(B_1)\Delta\pi(B_2)$ relatively compact implies*

$\overline{B}_1 = \overline{B}_2$.

Proof. Let $\pi(B_1)\Delta\pi(B_2)$ be relatively compact and suppose there exists $x \in B_1 \setminus \overline{B}_2$. Choose $B \in \mathcal{B}$ with $x \in B \subset \overline{B} \subset B_1 \setminus \overline{B}_2$. Since $B \subset B_1$, by $ESH2$), it follows, there is a compact subset G of X such that $\pi(B) \subset \pi(B_1) \cup G$, hence we have

$$\pi(B) \setminus \pi(B_2) \subset (\pi(B_1) \cup G) \setminus \pi(B_2). \tag{1}$$

Now, by $ESH3$), $\overline{B} \cap \overline{B_2} = \emptyset$ implies that $\pi(B) \cap \pi(B_2)$ is relatively compact, and so from

$$\emptyset \neq \pi(B) = (\pi(B) \setminus \pi(B_2)) \cup (\pi(B) \cap \pi(B_2))$$

we deduce that $\pi(B) \setminus \pi(B_2)$ is not relatively compact. Then, by (1), we get that $(\pi(B_1) \cup G) \setminus \pi(B_2)$ is also nonrelatively compact. Since G is compact, so $\pi(B_1) \setminus \pi(B_2)$. This contradicts the hypothesis of relative compactness made on $\pi(B_1)\Delta\pi(B_2)$. Therefore, $B_1 \setminus \overline{B}_2 = \emptyset$, or $B_1 \subset \overline{B}_2$. Similarly, one has $B_2 \subset \overline{B}_1$, and so we conclude that $\overline{B}_1 = \overline{B}_2$. ♢

The converse of Prop. 1.5 holds under an additional hypothesis on π. Such hypothesis implies zero-dimensionality of $\alpha X \setminus X$ and it is satisfied by any ESH which induces $\beta\mathbf{N}$.

Proposition 1.6. *Let $\pi : \mathcal{B} \to \mathcal{N}_X$ be an ESH which induces a compactification αX, satisfying the following property: for all $B \in \mathcal{B}$ there is a clopen U in $\alpha X \setminus X$ such that $\pi(B)\Delta\pi(U)$ is relatively compact.*

Then for all $B_1, B_2 \in \mathcal{B}$, if $\overline{B}_1 = \overline{B}_2$ then $\pi(B_1)\Delta\pi(B_2)$ is relatively compact. Moreover $\overline{B}$ is clopen for all $B \in \mathcal{B}$.

Proof. Suppose $B_1, B_2 \in \mathcal{B}$. Let U_1, U_2 be clopen subsets of $\alpha X \setminus X$ for which both $\pi(B_1)\Delta\pi(U_1)$ and $\pi(B_2)\Delta\pi(U_2)$ are relatively compact.

By Prop.1.5, we have $\overline{B}_1 = U_1$ and $\overline{B}_2 = U_2$ and if $\overline{B}_1 = \overline{B}_2$ it follows $U_1 = U_2$. Then the relative compactness of $\pi(B_1)\Delta\pi(B_2)$ follows from the one's of $\pi(B_1)\Delta\pi(U_1)$ and $\pi(B_2)\Delta\pi(U_2)$. ♢

Next corollary, together with Prop.1.4, characterizes the ESH which induce $\beta\mathbf{N}$.

