UNIFORMLY APPROACHABLE FUNCTIONS AND SPACES (*)

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Sommario. - *Le funzioni uniformemente approssimabili (UA) (introdotte in* [DP] *in una forma più debole) sono una naturale generalizzazione delle funzioni uniformemente continue e perfette. In questa nota si studiano le funzioni UA e gli spazi UA ovvero quegli spazi uniformi in cui ogni funzione reale continua è UA. Tali spazi comprendono propriamente gli spazi UC (spazi di Atsuji). Si caratterizzano inoltre i sottospazi di* $\mathbf{R}$ *che sono debolmente UA e si fornisce una nuova caratterizzazione degli spazi UC. Si prova infine un risultato topologico che implica, sotto l'ipotesi del continuo, l'esistenza di un insieme* $M \subseteq \mathbf{R}^n$ *tale che se* $f, g \in C(\mathbf{R}^n, \mathbf{R})$ *sono non costanti su ogni aperto e* $g(M) \subseteq f(M)$*, allora* $f = g$*.*

Summary. - *Uniformly approachable (UA) functions (introduced in [?] in a weaker form) are a common generalization of uniformly continuous functions and perfect functions. We study UA-functions and UA-spaces, i.e. those uniform spaces in which every real valued continuous function is UA. Such spaces properly include the UC-spaces (Atsuji spaces). We characterize the weakly-UA subspaces of* $\mathbf{R}$ *and give a new characterization of the UC spaces. We prove a topological result which implies, under the continuum hypothesis, the existence of a set* $M \subseteq \mathbf{R}^n$ *such that if* $f, g \in C(\mathbf{R}^n, \mathbf{R})$ *are not constant on any open set and* $g(M) \subseteq f(M)$*, then* $f = g$*.*

Work partially supported by the research projects 60% and 40% of the Italian Ministero dell'Università e della Ricerca Scientifica e Tecnologica.

(*) Pervenuto in Redazione il 17 ottobre 1994.

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1. Introduction.

UC-spaces (in which every continuous real valued function is uniformly continuous) were introduced by Atsuji as a natural generalization of compact space (see [?, ?] or [?, ?] for further generalizations of UC). Weakly uniformly approachable functions and spaces (briefly, WUA functions and spaces) were introduced in [DP]. In a WUA space X every continuous function $f: X \to \mathbf{R}$ is WUA, namely it can be approximated, in the sense of Definition 2.1, by uniformly continuous functions. WUA functions (in fact UA functions) are a common generalization of uniformly continuous functions and perfect functions (Theorem 5.2). Every UC space is obviously WUA, but there are interesting WUA spaces, for instance $\mathbf{R}$, which are not UC (however $\mathbf{R}^n$ is not WUA for $n > 1$).

In this paper we develop some basic tools for the study of WUA spaces (see Remark 10.3). In this way we arrive at a characterization of the WUA subspaces of $\mathbf{R}$ (Theorem 7.4) and some subspaces of $\mathbf{R}^2$ (§10).

In most examples the WUA-spaces that we consider have the stronger (and more natural) property of being UA (Definition 2.4). We have the strict inclusions $UC \subset UA \subset WUA$. While there are examples of WUA spaces which are not UA (for instance the real line minus a point), we do not know whether UA and WUA coincide for connected spaces. A nice property of UA-spaces is that if we "glue" in an appropriate manner two UA-spaces along a compact subspace, the result is UA (Theorem 11.1). Other closure properties of UA-spaces with respect to unions are considered in §11. For instance we prove that under some quite restrictive conditions, the union of countably many compact spaces is UA (Theorem 11.4).

In Theorem 12.1 we characterize UC spaces as those metric spaces where every bounded UA function is uniformly continuous.

The original motivation for WUA spaces comes from the study of closure operators on the category **Unif** of uniform spaces (in the sense of [?] or §13). In fact if $f: X \to \mathbf{R}$ is WUA, then f is *totally continuous*, i.e. it is continuous not only with respect to the Kuratowski closure operator, but also with respect to any closure operator of **Unif** ([?]). This shows that the class of all uniformly continuous functions $f: X \to \mathbf{R}$ cannot be "topologized", in the sense that it cannot be characterized as the class of all continuous functions with respect to some closure operator (at least taking $X = \mathbf{R}^n$). This is to be constrasted with a result of Ciesielski [?] which shows that many classes of functions from $\mathbf{R}$ to $\mathbf{R}$ can be topologized, e.g. the linear functions, the polynomials, or the analytic functions.

A crucial tool in the study of UA and WUA spaces, is a quite general topological result (Theorem 8.1) which can be of independent interest. When applied to $\mathbf{R}$ it implies for example that there is a subset M of $\mathbf{R}$, such that if $g: \mathbf{R} \to \mathbf{R}$ is a non-constant continuous function with $g(M) \subseteq M$, then g coincides with the identity function on an open interval and it is constant outside (see Example 8.7). Moreover, under the continuum hypothesis, there exists $M \subseteq \mathbf{R}^n$, such that if $f, g \in C(\mathbf{R}^n, \mathbf{R})$ are non-constant on any open set and $g(M) \subseteq f(M)$, then $f = g$.

The study of UA and WUA subsets of $\mathbf{R}^n$ is considerably more complicated than the study of the corresponding subsets of $\mathbf{R}$. We give some examples and results in §10 which should convey some idea of the difficulties involved. Further results in this direction will be given in a forthcoming paper with J. Pelant [?]. A list of open questions is given in §14.

2. Definitions and preliminary results.

Given a uniform space X we denote by $C(X)$ the set of continuous functions $f: X \to \mathbf{R}$. We use the abbreviation "f is u.c." for "f is uniformly continuous". We will make frequent use of Katětov's theorem: if X is a uniform space, F is a closed subset of X, and $[a, b]$ is a compact interval of $\mathbf{R}$, then any u.c. function $f: F \to [a, b]$ can be extended to a u.c. function $f: X \to [a, b]$ (see [?]).

Definition 2.1. Let X be a uniform space. We say that a map $f : X \longrightarrow \mathbf{R}$ is:

1. *Uniformly approachable* (briefly, *UA-function*) if for each compact subset K of X and for each $M \subseteq X$ there is a uniformly continuous function $g : X \to \mathbf{R}$ such that $g(x) = f(x)$ for each $x \in K$ and $g(M) \subseteq f(M)$. In this case we say that g is a *(K, M)-approximation* of f.
2. *Weakly uniformly approachable* (briefly, *WUA-function*) if for each $x \in X$ and for each $M \subseteq X$ there is a uniformly continuous function $g : X \to \mathbf{R}$ such that $g(x) = f(x)$ and $g(M) \subseteq f(M)$. In this case we say that g is a *(x, M)-approximation* of f.

We give several trivial properties of these two notions which show that they should be considered as a special kind of continuity placed between the usual continuity and the uniform one.

FACT 2.2. ([?]) If f is WUA, then f is continuous.

Proof. Let $f : X \to \mathbf{R}$ be a WUA map. To show that f is continuous take $M \subseteq X$. For $x \in \overline{M}$, one may choose a u.c. function $g : X \to \mathbf{R}$ with $g(x) = f(x)$ and $g(M) \subseteq f(M)$. Since g is continuous, one has $f(x) = g(x) \in g(\overline{M}) \subseteq \overline{g(M)} \subseteq \overline{f(M)}$. Therefore $f(\overline{M}) \subseteq \overline{f(M)}$. This proves that f is continuous. ♢

PROPOSITION 2.3. *Let $f \in C(X)$. We have:*

1. *Every $(K, M \setminus K)$-approximation is also a (K, M)-approximation, hence it suffices to check the existence of (K, M)-approximations for disjoint K and M.*
2. *If $K \cap \overline{M} = \emptyset$, then f has a (K, M)-approximation. Hence it suffices to check the existence of (K, M)-approximations for disjoint K and M with $K \cap \overline{M} \neq \emptyset$.*
3. *If $f(x) \in f(M)$ (in particular, if $x \in M$) then f has a (x, M)-approximation.*

Proof. 1. is clear.

2. Suppose $K \cap \overline{M} = \emptyset$. If $M \neq \emptyset$ take any point $m \in M$ and set $g_1(\overline{M}) = f(m)$, $g_1(x) = f(x)$ for each $x \in K$. The function $g_1 : K \cup \overline{M} \to \mathbf{R}$ is uniformly continuous. Now Katětov's theorem allows us to extend g_1 to a u.c. function $g : X \to \mathbf{R}$ which is obviously a (K, M)-approximation of f. If $M = \emptyset$ apply Katětov's theorem to $f|_K$.

3. Assume that $f(x) \in f(M)$. Then the constant function $g = f(x)$ is an (x, M)-approximation of f. ♢

Now we recall the following well-known notion: a uniform space X is a *UC space* (also *Atsuji space*) if every continuous function $X \to \mathbf{R}$ is uniformly continuous (see [A1] and [A2] for various characterizations of these spaces[1]). In analogy with UC spaces we introduce the following generalizations.

1 A metric space X is UC if the set X' of non-isolated points of X is compact and for each $\varepsilon > 0$ the set $D_\varepsilon = \{x \in X : d(x, X') > \varepsilon\}$ is *uniformly* discrete (i.e. the distances $d(x, y)$, for $x \neq y$ in D_ε, have a positive lower bound).