Corollary 1.7. *Let $\pi : \mathcal{B} \to \mathcal{N}$ be an ESH which induces $\beta\mathbf{N}$. Then $\mathcal{B}^* \subset \mathcal{B} \subset \mathcal{B}^\sim$ and $\pi(B)\Delta\pi(\overline{B})$ is finite for all $B \in \mathcal{B}$.*

Proof. If π induces $\beta\mathbf{N}$, then $\pi_0 = \pi|\mathcal{B}^*$ is also an ESH which induces again

$\beta\mathbf{N}$. Let $B \in \mathcal{B}$. If we put $A = \pi(B)$, then, by Prop.1.3, we get $\pi_0(A^*) \sim A$, that is, $\pi(A^*) \sim \pi(B)$. Hence $\pi(A^*)\Delta\pi(B)$ is finite and π satisfies the hypothesis of Prop. 1.6. Therefore, $\mathcal{B} \subset \mathcal{B}^{\sim}$ and $\pi(B)\Delta\pi(\overline{B})$ is finite for all $B \in \mathcal{B}$. $\diamondsuit$

REMARK. We have already seen in Prop. 1.2 that if $K = \alpha X \setminus X$ is a 0-dimensional remainder of a space X and π and π' are ESH defined on the family of the clopen subsets of K, then $\pi(B)\Delta\pi'(B)$ is relatively compact for all clopen subsets B of K. From Cor. 1.7, we have the same conclusion if π and π' are two ESH, which induce $\beta\mathbf{N}$, and are defined on the same basis. In [1,p. 858] an example is given of two ESH π and π', defined on the same basis $\mathcal{B}$, inducing a compactification αX of a discrete space X, such that $\pi(U)\Delta\pi'(U)$ is, in general, not relatively compact for U belonging to $\mathcal{B}$.

A space Y is said to be extremally disconnected if every open set has open closure. It is known that $\beta\mathbf{N}\setminus\mathbf{N}$ is not extremally disconnected (see, for example, [7], ex 6w) and so the previous corollary implies the following result.

THEOREM 1.8. *There is no ESH defined on the family of all the open subsets of $\beta\mathbf{N} \setminus \mathbf{N}$ which induces $\beta\mathbf{N}$.*

2. Our goal in this section is to extend Theorem 1.8 to non-pseudocompact spaces. The following theorem is a result of Van Douwen's (see [4], Open retraction lemma).

THEOREM 2.1. (Van Douwen [4]) *Let X be a (non-compact locally compact) non-pseudocompact space. Then there are a regularly closed F in $X^* = \beta X \setminus X$ and a homeomorphic copy H of $\mathbf{N}^* = \beta\mathbf{N} \setminus \mathbf{N}$ such that:*

1) $H \subset Int_{X^*} F$

2) there is an open retraction $r : F \to H$.

Following the proof in [4] it is easily seen that the copy H of $\mathbf{N}^*$ contained in X^* can be chosen such that $H = (Cl_{\beta X} N) \setminus X$ where $N \subset X$ is a C^*-embedded copy of $\mathbf{N}$. In the following, we denote by $\mathcal{T}_K$ the family of the open subsets of a topological space K.

THEOREM 2.2. *If X is non-pseudocompact, there is no ESH defined on the family of all the open subsets of $\beta X \setminus X$ which induces βX.*

Proof. From Van Douwen's open retraction Lemma, there are a (closed)

C^*-embedded copy N of $\mathbf{N}$ in X, a closed subset F of $X^* = \beta X \setminus X$ with $H = \beta N \setminus N \subset Int_{X^*}F$ and an open retraction $r : F \to H$.

Let $O = Int_{X^*}F$ and $s = r|O$. Now, suppose there is an ESH π defined on the family of all the open subsets of $\beta X \setminus X$ which induces βX. First of all, we note that, if A is a nonempty open subset of $H = \beta N \setminus N$, then $s^{-1}(A)$ is an open subset of $\beta X \setminus X$ such that $F = \pi(s^{-1}(A)) \cap N$ is infinite. In fact, suppose not. Then, if $x \in A$, the set $s^{-1}(A) \cup \big(\pi(s^{-1}(A)) \setminus F\big)$ would be an open neighborhood of x in βX, disjoint from N. A contradiction with $x \in A \subset Cl_{\beta X}N$. Of course, $\pi(s^{-1}(\emptyset)) \cap N = \emptyset$. Hence, we can consider the map $\pi' : \mathcal{T}_H \to \mathcal{N}_N$ defined by $\pi'(A) = \pi(s^{-1}(A)) \cap N$. Now, we show that π' is an ESH which induces βN.