DEFINITION 2.4. A uniform space X is:

1. a *UA space* if each continuous function $f: X \to \mathbf{R}$ is uniformly approachable;
2. a *WUA space* if each continuous function $f: X \to \mathbf{R}$ is weakly uniformly approachable.

Clearly every compact space is UA (and a fortiori WUA). Using part 2 of Proposition 2.3 it is easy to see that each discrete metric space X is UA, while it is UC iff it is uniformly discrete. This first example shows that our generalizations of UC spaces behave differently from UC spaces even in the simplest case of discrete spaces. We will see below that this distinction remains present also with respect to completeness (UC spaces are complete, while UA need not be complete, see Corollary 9.2) and other properties. The following problem set in [DP] will be one of the main objectives of this paper:

Characterize the (metric) UA (WUA) functions and spaces.

Actually in [?] only WUA spaces are considered (there called UA). A useful test to prove that a space X is not UA (WUA), is given by the next lemma.

LEMMA 2.5. *If X is a normal UA (WUA) space then any closed subset of X is UA (resp. WUA).*

Proof. Let F be a closed subset of X. Then any continuous function on F can be extended to a continuous function on X. $\diamondsuit$

It is natural to call a uniform space X such that every point of X has a UA (WUA) neighborhood *locally* UA (resp. *locally* WUA). Then every point of such a space has a base of UA (resp. WUA) neighborhoods according to Lemma 2.5. Example 3.3 points out that neither UA nor WUA are local properties, by showing that a locally compact space need not be even WUA.

3. First examples.

We will show that $\mathbf{R}$ is UA while $\mathbf{R}^2$ is not. A uniform space X is *uniformly-connected* if every uniformly continuous function of X into the discrete space $\{0,1\}$ is constant.

LEMMA 3.1. *A uniformly connected UA space is necessarily connected.*

Proof. Assume X is a UA space and $X = A_1 \cup A_2$ is a partition of X into non-empty closed disjoint sets. Let $f\colon X \to \{0,1\}$ be the characteristic function of the set A_1. Then f is continuous. Fix any $a_1 \in A_1$, $a_2 \in A_2$ and set $K = \{a_1, a_2\}$. Take a (K,X)-approximation g of f. Then $g : X \to \{0,1\}$ is a non-constant uniformly continuous function, hence X is not uniformly connected.

$\diamondsuit$

COROLLARY 3.2. *If a normal uniform space X contains a closed subspace which is uniformly connected and not connected, then X is not UA.*

EXAMPLE 3.3. The following subspaces of $\mathbf{R}^2$ are not UA:

- The circle minus a finite non-empty set of points.
- The metric subspace X of $\mathbf{R}^2$ consisting of the two hyperbolas $xy = 1$, $xy = 2$ and $x > 0, y > 0$.

By Example 3.3 $\mathbf{R}^2$ is not UA since it has X as a closed subspace.

M. Burke proved the following stronger result.

FACT 3.4.([?]) $\mathbf{R}^2$ is not a WUA space since the multiplication $m(x,y) = xy$ is not WUA. This is witnessed by the subset $M = \{(x,y) \in \mathbf{R}^2 : 0 < x, y < \infty, (xy)^{-1} \in \mathbf{N}\}$ and the point $(0,0) \in \mathbf{R}^2$.

Burke's proof that $\mathbf{R}^2$ is not WUA will be generalized in Section 6 to yield a result about arbitrary uniform spaces. It can be shown that the spaces in Example 3.3 are not even WUA (see Proposition 10.1). This yields another proof that $\mathbf{R}^2$ is not WUA.

PROPOSITION 3.5. $\mathbf{R}$ *with the uniformity induced by the usual metric is UA.*

Proof. Let $f: \mathbf{R} \to \mathbf{R}$ be a continuous function. Let $K \subseteq \mathbf{R}$ be a compact set and let $M \subseteq \mathbf{R}$ be an arbitrary subset. Then K is contained in some interval $[a, b]$. If the set $(-\infty, a) \cap M$ is non-empty, let a' be an element of this set. Otherwise let a' be an arbitrary real $< a$. Analogously, if the set $(b, \infty) \cap M$ is non-empty, let b' be an element of this set, otherwise let b' be an arbitrary real $> b$. Now define $g(z) = f(z)$ for $z \in [a', b']$, $g(z) = f(a')$ for $z < a'$ and $g(z) = f(b')$ for $z > b'$. Then g is uniformly continuous, coincides with f on K, and satisfies $g(M) \subseteq f(M)$. ♢

Actually one can prove a stronger result by characterizing the UA metrics compatible with the euclidean topology of $\mathbf{R}$ (see Theorem 9.1).

4. Truncations.

The function g used in the proof of Proposition 3.5 turns out to be of great importance in our study of uniform approximation. This is why we give the following more general definition.

DEFINITION 4.1.

1. Let X be a topological space and let $f, g \in C(X)$. We say that g is a *truncation* of f if g is constant on each connected component of $\{x \mid f(x) \neq g(x)\}$.
2. Let $a, b \in [-\infty, +\infty]$ be two extended real numbers. We say that g is a (a, b)-truncation of f, if $g(x) = f(x)$ when $a < f(x) < b$, $g(x) = a$ when $f(x) \leq a$, and $g(x) = b$ when $f(x) \geq b$.

Clearly every (a, b)-truncation is a truncation. We denote by $r_{a,b}$ the (a, b)-truncation of the identity of $\mathbf{R}$ and by $f_{a,b}$ the (unique) (a, b)-truncation of f. (Obviously, $f_{a,b} = r_{a,b} \circ f$.)

EXAMPLE 4.2. Let f be an increasing continuous function on $\mathbf{R}$. Then every truncation of f is an (a, b)-truncation (for some $a, b \in [-\infty, \infty]$).

Thus the truncations of the strictly increasing continuous functions on $\mathbf{R}$ can be classified by giving two extended real numbers $a, b \in [-\infty, +\infty]$. On the other hand the sin function has truncations which are not of the above form.

Note that if g is a truncation of f and $\mathcal{U}$ is a connected component of $\mathcal{O} = \{x \mid f(x) \neq g(x)\}$, then g is constant on the closure of $\mathcal{U}$ and coincides with f on $\partial\mathcal{U}$.

The characterization of the truncations of the strictly increasing functions on $\mathbf{R}$ (Example 4.2) can be extended as follows.

DEFINITION 4.3. (Pseudo-monotone functions) Let X be a topological space. We say that $f \in C(X)$ is *pseudo-monotone*, if for every real number c, the sets $\{x \mid f(x) > c\}$ and $\{x \mid f(x) < c\}$ are connected.

LEMMA 4.4. *If $f \in C(X)$ is pseudo-monotone, then the only truncations of f are (a,b)-truncations for some $a, b \in [-\infty, +\infty]$.*

Proof. Let g be a truncation of f and let $\mathcal{O} = \{x \mid f(x) \neq g(x)\}$. $\mathcal{O}$ is the union of two disjoint open sets $\mathcal{O}^- = \{x \mid f(x) < g(x)\}$ and $\mathcal{O}^+ = \{x \mid f(x) > g(x)\}$. We claim that $\mathcal{O}^+$ and $\mathcal{O}^-$ are (empty or) connected. Granted this, g is constant on $\mathcal{O}^-$ and $\mathcal{O}^+$. Set $a := -\infty$ if $\mathcal{O}^- = \emptyset$, otherwise $a := g(x)$ for any $x \in \mathcal{O}^-$; analogously, $b := +\infty$ if $\mathcal{O}^+ = \emptyset$, otherwise $b := g(x)$ for any $x \in \mathcal{O}^+$. Then $\mathcal{O}^- = \{x \mid f(x) < a\}$ and $\mathcal{O}^+ = \{x \mid f(x) > b\}$, so that $\mathcal{O}^- \cap \mathcal{O}^+ = \emptyset$ yields $a \leq b$. It is easy to see now that g coincides with the (a,b)-truncation of f. To prove the claim suppose for a contradiction that one of $\mathcal{O}^-$ and $\mathcal{O}^+$, say $\mathcal{O}^+$, is not connected, and let $\mathcal{U}_1$ and $\mathcal{U}_2$ be distinct connected components of $\mathcal{O}^+$. It follows that $\mathcal{U}_1$ and $\mathcal{U}_2$ are also connected components of $\mathcal{O}$. Since g is a truncation of f, there are constants c_1 and c_2 such that $g = c_i$ on $\mathcal{U}_i$ and $g = f = c_i$ on $\partial\mathcal{U}_i$ $(i = 1, 2)$. Without loss of generality suppose that $c_1 \leq c_2$ and let $[f > c_1]$ be the open set $\{x \mid f(x) > c_1\}$. Since $[f > c_1] \cap \partial\mathcal{U}_1 = \emptyset$, $[f > c_1]$ can be written as $([f > c_1] \cap \mathcal{U}_1) \cup ([f > c_1] \cap X \setminus \overline{\mathcal{U}_1})$. The first member of this union coincides with $\mathcal{U}_1$ and therefore is a non-empty open set. The second one is an open set containing $\mathcal{U}_2$, and therefore it is also a non-empty open set. It follows that $[f > c_1]$ is not connected, contradicting the hypothesis that f is pseudo-monotone. $\diamondsuit$

5. Perfect functions on $\mathbf{R}^n$ are UA.

A continuous map $f : X \to Y$ between topological spaces is called *perfect* if it sends closed sets to closed sets and inverse images of points of Y are compact subsets of X. Then also inverse images of compact sets are compact (cf. [E, P3.X]). It is easy to see that in case Y is locally compact, a continuous map $f : X \to Y$ is perfect iff the inverse image under f of every compact subset of Y is a compact subset of X. In the sequel we consider perfect functions $f\colon X \to \mathbf{R}$.

It is easy to find perfect continuous functions which are not uniformly continuous. For example take for $n \geq 1$ and $X = \mathbf{R}^n$ the function f defined by $f(x_1, ..., x_n) = a_1 x_1^{k_1} + ... + a_n x_n^{k_n}$, where each a_i is a positive real number and each k_i is an even natural number.

LEMMA 5.1. *If $f\colon \mathbf{R}^n \to \mathbf{R}$ is perfect, then every (a, b)-truncation $f_{a,b}$ of f with $a, b \in \mathbf{R}$ (i.e. with $a \neq -\infty$ and $b \neq +\infty$) is uniformly continuous.*

Proof. Assume $g = f_{a,b}$ is not uniformly continuous. Then $a \neq b$ and there exist $\varepsilon > 0$ and two sequences $\{x_m\}$ and $\{y_m\}$ in $\mathbf{R}^n$ such that

$$d(x_m, y_m) < 1/m \text{ and } |g(x_m) - g(y_m)| \geq \varepsilon. \tag{1}$$

By the continuity of g neither of the sequences has a convergent subsequence. Since the closure K of the open set $g^{-1}(a, b) = f^{-1}(a, b)$ is compact, it follows that only finitely many points x_m and y_m belong to K. By the definition of g this means that for all but finitely many m's, $g(x_m)$ and $g(y_m)$ belong to $\{a, b\}$. By (1) we can assume that $\{g(x_m), g(y_m)\} = \{a, b\}$. Let C_m be the segment $[x_m, y_m]$ in $\mathbf{R}^n$. Then there exists a point $z_m \in C_m$ such that $g(z_m) = (a + b)/2$. Since $K_1 = f^{-1}((a + b)/2)$ is compact and $f(z_m) = g(z_m) \in K_1$, we can find a converging subsequence of $\{z_m\}$. This will produce a converging subsequence of $\{x_m\}$ (and $\{y_m\}$), a contradiction. ♢

THEOREM 5.2. *Every perfect continuous function $f\colon \mathbf{R}^n \to \mathbf{R}$ is UA.*

Proof. Fix $K \subseteq \mathbf{R}^n$ compact, and $M \subseteq \mathbf{R}^n$. Let $a, b \in \mathbf{R}$ be such that $f(K) \subseteq [a, b]$. If possible choose a and b in $f(M)$, namely choose $a, b \in \mathbf{R}$ such that $f(K) \subseteq [a, b]$ and:

i) either $a \in f(M)$ or there are no points of $f(M)$ smaller than a;

ii) either $b \in f(M)$ or there are no points of $f(M)$ greater than b.

Let $g = f_{a,b}$. Clearly $g_{|K} = f_{|K}$. By our choice of a and b, $g(M) \subseteq f(M)$. By Lemma 4.4, g is uniformly continuous. ♢

REMARK 5.3. The proof of Theorem 5.2 works in a much more general situation, for example with $\mathbf{R}^n$ substituted by any metric space X with "short connecting sets" - this means that for every pair of points x, y we can choose a connected set $A(x, y)$ containing x and y in such a way that the diameter of $A(x, y)$ converges to 0 as $d(x, y)$ goes to 0. Note that there are connected and locally arcwise connected metric spaces X for which Theorem 5.2 fails, for instance the circle minus one point. So the assumption of having "short connecting sets" is necessary.

Theorem 5.2 does not not permit the approximation of bounded functions. Actually, if a bounded function f is perfect, then its domain is compact, so f is uniformly continuous. To give the reader a feeling of the properties of UA functions, we characterize the UA quadratic forms. We need the following:

PROPOSITION 5.4. *If $\pi: X \times Y \to X$ is the projection on the first coordinate and $f: X \to \mathbf{R}$ is UA, then $f \circ \pi: X \times Y \to \mathbf{R}$ is UA.*

Proof. Given two subsets K and M of $X \times Y$, with K compact, let $g: X \to \mathbf{R}$ be a $(\pi(K), \pi(M))$-approximation of f. Then $g \circ \pi: X \times Y \to \mathbf{R}$ is a (K, M)-approximation of $f \circ \pi$. $\diamondsuit$

LEMMA 5.5. *Let $f(x, y) = ax^2 + bxy + cy^2$, $a, b, c \in \mathbf{R}$, be a quadratic form in $\mathbf{R}^2$. Then the following are equivalent:*

a) $f: \mathbf{R}^2 \to \mathbf{R}$ *is* UA;

b) $f: \mathbf{R}^2 \to \mathbf{R}$ *is* WUA;

c) $\Delta = b^2 - 4ac \leq 0$.

Proof. a) $\to$ b) is obvious. To prove the implication b) $\to$ c) assume that $\Delta > 0$. Then f factorizes in distinct linear factors and after an appropriate linear substitution one can assume that $f(x, y) = xy$. Now f is not UA by Example 3.4.

To prove the last implication c) $\to$ a) assume that $\Delta = b^2 - 4ac \leq 0$. Then after an appropriate linear substitution one can assume that either $f(x, y) = x^2$ or $f(x, y) = x^2 + y^2$. In the first case f is UA as a function depending on only one of the variables (by Proposition 3.5 and Proposition 5.4). In the second case the function is perfect, so we can apply Theorem 5.2 to conclude that f is UA. $\diamondsuit$

COROLLARY 5.6. *A quadratic form f is UA if and only if every (a,b)-truncation of f, with $a,b \in \mathbf{R}$, is uniformly continuous.*

Proof. After a suitable linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ we can reduce to the case in which f is either of the form xy, or x^2+y^2, or x^2 (we use the fact that $(f \circ T)_{a,b} = f_{a,b} \circ T$). Clearly $(x^2+y^2)_{a,b}$ and $(x^2)_{a,b}$ are uniformly continuous for every $a,b \in \mathbf{R}$. On the other hand $(xy)_{0,1}$ is not uniformly continuous. ♢

One can use Lemma 5.5 to show that the sum of two UA functions (quadratic forms) need not be even WUA: just take $f_1(x,y) = x^2$ and $f_2(x,y) = -y^2$. Note that f_1 is UA although it is neither perfect nor uniformly continuous (indeed, the sum of a UA and a u.c. function is always UA). We also obtain further examples of non-perfect UA functions which are not uniformly continuous: $f(x_1,\ldots,x_n) = a_1x_1^{k_1} + \ldots + a_m x_m^{k_m}$ for $1 < m < n$, positive real numbers a_i, and even natural numbers k_i. These functions are UA by Proposition 5.4 since they can be obtained by composition of a projection and a perfect function.

6. Spaces containing pseudo-hyperbolas are not WUA.

Lemma 2.5 tells us that to show that a normal uniform space X is not UA (WUA), it suffices to find a closed subspace which is not UA (resp. WUA). For instance in the case of $\mathbf{R}^2$ one can take as a closed subspace the union of the x-axis, the y-axis, and the family of hyperbolas $H_n = \{(x,y) \in \mathbf{R}^2 \mid xy = 1/n\}$ for $n \in \mathbf{N}$. This example can be generalized as follows.

DEFINITION 6.1. Let X be a uniform space. A family of *pseudo-hyperbolas* in X is given by a countable family $\{H_n\}$ of disjoint subsets of X such that for every $n \in \mathbf{N}$:

1. H_n is closed and uniformly connected;
2. $H_n \cup H_{n+1}$ is uniformly connected;
3. $H_n \cap \overline{\bigcup_{m>n} H_m} = \emptyset$;
4. the set $H = \bigcup_n H_n$ is not closed in X.

EXAMPLE 6.2. A family of pseudo-hyperbolas in $\mathbf{R}^2$ is given by the sets $H_n = \{(x,y) \mid (xy)^{-1} = n\}$.

THEOREM 6.3. *If a normal uniform space X has a family of pseudo-hyperbolas, then X is not WUA.*

Proof. Let the family of pseudo-hyperbolas be given by the closed sets $H_n \subseteq X$. Let $H = \bigcup_n H_n$. Fix a point $p \in \overline{H} \setminus H$. Define $f\colon \overline{H} \to [0,1]$ as follows:

a) The restriction of f to H_n is the constant function $1/n$;

b) For $x \in \overline{H} \setminus H$, $f(x) = 0$.

We claim that f is continuous on $\overline{H}$. We first show that f is continuous at every point $x \in H_n$. To this aim it suffices to find a neighborhood of x where f is constant. Since $H_n \cap \overline{\bigcup_{m>n} H_m} = \emptyset$, there is a neighborhood of x not intersecting $\overline{\bigcup_{m>n} H_m}$. Moreover since each H_m is closed and the H_m's are disjoint, there is a neighborhood of x not intersecting the closed set $H_0 \cup H_1 \cup \ldots \cup H_{n-1}$. On the intersection of the two neighborhoods f is the constant $1/n$.