We have $\pi'(H) = \pi(s^{-1}(H)) \cap N = \pi(O) \cap N$. Since

$$T = (O \cup \pi(O)) \cap \beta N = (\beta N \setminus N) \cup (\pi(O) \cap N)$$

is open in βN, then

$$\beta N \setminus T = N \setminus (\pi(O) \cap N) = N \setminus \pi'(H)$$

is closed in βN, hence compact. This proves $ESH1$).

Let A, B be open subsets of H. Then

$$\begin{aligned} E &= \pi(s^{-1}(A) \cup s^{-1}(B))\Delta(\pi(s^{-1}(A)) \cup \pi(s^{-1}(B))) \\ &= \pi(s^{-1}(A \cup B))\Delta(\pi(s^{-1}(A)) \cup \pi(s^{-1}(B))) \end{aligned}$$

is relatively compact in X.

Hence

$$\begin{aligned} G &= E \cap N \\ &= (\pi(s^{-1}(A \cup B)) \cap N)\Delta((\pi(s^{-1}(A)) \cap N) \cup (\pi(s^{-1}(B)) \cap N)) \\ &= \pi'(A \cup B)\Delta(\pi'(A) \cup \pi'(B)) \end{aligned}$$

is also relatively compact in X. Thus, $Cl_X G$ is compact and, since N is closed in X, we have $Cl_X G = Cl_N G$. It follows that $\pi'(A \cup B)$ $\Delta(\pi'(A) \cup \pi'(B))$ is relatively compact in N and so $ESH2$) holds.

To prove $ESH3$), let A, B be open subsets of H such that $Cl_H A \cap Cl_H B = \emptyset$. Then $r^{-1}(Cl_H A) \cap r^{-1}(Cl_H B) = \emptyset$ and so we have $Cl_F(r^{-1}(A)) \cap Cl_F(r^{-1}(B)) = \emptyset$. It follows that $Cl_{X^*}(r^{-1}(A)) \cap Cl_{X^*}(r^{-1}(B)) = \emptyset$, since F is closed in X^*. Then, it is also $Cl_{X^*}(s^{-1}(A)) \cap Cl_{X^*}(s^{-1}(B)) = \emptyset$ and so $\pi(s^{-1}(A)) \cap \pi(s^{-1}(B))$ is relatively compact in X. Therefore, also

$$(\pi(s^{-1}(A)) \cap N) \cap (\pi(s^{-1}(B)) \cap N)$$

is relatively compact in X, hence in N, which is closed in X. That is, $\pi'(A) \cap \pi'(B)$ is relatively compact in N. This proves $ESH3$).

Finally, if A is an open subset of H and $F \subset N$ is finite, then

$$A \cup (\pi'(A) \setminus F) = (s^{-1}(A) \cup (\pi(s^{-1}(A)) \setminus F)) \cap \beta N$$

which is clearly open in βN. Hence, we would have $\beta N = N \cup_\pi (\beta N \setminus N)$, where π is an ESH defined on the family of all the open subsets of $\beta N \setminus N$, contradiction. $\diamondsuit$

REMARK. If X is a pseudocompactum it may happen that X contains a C^*-embedded copy N of $\mathbf{N}$, and there exist a closed subset F in $X^* = \beta X \setminus X$ with $H = \beta N \setminus N \subset Int_{X^*} F$ and a retraction $r : F \to H$. In this case, no ESH $\pi : \mathcal{T}_{X^*} \to \mathcal{N}_X$, inducing βX, can be constructed.

An example is the pseudocompact space $\Lambda = \beta\mathbf{R} \setminus (\beta\mathbf{N} \setminus \mathbf{N})$, where we can choose $F = H = \beta\Lambda \setminus \Lambda = \beta\mathbf{N} \setminus \mathbf{N}$ and $r = 1_F$. Note that $\beta\Lambda$ is an ESH-compactification , because $\beta\Lambda \setminus \Lambda = \beta\mathbf{N} \setminus \mathbf{N}$ is 0-dimensional.