It remains to show that f is continuous at every point $x \in \overline{H} \setminus H$. Let $m \in \mathbf{N}$. Then $F = \bigcup_{n \leq m} H_n$ is closed and $x \notin F$. Hence there exists a neighborhood U of x missing F. Obviously $|f(u)| < 1/m$ for each $u \in U$. Hence f is continuous at x.

Since X is normal and $\overline{H}$ is closed, we can extend f to a continuous function $f\colon X \to [0,1]$. Suppose for a contradiction that X is WUA and let g be a uniformly continuous function on X such that $g(p) = f(p) = 0$ and $g(H) \subseteq f(H)$. Since H_n is uniformly connected, $g(H_n)$ is uniformly connected. On the other hand, $g(H_n) \subseteq f(H) = \{1/n \mid n \in \mathbf{N}\}$. Thus $g(H_n)$ is a singleton, hence g is constant on H_n. Now since $H_n \cup H_{n+1}$ is uniformly connected and g is uniformly continuous, g is constant on the whole of H. It follows that g is constant on $\overline{H}$ and therefore $g = 0$ on $\overline{H}$ (as $g(p) = f(p) = 0$). Thus $g(H) \not\subseteq f(H)$ since $0 \in g(H)$ and $0 \notin f(H)$. Contradiction. ♢

7. Subsets of R.

We know that $\mathbf{R}$ itself is UA. It follows from Lemma 2.5 that any closed subset of $\mathbf{R}$ is UA. However many subsets of $\mathbf{R}$ contain pseudo-hyperbolas and therefore are not even WUA (by Theorem 6.3). We will show that pseudo-hyperbolas are sufficient to characterize the WUA subsets of $\mathbf{R}$.

LEMMA 7.1. *Let X be a subset of $\mathbf{R}$ with the uniformity induced by the usual metric and let x belong to the interior of X. Then for every $M \subseteq X$ and every $f \in C(X)$ there exists a (x, M)-approximation of f.*

Proof. First note that it suffices to find an $(x, M \cap [x, +\infty))$-approximation of $f^+ = f_{|X \cap [x,+\infty)}$ and an $(x, M \cap (-\infty, x])$-approximation of $f^- = f_{|X \cap (-\infty,x]}$, for then we can glue together the two approximations. Without loss of generality we consider f^+. By Proposition 2.3 we can assume that $x \notin M$ and $x \in \overline{M \cap [x, +\infty)}$. Hence there exists $m \in M$ such that $x < m$ and $[x, m] \subseteq X$ (as x is in the interior of X). Define $g\colon [x, +\infty) \to \mathbf{R}$ by setting $g = f$ on $[x, m]$ and $g(t) = f(m)$ for $t > m$. Then g is an $(x, M \cap [x, +\infty))$-approximation of f^+. ♢

COROLLARY 7.2. *Every open subset of $\mathbf{R}$ is WUA.*

EXAMPLE 7.3. The space $X = \mathbf{R} \setminus \{0\}$ is WUA, but not UA. In fact, being uniformly connected and not connected, X cannot be UA by Lemma 3.1 . By Corollary 7.2 X is WUA as it is an open subset of $\mathbf{R}$. More generally, one can show that an open subset of $\mathbf{R}$ is UA iff it is regular open (i. e. coincides with the interior of its closure).

We come now to a characterization of the WUA subspaces of $\mathbf{R}$.

THEOREM 7.4. *For a subspace X of $\mathbf{R}$ the following are equivalent:*

a) X is WUA;

b) X does not contain pseudo-hyperbolas;

c) for every interval Δ of $\mathbf{R}$ such that $X \cap \Delta$ is dense in Δ, $X \cap \Delta$ is an open subset of Δ.

Proof. a) $\to$ b) is done in Theorem 6.3.

b) $\to$ c). Suppose that $X \cap \Delta$ is dense in the interval Δ and $X \cap \Delta$ is not open in Δ. Then there is a point $p \in X \cap \Delta$ and a converging sequence $x_n \to p$ with $x_n \in \Delta \setminus X$. By taking a subsequence we can assume that x_n is strictly increasing or strictly decreasing. Suppose without loss of generality that x_n is strictly increasing, i.e. $x_n < x_{n+1}$. Let $H_n = X \cap (x_n, x_{n+1})$.

Then the sets H_n constitute a system of pseudo-hyperbolas. So X is not WUA.

c) $\to$ a). Assume that for every open interval Δ of $\mathbf{R}$ such that $X \cap \Delta$ is dense in Δ, $X \cap \Delta$ is an open subset of Δ. Let $f: X \to \mathbf{R}$ be a continuous function, $x \in X$ and $M \subseteq X$. To get an (x, M)-approximation of f it suffices to find an $(x, M \cap [x, +\infty))$-approximation of f restricted to $X \cap [x, +\infty)$ and an $(x, M \cap (-\infty, x])$-approximation of f restricted to $X \cap (-\infty, x]$. Without loss of generality we consider $f_{|X \cap [x,+\infty)}$.

By Proposition 2.3 we can assume $x \notin M$ and $x \in \overline{M \cap [x, +\infty)}$. If there is an interval $[x, b)$ entirely contained in X, we argue as in Lemma 7.1. So we can assume that there is no such an interval. But then by our assumptions on X, it follows that X is not dense in $[x, b) \cap X$ for every $b > x$. Hence there exist two decreasing converging sequences $a_n \to x$ and $b_n \to x$ in $\mathbf{R}$ such that for each n, $b_{n+1} < a_n < b_n$ and the interval $[a_n, b_n]$ is disjoint from X. Since $x \in \overline{M \cap [x, +\infty)}$, there exists a sequence $u_n \in f(M \cap [x, +\infty))$ converging to $f(x)$. Define $g: X \cap [x, +\infty) \to \mathbf{R}$ as a step-wise function which assumes the constant value u_{n+1} on $X \cap [b_{n+1}, a_n]$, assumes the value u_0 on $X \cap [b_0, +\infty)$, and coincides with f at x. Then $g(x) = f(x)$ and $g(M \cap [x, +\infty)) \subseteq f(M \cap [x, +\infty))$. To prove that g is u.c. it suffices to note that g is continuous at x and uniformly continuous on $[c, +\infty)$ for any $c > x$. ♢

The above characterization of WUA subsets of $\mathbf{R}$ provides the following "local" criterion (which fails for subsets of $\mathbf{R}^2$).

Corollary 7.5. *Let X be a subset of $\mathbf{R}$. Then the following are equivalent:*

1. *X is WUA;*
2. *X is locally WUA.*

Example 7.6. The rational numbers form a non-WUA space according to Theorem 7.4. An example of a continuous function $f: \mathbf{Q} \to \mathbf{R}$ which cannot be approximated by a uniformly continuous function can be given as follows. Consider an increasing sequence of irrational numbers a_n converging to 0. Define $f: \mathbf{Q} \to \mathbf{R}$ by: $f(x) = 1/n$ if $a_n < x < a_{n+1}$, $f(x) = 0$ on the remaining rational numbers. The sets $H_n = \{x \in \mathbf{Q} \mid a_n < x < a_{n+1}\}$ form a system of pseudo hyperbolas, and reasoning as in Theorem 6.3 we see that f cannot be approximated.

By Corollary 7.5 locally compact subsets of $\mathbf{R}$ are WUA. As the following example shows the converse is not true (for the set X). It also shows that the property WUA may be destroyed by adding only a single point.

EXAMPLE 7.7. For $n \in \mathbf{N}$ set $\Delta_n = (1/(n+1), 1/n)$, $Y = \cup_{n=1}^{\infty} \Delta_{2n}$, $Z = \cup_{n=1}^{\infty} \Delta_n$ and $X = \{0\} \cup Y$. Then: i) Z is WUA but not UA, while $\{0\} \cup Z$ is not WUA, ii) X and Y are UA.

Proof. i) follows from Corollary 7.2 and Lemma 3.1 since Z is uniformly connected, but not connected.

ii) To check that X is UA let $f \in C(X)$ and take a compact subset K of X and $\emptyset \neq M \subseteq X$. Then, for each n, $K_n = K \cap \Delta_{2n}$ is a compact subset of Δ_{2n}. It is not restrictive to assume that $0 \in K$. For each n there exist reals $1/(2n+1) < a_n \leq b_n < 1/(2n)$ such that $K_n \subseteq [a_n, b_n]$. As next step we "enlarge" each compact K_n to a compact interval $[a_n', b_n'] \subseteq \Delta_{2n}$ containing the interval $[a_n, b_n]$ and having the property that either $a_n' \in M$ or $a_n' = a_n$ and $(1/(2n+1), a_n] \cap M = \emptyset$. Define b_n' analogously. Define g to agree with f on the set $\{0\} \cup \bigcup_{n=0}^{\infty} [a_n', b_n']$, $g(z) = f(a_n')$ for each $z \in (1/(2n+1), a_n']$ and $g(z) = f(b_n')$ for each $z \in [b_n', 1/(2n))$. It is easy to see that the restriction of the function g on the *compact* set $X' = \{0\} \cup \bigcup_{n=0}^{\infty} [a_n', b_n']$ is continuous, so u.c. Elsewhere the function is locally constant, so that again the restriction on $X \setminus X'$ is u.c. Since the sequence of these constants converges to $f(0)$ it is easy to conclude that g is the desired approximation.

It is much easier to see that Y is UA. In fact, fix $f \in C(Y)$ and $M \subseteq Y$. Now each compact $K \subseteq Y$ meets only finitely many intervals Δ_n. On these intervals we carry out the same construction as in the case of X. On the remaining intervals we let g assume a constant value in $f(M)$. $\diamondsuit$

8. The magic set.

We prove a topological result which can be used to show that many subsets of $\mathbf{R}^2$ are not UA (actually not even WUA). Our result applies to any separable topological space. We recall that the *fibers* of a function $f\colon X \to Y$ are the counterimages $f^{-1}(y)$ of the points $y \in Y$.

THEOREM 8.1. *Let X be a separable topological space. Then there is a set $M \subseteq X$ such that for every $f, g \in C(X)$, if f has countable fibers and $g(M) \subseteq f(M)$, then g is a truncation of f. Moreover if H is a countable* *subset of X, we can choose M so that $M \cap H = \emptyset$.*

Proof. Let $\mathcal{C}$ be the set of all pairs of continuous functions $(f, g) \in C(X) \times C(X)$ such that f has countable fibers and g is not a truncation of f. Since X is separable, $\mathcal{C}$ has cardinality $2^{\aleph_0}$. Thus we can write $\mathcal{C} = \{(f_\alpha, g_\alpha) \mid \alpha < 2^{\aleph_0}\}$. We must prove that there is $M \subseteq X$ such that for every $(f, g) \in \mathcal{C}$, $g(M) \nsubseteq f(M)$. We construct $M \subseteq X$ by stages. At stage $\alpha < 2^{\aleph_0}$ we will put a new element m_α in M. The idea is that at stage α we want to "kill" (f_α, g_α). The definition of $m_\alpha \in X$ is done by transfinite induction on $\alpha < 2^{\aleph_0}$. Suppose that for each $\beta < \alpha$ we have already defined $m_\beta \in X$. We need to define m_α. Consider the function g_α of our enumeration. Since g_α is not a truncation of f_α, there is a connected component $\mathcal{U}_\alpha$ of $\{x \mid f_\alpha(x) \neq g_\alpha(x)\}$, such that g_α is non-constant on $\mathcal{U}_\alpha$. The image $g_\alpha(\mathcal{U}_\alpha)$ is a non-trivial connected set of $\mathbf{R}$, so it has the cardinality of the continuum. Choose m_α so that the following conditions hold:

- $m_\alpha \in \mathcal{U}_\alpha$ $\quad (1)_\alpha$
- $m_\alpha \notin H$ $\quad (2)_\alpha$
- $m_\alpha \notin \bigcup_{\gamma < \alpha} f_\gamma^{-1}(g_\gamma(m_\gamma))$ $\quad (3)_\alpha$
- $g_\alpha(m_\alpha) \notin \bigcup_{\gamma < \alpha} f_\alpha(m_\gamma)$ $\quad (4)_\alpha$

Note that m_α exists as desired because: i) $|\mathcal{U}_\alpha| \geq 2^{\aleph_0}$, ii) $|H| \leq \aleph_0$, iii) the set on the RHS of (3) has cardinality $< 2^{\aleph_0}$ (since f_γ has countable fibers), iv) $|g_\alpha(\mathcal{U}_\alpha)| \geq 2^{\aleph_0}$ (as it is a non-trivial connected subset of $\mathbf{R}$).

Let $M = \{m_\alpha \mid \alpha < 2^{\aleph_0}\}$. It is clear that $M \cap H = \emptyset$. To finish the proof it suffices to show that $g_\alpha(m_\alpha) \notin f_\alpha(M)$. Suppose for a contradiction that:

$$g_\alpha(m_\alpha) = f_\alpha(m_\gamma) \tag{2}$$

There are three cases:

Case 1. Suppose $\alpha = \gamma$. Then $g_\alpha(m_\alpha) = f_\alpha(m_\alpha)$ contradicting $m_\alpha \in \mathcal{U}_\alpha$.

Case 2. Suppose $\alpha < \gamma$. By (2), $m_\gamma \in f_\alpha^{-1} g_\alpha(m_\alpha)$, contradicting $(3)_\gamma$.

Case 3. Suppose $\alpha > \gamma$. By (2), $g_\alpha(m_\alpha) = f_\alpha(m_\gamma)$, contradicting $(4)_\alpha$.
◊

We call the set M of Theorem 8.1 a *magic set*. We give now an application to the study of UA spaces.

Corollary 8.2. *Let X be a separable uniform space and suppose that there exists $K \subseteq X$ compact and $f \in C(X)$ with countable fibers such that f has no uniformly continuous truncations g which agree with f on K. Then X is not UA.*

Proof. Fix a magic set $M \subseteq X$. If X is UA, then it has a (K, M)-approximation g. Then g is a u.c. truncation of f which agrees with f on K. ♢

The set H in Theorem 8.1 is needed to give the following application to WUA spaces.

Corollary 8.3. *Let X be a separable uniform space and suppose that there exists $f \in C(X)$ with countable fibers without non-constant uniformly continuous truncations. Then X is not WUA.*

Proof. Let $f \in C(X)$ be as stated in the hypothesis. Let $x_0 \in X$, let $y_0 = f(x_0)$, and let $H = f^{-1}(y_0)$. By Theorem 8.1 there exists a set $M \subseteq X$ such that $M \cap H = \emptyset$ and such that every function $g \in C(X)$ with $g(M) \subseteq f(M)$ is a truncation of f. Since $M \cap H = \emptyset$, $y_0 \notin f(M)$. It follows that there cannot be a $(\{x_0\}, M)$-approximation of f. In fact suppose that g is such an approximation. Then g is a uniformly continuous truncation of f, hence it is constant. But since $g(x_0) = f(x_0) = y_0$, g must be the constant function with value y_0. This is absurd since $g(M) \subseteq f(M)$ and $y_0 \notin f(M)$. ♢

Remark 8.4. Note that many spaces do not admit continuous functions with countable fibers, but we can still apply the above result by considering instead of the space itself a suitable closed subspace admitting such functions (by Lemma 2.5 if a closed subspace is not WUA, then the space itself is not WUA). Note also that in the above corollary we have actually proved a stronger result: not only X is not WUA, but f does not admit (x, M)-approximations for *every* $x \in X$.

Assuming the continuum hypothesis CH, a slight modification of the proof of Theorem 8.1 yields the following result.

Theorem 8.5. (CH) *Let X be a separable Baire space (e.g. $X = \mathbf{R}^n$). Then there is a set $M \subseteq X$ such that for every $f, g \in C(X)$ non constant* *on each open set, if* $g(M) \subseteq f(M)$*, then* $f = g$*.*

Proof. Let $\mathcal{C} = \{(f_\alpha, g_\alpha) \mid \alpha < 2^{\aleph_0}\}$ be the set of all pairs of continuous functions $(f, g) \in C(X) \times C(X)$ such that f and g have nowhere dense fibers and $f \neq g$. Let $\mathcal{U}_\alpha = \{x \in X \mid f_\alpha(x) \neq g_\alpha(x)\}$. Since g_α has nowhere dense fibers, g_α is non-constant on $\mathcal{U}_\alpha$. Now define $M = \{m_\alpha \mid \alpha < 2^{\aleph_0}\}$ in such a way that m_α satisfies the clauses $(1)_\alpha$, $(3)_\alpha$ and $(4)_\alpha$ of Theorem 8.1. m_α exists as desired since in a Baire space the union of countably many nowhere dense sets has empty interior. ♢

The continuum hypothesis is not needed if we assume that f and g have countable fibers. So we have:

PROPOSITION 8.6. *Let* X *be a separable locally connected Tychonoff space without isolated points. Then there is a set* $M \subseteq X$ *such that for every* $f, g \in C(X)$ *with countable fibers, if* $g(M) \subseteq f(M)$*, then* $f = g$*.*

The proof is a trivial modification of the previous one. The assumption that X is a locally connected Tychonoff space is only needed to ensure that the non-empty open sets have cardinality $\geq 2^{\aleph_0}$.

EXAMPLE 8.7. There exists $M \subseteq \mathbf{R}$ such that if $g \in C(\mathbf{R})$ and $g(M) \subseteq M$, then g coincides with the identity on an open interval and is the identity outside (apply Theorem 8.1 with $f =$ the identity function). In particular if g has countable fibers, then g is the identity.

Note that in Example 8.7 M must be dense in $\mathbf{R}$. Also note that one cannot take for M the rational or the irrational numbers (for $g(x) = |x|$).

9. R with other metrics.

THEOREM 9.1. *Let* d *be a metric on* $\mathbf{R}$ *compatible with the euclidean topology of* $\mathbf{R}$*. Then the following are equivalent:*

1. $(\mathbf{R}, d)$ *is not* UA*;*
2. $(\mathbf{R}, d)$ *is not* WUA*;*
3. $\exists\{x_n\} \to -\infty$, $\exists\{y_n\} \to +\infty$ *with* $d(x_n, y_n) \to 0$*.*

Proof. $2 \to 1$ is obvious.