In passing, we note that, by a proof similar to that of Thm.2.2, one can prove the following.

PROPOSITION 2.3. *Let $\pi : \mathcal{B} \to \mathcal{N}_X$ be an ESH which induces a compactification αX and let Y a closed non compact subspace of X such that $\alpha Y = Cl_{\alpha X} Y$. Suppose there exist a closed subset F of $K = \alpha X \setminus X$ with $H = \alpha Y \setminus Y \subset O = Int_K F$ and a retraction $r : F \to H$. Let $s = r|O$. Suppose further that $B \in \mathcal{B}_Y = \{U \cap Y : U \in \mathcal{B}\}$ implies $s^{-1}(B) \in \mathcal{B}$. Then the map $\pi' : \mathcal{B}_Y \to \mathcal{N}_Y$ defined by $\pi'(B) = \pi(s^{-1}(B)) \cap Y$ is an ESH which induces αY.*

It is known that a compactification αX is singular iff $K = \alpha X \setminus X$ is a retract of αX. Now, since a singular compactification αX can be induced by an ESH $\pi : \mathcal{T}_K \to \mathcal{N}_X$, then, from Thm. 2.2, we obtain the well known result (see [3]).

COROLLARY 2.4. *If $\beta X \setminus X$ is a retract of βX then X is pseudocompact.*

A weakly singular compactification αX (see [5]) can be defined as a compactification such that the remainder $\alpha X \setminus X$ is a neighborhood retract. In this case, if F is a compact subset of X and $r : \alpha X \setminus F \to \alpha X \setminus X$ is a retraction of $\alpha X \setminus X$, then $\pi : \mathcal{T}_K \to \mathcal{N}_X$ defined by $\pi(U) = r^{-1}(U) \cap X$ is an ESH which induces αX. All compactifications with finite remainder are weakly singular but, in general, not singular, as, for example, the two-point compactification of $\mathbf{R}$.

It is an open question if the existence of an ESH $\pi : \mathcal{T}_{X^*} \to \mathcal{N}_X$ with $X^* = \beta X \setminus X$, inducing βX, implies βX weakly singular.

Now, we note that, if αX is a compactification with extremally disconnected remainder $K = \alpha X \setminus X$, then αX is induced by an ESH defined on $\mathcal{T}_K$. In fact, an ESH $\pi_0 : \mathcal{B}^* \to \mathcal{N}_X$, which induces αX, can be extended to $\mathcal{T}_K$ by setting $\pi(U) = \pi_0(\overline{U})$ for all $U \in \mathcal{T}_K$ (see Prop.1.4).

Hence, by Thm.2.2, we get that, if $\beta X \setminus X$ is extremally disconnected, then X is pseudocompact.

If αX is a singular compactification and $\gamma X \leq \alpha X$, then γX is again a singular compactification (see [8] and [2]). The following result holds for ESH-compactifications.

PROPOSITION 2.4. *Let $\alpha X = X \cup_\pi K$ be an ESH-compactification with $\pi : \mathcal{B} \to \mathcal{N}_X$ and let $\gamma X \leq \alpha X$. Suppose there exists a basis $\mathcal{D}$ for $\gamma X \setminus X$ closed with respect to finite unions such that, if $g : \alpha X \to \gamma X$ is the canonical quotient map, then $g^{-1}(D) \in \mathcal{B}$ for all $D \in \mathcal{D}$. Then γX is an ESH-compactification .*

Proof. It is sufficient to consider $\pi' : \mathcal{D} \to \mathcal{N}_X$ defined by $\pi'(D) = \pi(g^{-1}(D))$.

COROLLARY 2.5. *If $\beta X \setminus X$ is extremally disconnected then every compactification of X is an ESH-compactification .*

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