To prove $3 \to 2$ define the function $f: \mathbf{R} \to \mathbf{R}$ to be the identity. Now f is continuous and strictly increasing, so that by Example 4.2 every truncation of f is an (a, b)-truncation. Hence f has no uniformly continuous truncations beyond the constant ones. Therefore, by Corollary 8.3, f is not WUA.

$1 \to 3$. It suffices to see that if 3 fails then X is UA. In fact, let $f: \mathbf{R} \to \mathbf{R}$, $n \in \mathbf{N}$ and $M \subseteq \mathbf{R}$. To find an $([-n, n], M)$-approximation of f choose $a, b \in \mathbf{R}$ such that $a \leq -n$, $n \leq b$ and such that: i) either $a \in f(M)$ or there are no points of $f(M)$ smaller than a; ii) either $b \in f(M)$ or there are no points of $f(M)$ greater than b. Define g to be the truncation of f which agrees with f on $[a, b]$. Then g is constant on $(-\infty, a]$ and on $[b, +\infty)$. If g were u.c., then g would an $([a, b], M)$-approximation of f. If g were not u.c., then obviously $f(a) \neq f(b)$. Then for some $\varepsilon > 0$ $\exists \{x_n\}$, $\exists \{y_n\}$ with $d(x_n, y_n) \to 0$ and

$$|g(x_n) - g(y_n)| \geq \varepsilon. \tag{3}$$

By the continuity of g no subsequence of these sequences is convergent. Thus every compact interval contains only finitely many of these points. Taking subsequences we may assume that both sequences diverge to $-\infty$ or $+\infty$. By (3) one of them diverges to $+\infty$ and the other one to $-\infty$. $\diamondsuit$

Corollary 9.2. *$\mathbf{R}$ has one metric for which it is UA and complete (the usual one), one metric for which it is UA and not complete (the one induced by the homeomorphism $f : \mathbf{R} \to (-\pi/2, \pi/2)$, $f(x) = \operatorname{arctg} x$) and one metric for which it is not even WUA (the one induced by a homeomorphism of $\mathbf{R}$ with the unit circle minus one point).*

The proof of Theorem 9.1 works also for uniformities $\mathcal{U}$ compatible with the euclidean topology of $\mathbf{R}$. Then clause 3 becomes: for each entourage $U \in \mathcal{U}$ and each natural n there exist $x < -n$ and $y > n$ with $(x, y) \in U$.

10. Subsets of $\mathbf{R}^n$.

We have seen that a subset of $\mathbf{R}$ is WUA iff it has no pseudo-hyperbolas. This fails for subsets of $\mathbf{R}^2$ as the following proposition shows.

Proposition 10.1. The following subspaces of $\mathbf{R}^2$ are not WUA (and yet they do not have pseudo-hyperbolas).

1. The space X_1 consisting of the unit circle minus a non-empty finite set of points.

2. The space X_2 consisting of the union of the two hyperbolas $xy = 1$ and $xy = 2$ with $x, y \geq 0$.

3. The space $X_3 =$ Ladder A depicted in Figure 1.

Proof. We leave to the reader the easy verification that X_1, X_2 and X_3 have no pseudo-hyperbolas. To prove that they are not WUA we apply Theorem 8.3. So we must find, on each of the above spaces, a continuous function with countable fibers and without non-constant uniformly continuous truncations.

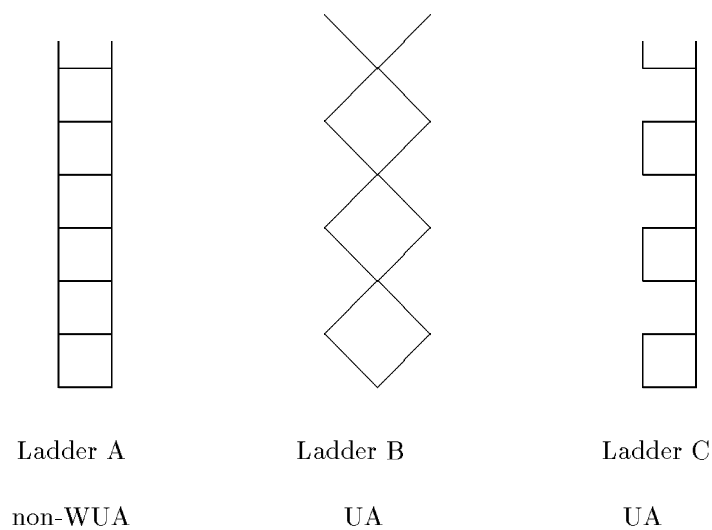

Figure 1: Three subsets of $\mathbf{R}^2$
(The pictures are meant to be infinitely prolonged upwards)

1) We can identify X_1 as a cofinite subset of the set of complex numbers $e^{i\theta}$ with $0 < \theta < 2\pi$. Define $f\colon X_1 \to \mathbf{R}$ by $f(e^{i\theta}) = \theta$. Then f is as desired.

2) Define $f\colon X_2 \to \mathbf{R}$ as follows. If $xy = 1$ set $f(x, y) = e^x$. If $xy = 2$ set $f(x, y) = -e^{-x}$. This works.

3) Define $f\colon X_3 \to \mathbf{R}$ as follows. First identify X_3 as the subspace of $\mathbf{R}^2$ consisting of the union of the two vertical axes $x = 0$ and $x = 1$, together

with the horizontal segments $I_n = \{(x,n) \in \mathbf{R}^2 \mid 0 \leq x \leq 1\}$ $(n \in \mathbf{N})$. Define $f(0,y) = e^y$, $f(1,y) = -e^y$. This defines f on the two axes of the ladder. On each horizontal segment I_n, f is linear. This defines uniquely f since we have already defined f on the extrema of the horizontal segments I_n (since they lie on the axes). f is pseudo-monotone, so each truncation of f is an (a,b)-truncation. It is easy to verify that any non-constant such truncation is not uniformly continuous. Since f has countable fibers, Theorem 8.3 applies and X_3 is not WUA. ♢

COROLLARY 10.2. *There are closed subsets of* $\mathbf{R}^2$ *which are not* WUA *and yet have no pseudo-hyperbolas.*

Proof. Take the spaces X_2 and X_3 of the previous proposition.

♢

REMARK 10.3. We have so far four criteria for showing that a normal uniform space X is not UA:

1. X has a closed subset which is uniformly connected and not connected.
2. X has a family of pseudo-hyperbolas (this entails that X is not even WUA).
3. X has a separable closed subspace Y and a function $f \in C(Y)$ with fibers of cardinality $\leq \aleph_0$ and without non-constant uniformly continuous truncations (this entails that X is not even WUA).
4. X has a compact subspace $K \subseteq X$ and a function $f \in C(X)$ with countable fibers such that f has no uniformly continuous truncations g which agree with f on K.

For separable spaces the last two criteria seem to be stronger. Criterion 1 suffices for showing that the spaces X_1, X_2 and X_3 of Proposition 10.1 are not UA [2]. An application of criterion 3 yields the stronger result that these spaces are not even WUA. An interesting property of the space X_3 is that it is not UA under every metric compatible with its topology (see [?]). This can be used to show that any non-compact metrizable manifold

2 We thank an anonymous referee for the observation that criterion 1 could be applied to the space X_3.

of dimension > 1 is not UA since it has a closed subspace homeomorphic to ladder A (see [?]). The reader may easily check (applying Remark 5.3) that every perfect function of X_3 is UA, while X_1 and X_2 do not have this property.

In Section 11 we will show that the spaces ladder B and ladder C of Figure 1 are UA.

11. Unions of UA spaces.

The property of being UA is not preserved under the taking of subspaces and continuous images. On the other hand the next theorem shows that UA spaces behave well under "gluing" along a compact set, provided we make the gluing in a careful manner. We do not know whether a similar result holds for WUA spaces even if we glue along a single point.

THEOREM 11.1. *Let X be a uniform space which can be written as $X = X_1 \cup X_2$ where X_1 and X_2 are UA. Suppose that:*

1. *$X_1 \cap X_2$ is compact;*
2. *if $g_1 : X_1 \to \mathbf{R}$ and $g_2 : X_2 \to \mathbf{R}$ are uniformly continuous functions which agree on the common domain, then their union is uniformly continuous.*

Then X is UA.

Proof. Let $K \subseteq X$ be compact and let $M \subseteq X$. Given $f \in C(X)$ we want to find a (K, M)-approximation g of f. Let $M_i = M \cap X_i$ and $K_i = (K \cap X_i) \cup (X_1 \cap X_2)$ $(i = 1, 2)$. Note that K_i is compact. In X_i there exists a (K_i, M_i)-approximation g_i of $f_{|X_i}$. Then $g = g_1 \cup g_2$ is a (K, M)-approximation of f. ♢

The above theorem says that the pushout (in the category of uniform spaces) of two UA spaces along a compact space is UA. It can be shown that we cannot weaken the condition that $X_1 \cap X_2$ is compact, to the condition that $X_1 \cap X_2$ is UA. In fact the space "Ladder A" of Figure 1 is not UA and yet it is the union of two UA spaces whose intersection is a uniformly discrete subspace.

QUESTION 11.2. Under which conditions a countable union of compact sets is UA?

In the metric case Theorem 11.4 gives a sufficient condition.

DEFINITION 11.3. Define

$$Osc(g) = \sup_{x \in dom(g)} g(x) - \inf_{x \in dom(g)} g(x) .$$

THEOREM 11.4. *Let $X = (X, d)$ be a metric space which can be written as a countable union $X = \bigcup_{i\in\mathbf{N}} X_i$ of compact sets X_n. Suppose that:*

1. *for all n the set X_n intersects X_{n+1} in exactly one point, and does not intersect X_m for $m > n+1$;*
2. *for all $n \in \mathbf{N}$, $\bigcup_{i\geq n} X_i$ is a closed subset of X.*

Then X is UA.

Proof. Let $f \in C(X)$, let K be a compact subset of X and let M be an arbitrary subset of X. We must find a uniformly continuous function $g: X \to \mathbf{R}$ such that $g = f$ on K, and $g(M) \subseteq f(M)$. From hypothesis 1. it follows:

CLAIM. If $g: X \to \mathbf{R}$ is a continuous function which is uniformly continuous on $\bigcup_{i\geq n} X_i$ for some n, then g is uniformly continuous on the whole of X.

CLAIM. If $g: X \to \mathbf{R}$ is a continuous function with the property that $Osc(g_{|X_n}) \leq 1/2^n$ for every sufficiently large n, then g is uniformly continuous.

CLAIM. Any compact subset of $X = \bigcup_i X_i$ is contained in the union of finitely many X_i's.

We leave the verification of the above claims to the reader. Going back to the proof let $p_n \in X$ be such that $X_{n-1} \cap X_n = \{p_n\}$ $(n > 0)$. Since K is compact there exists $n_0 > 0$ such that $K \subseteq X_0 \cup X_1 \cup \ldots \cup X_{n_0}$. We can assume that K is disjoint from M and intersects the closure of M. We consider the following cases:

Case 1. If for some $n > n_0$, $f(p_n) \in f(M)$, we define $g: X \to \mathbf{R}$ by: $g = f$ on $X_0 \cup X_1 \cup \ldots \cup X_n$ and g is the constant $f(p_n)$ on the rest of X. Then g is as desired.

From now on we assume that for every $n > n_0$ $f(p_n) \notin f(M)$. Under this assumption we distinguish the following cases:

Case 2. There exists $z \in f(M)$ and $n > n_0$ with $f(p_n) < z < f(p_{n+1})$ or $f(p_n) > z > f(p_{n+1})$. Suppose $f(p_n) < z < f(p_{n+1})$ (the other case being similar). We define $g: X \to \mathbf{R}$ as follows. $g(x) = f(x)$ for x belonging to $X_0 \cup \ldots \cup X_{n-1}$ and for those $x \in X_n$ with $f(x) < z$. In the remaining cases $g(x) = z$. Clearly g is continuous and $g(M) \subseteq f(M)$ (as $z \in f(M)$). Since g is constant on $\bigcup_{i>n+1} X_i$, g is uniformly continuous.

Case 3. There exists $n > n_0$ with $f(p_n) \neq f(p_{n+1})$ but case two fails. We can assume $f(p_n) < f(p_{n+1})$ (the other case being similar). Choose $y \in f(M)$. Let $A = X_0 \cup \ldots X_{n-1} \cup \{x \in X \mid f(x) \leq f(p_n)\}$ and $B = \bigcup_{i \geq n+1} X_i \cup \{x \in X \mid f(x) \geq f(p_{n+1})\}$. Then A is compact and B is a closed set at distance > 0 from A. Define $g = f$ on A and let g assume the constant value y on B. Then g is u.c. on $A \cup B$ and therefore it can be extended to a u.c. function on the whole of X. In the case that we are considering, M is disjoint from $X \setminus (A \cup B)$. So regardless of how we extend g from $A \cup B$ to X, we have $g(M) \subseteq f(M)$ (since its restriction to $A \cup B$ has this property).

Case 4. Assume $\exists a \forall n > n_0\ f(p_n) = a$. Moreover suppose that there are points of $f(M)$ arbitrarily close to a both greater than and smaller than a. This means that for each $n \in \mathbf{N}$ we can find real numbers $x_n < a < y_n$ with $x_n, y_n \in f(M)$ and $(y_n - x_n) < 1/2^n$. Let f_n be the (x_n, y_n)-truncation of f. Define $g: X \to \mathbf{R}$ so that g coincides with f on $X_0 \cup \ldots \cup X_{n_0}$ and $g = f_n$ on X_n for all $n > n_0$.
g is continuous since it is a union of continuous functions defined on the various X_n's which take the same value a at the points p_n. Moreover since $x_n, y_n \in f(M)$, $g(M) \subseteq f(M)$. To see that g is uniformly continuous it suffices to note that for $n > n_0$ $Osc(g_{|X_n}) < 1/2^n$.

Case 5. Assume $\exists a \forall n > n_0\ f(p_n) = a$ and suppose that there are points of $f(M)$ arbitrarily close to a, but Case 4 fails. Without loss of generality suppose that $a = 0$ and there is a sequence of points $u_n \in f(M)$ converging to 0 from below, but there is no such sequence converging to 0 from above. We can assume $|u_n| < 1/2^n$. Fix $n > n_0$ and let $M^+ = \{x \in M \cap X_n \mid f(x) > 0\}$ and let $[f \leq 0] = \{x \in X_n \mid f(x) \leq 0\}$. From our assumptions it follows that $\overline{M^+}$ and $[f \leq 0]$ are closed subsets of X_n at distance > 0. Therefore there exists a continuous function $g_n: X_n \to [u_n, 0]$ such that $g_n(x) = \max\{f(x), u_n\}$ if $f(x) \leq 0$ and $g_n(x) = u_n$ if $x \in \overline{M^+}$. Note that all the g_n's assume the same value 0 at the points p_n, so the union $g' = \bigcup_{n > n_0} g_n$ is a continuous function. Finally define $g = f$ on $X_0 \cup \ldots \cup X_{n_0}$ and $g = g'$ on $\bigcup_{n > n_0} X_n$. g is uniformly continuous since $Osc(g_{|X_n}) \leq |u_n| < 1/2^n$ for $n > n_0$. From the definition of g it follows that the only points $x \in X$ in which $g(x)$ is different both from $f(x)$ and from one of the constants $u_n \in f(M)$ are points not belonging to M. It follows that $g(M) \subseteq f(M)$.

Case 6. $\exists a \forall n > n_0 f(p_n) = a$ and $a \notin \overline{f(M)}$. It follows in particular that $p_{n_0+1} \notin \overline{M}$. Fix $b \in f(M)$ and define $g = f$ on $X_0 \cup \ldots X_{n_0}$ and

$g(x) = b$ for $x \in (\overline{M} \cap X_{n_0+1}) \cup \bigcup_{n>n_0+1} X_n$. By Katětov theorem we can extend g to a u.c. function on X, and this will be a (K, M)-approximation of f.

Since the above cases exhaust all the possible cases, the proof of the theorem is now complete. ♢

Note that the hypothesis that $X_n \cup X_{n+1}$ consists of a single point cannot be weakened. Ladder A (Figure 1) provides an example of a non-UA space which can be written as a countable union of compact spaces X_n with $X_n \cap X_{n+1}$ consisting of two points (and all the other hypothesis of Theorem 11.4 are satisfied).

COROLLARY 11.5. *The spaces ladder B and ladder C of Figure 1 are UA.*

A simple application of 11.4 gives also a new proof of the fact that $\mathbf{R}$ is UA.

PROPOSITION 11.6. *Theorem 11.4 holds also if we replace compactness of X_n by UA, asking that for each $n \in \mathbf{N}$ the union $(X_1 \cup \ldots \cup X_n) \cup X_{n+1}$ is a pushout.*

Proof. The third claim in the proof of Theorem 11.4 remains true also in this case. The other two claims work also in this more general case if we require the continuous function g to be uniformly continuous on each X_n. Now an intermediate step has to be carried out before arguing as in the proof of Theorem 11.4: using UA-ness the function $f \in C(X)$ has to be replaced by $h \in C(X)$ such that for each n the restriction $h|_{X_n}$ is a $(K \cap X_n) \cup \{p_n, p_{n+1}\}, M \cap X_n)$-approximation of the restriction $f|_{X_n}$. Now the proof continues as in Theorem 11.4. ♢

We do not know a characterization of the UA subsets of $\mathbf{R}$. The next corollary gives some partial information and allows us to reduce the study of arbitrary UA subspaces of $\mathbf{R}$ to UA subspaces of the compact interval $[0, 1]$. It follows immediately from Proposition 11.6.

COROLLARY 11.7. *A subspace X of $\mathbf{R}$ is UA iff for each $n \in \mathbf{Z}$ the subspace $X \cap [n, n+1]$ is UA.*

12. A new characterization of UC spaces.

Let **Unif** be the category of uniform spaces and u.c. maps. Let for $X \in$ **Unif**, $C_{ua}(X)$ ($C_{wua}(X)$) denote the set of UA (resp. WUA) functions $X \to \mathbf{R}$. Since every WUA-function is continuous by Fact 2.2 , we have the following chain of inclusions

$$C_u(X) \overset{(1)}{\subseteq} C_{ua}(X) \overset{(2)}{\subseteq} C_{wua}(X) \overset{(3)}{\subseteq} C(X). \tag{4}$$

Theorem 5.2 shows that the gap between $C_u(\mathbf{R}^n)$ and $C_{ua}(\mathbf{R}^n)$ is big for any $n > 0$. We note that the inclusion (3) is an equality iff X is a WUA space. Now $C_{wua}(X)$ is as big as possible. At this point one is tempted to try the "symmetric" property, namely the spaces X with $C_{ua}(X) = C_u(X)$, so that now $C_{ua}(X)$ is as small as possible. Evidently this occurs when X is UC, i. e. when all inclusions in (4) become equalities. Surprisingly, it can be shown that this trivial observation can be substantially sharpened: a metric space X is UC iff (1) is an equality (see Theorem 12.1 below). Our theorem was stimulated by an earlier result of M. Burke [?] where the stronger assumption $C_{wua}(X) = C_u(X)$ was used to get UC for a metric space X. The function f in our proof is the same as in [?], Theorem 6.

THEOREM 12.1. *A metric space X is UC iff every bounded uniformly approachable function is uniformly continuous, so that a space X with $C_u(X) = C_{ua}(X)$ is necessarily a UC space.*

Proof. If X is UC then obviously (1) is an equality in (4). Assume X is not UC. Then there exists a continuous non u.c. function $g: X \to \mathbf{R}$. Hence for some set $S = \{s_n : n \in \mathbf{N}\}$ and some $\varepsilon > 0$ we have $|g(s_{2n}) - g(s_{2n+1})| \geq \varepsilon$ while $d(s_{2n}, s_{2n+1}) \to 0$ as $n \to \infty$. Clearly S is closed and discrete (see for example [?]). Hence we can choose balls $U_n = B_{r_n}(s_n)$ so that $\{\overline{B_{2r_n}(s_n)} : n \in \mathbf{N}\}$ is a discrete family of sets. Let $U = \bigcup_n U_n$. Then $\overline{U} = \bigcup_n \overline{U_n}$.

We are going to use this data to define a UA non-uniformly-continuous function $f: X \to [0,1]$ as follows: f is 0 on $X \setminus \bigcup_n U_{2n-1}$, for $n \in \mathbf{N}$ $f(s_{2n-1}) = 1$ and the restriction $f|_{\overline{U_{2n-1}}}$ is uniformly continuous (take for example $f(x) = (1/r_{2n-1})(r_{2n-1} - d(x, s_{2n-1}))$ for $x \in U_{2n-1}$).

Note that f is not uniformly continuous since $f(s_{2n}) = 0$, $f(s_{2n+1}) = 1$. To show that f is UA consider a compact set K in X and a non-void set $M \subseteq X$.

Case 1: $K \cap \overline{U} = \emptyset$. If $f(m) = 0$ for some $m \in M$, then the constant 0 is a (K, M)-approximation of f. Otherwise $M \subseteq U$, so $K \cap \overline{M} = \emptyset$ by our assumption. By Proposition 2.3 there exists a (K, M)-approximation of f.

Case 2: $K \cap \overline{U} \neq \emptyset$. Let $D = \{k \in \mathbf{N} : K \cap \overline{B_{2r_k}(s_k)} \neq \emptyset\}$. Then D is finite since K is compact. Set $C = \bigcup_{k \in D} \overline{U_k}$ and note that $f_{|C \cup K}$ is u.c. Moreover, $F = \bigcup_{k \notin D} \overline{U_k}$ is closed, $K \subseteq X \setminus F$ and $C \cap F = \emptyset$, more precisely $d(C, F) > 0$.

a) If $f(m) = 0$ for some $m \in M$, then let g agree with f on C and be identically 0 elsewhere. Then g is a (K, M)-approximation of f and g is uniformly continuous.

b) If $f(m) > 0$ for every $m \in M$, then

$$M \subseteq \bigcup_{k \in \mathbf{N}} U_{2k+1}. \tag{5}$$

Our aim is to define a u.c. function $g: K \cup \overline{M} \to [0,1]$ which is a (K, M)-approximation of $f|_{K \cup \overline{M}}$. Then obviously any u.c. extension $g : X \to [0,1]$ of g, existing by Katětov's theorem, will be a (K, M)-approximation of f.

If $M \cap F = \emptyset$ then $\overline{M} \subseteq C$. So it suffices to recall that $g = f_{|C \cup K}$ is u. c. Suppose now that $M \cap F \neq \emptyset$ and fix $c \in f(M \cap F)$. Define $g : K \cup \overline{U} \to [0,1]$ as follows: define $g(z) = f(z)$ if $z \in K \cup C$ and $g(z) = c$ for $z \in F$. Since $K \cap F = \emptyset$ by the choice of C, we get, in view of the compactness of K, $d(K \cup C, F) = \min\{d(K, F), d(C, F)\} > 0$. This means that the function $g : K \cup C \cup F = K \cup \overline{U} \to [0,1]$ is u.c. Since $\overline{M} \subseteq \overline{U}$ by (5) we are through. ♢

13. WUA-functions are totally continuous.

We denote by 2^X the family of all subsets of X. We call *closure operator* on **Unif** a family $C = (c_X)_{X \in \mathbf{Unif}}$ of maps

$$c_X : 2^X \longrightarrow 2^X \qquad M \longmapsto c_X(M)$$

such that for every X in **Unif**

i) $M \subseteq c_X(M)$ for all $M \in 2^X$;

ii) $M \subseteq M' \in 2^X \Rightarrow c_X(M) \subseteq c_X(M')$;

iii) $f(c_X(M)) \subseteq c_Y(f(M))$ for all $f: X \to Y$ in **Unif** and $M \in 2^X$.

A closure operator C is *additive* if $c_X(A \cup B) = c_X(A) \cup c_X(B)$ always holds for $A, B \subseteq X$. The leading example is the usual Kuratowski closure operator K, more can be found in [?], [?].

Let U: **Unif** $\to$ **Set** be the usual forgetful functor. For a closure operator C of **Unif** we say that the map $f: U(X) \to U(Y)$ is *C-continuous*, if it satisfies $f(c_X(M)) \subseteq c_Y(f(M))$ for all $M \in 2^X$, f is *totally continuous* if it is C-continuous for each closure operator C. Obviously, totally continuous maps are continuous (being K-continuous). The next proposition clarifies the relation between this new notion of continuity and WUA functions. The proof is analogous to that of Fact 2.2.

Proposition 13.1. ([?], Corollary 4.4) *WUA functions are totally continuous.*

Let for $X \in \mathbf{Unif}$, $C_t(X)$ denote the set of totally continuous functions $f\colon X \to \mathbf{R}$ and let C_{ta} denote the set of all maps $f\colon X \to \mathbf{R}$ which are *C-continuous* w.r.t. all *additive* closure operators. Now we put together all we observed on various types of continuity for real-valued functions. By Proposition 13.1, we can add two inclusions to the chain (4) to get the following longer chain of inclusions

$$C_u(X) \overset{(1)}{\subseteq} C_{ua}(X) \overset{(2)}{\subseteq} C_{wua}(X) \overset{(3)}{\subseteq} C_t(X) \overset{(4)}{\subseteq} C_{ta}(X) \overset{(5)}{\subseteq} C(X). \qquad (6)$$

The recent results from [?] permit us to show that the inclusion (5) is actually an equality in some cases.

Theorem 13.1. *Let X be a metric space which is either zero-dimensional or connected and locally arcwise connected. Then*
$C_{ta}(X) = C(X)$.

Proof. Our proof is based on a result of [?]. To give it here we need the following definitions. The *discrete* (*trivial*) closure operator D (resp. T) of **Unif** is defined by setting $d_X(M) = M$ (resp. $t_X(M) = X$) for each $X \in \mathbf{Unif}$ and $M \subseteq X$ (see [?]). It is proved in [?] that on a space X as in our hypothesis K, D and T are the unique additive closure operators. Since every continuous map is obviously C-continuous for any of K, D and T, we are through.

◊

The above theorem suggests the following

Question 13.3. Is there a metric space X with $C_t(X) \neq C(X)$ (resp. $C_{ta}(X) \neq C(X)$) ?

14. Questions.

A general question is to characterize the UA and WUA spaces and functions. We list below more specific questions.

1. Can we prove Theorem 8.5 without the continuum hypothesis?
2. Let X be separable and let $f \in C(X)$. Consider the following two conditions: 1) for every compact $K \subseteq X$ there is a u.c. truncation g of f with $g_{|K} = f_{|K}$; 2) f is a UA function. Are these conditions equivalent? We know that 2) implies 1) if f has countable fibers.
3. Characterize the UA functions $f: \mathbf{R}^2 \to \mathbf{R}$.
4. Characterize the UA subsets of $\mathbf{R}$.
5. Characterize the topological spaces which admit a UA uniformity, and those which are UA under every uniformity compatible with their topology. Does the latter class of spaces include the UC spaces?
6. Do WUA and UA coincide for connected spaces?
7. Suppose that a uniform space X has a dense UA subspace. Does it follow that X is UA? (This fails for WUA according to Example 7.7.)
8. Let X be the pushout of two WUA spaces over a single point. Is X WUA? (This holds for UA by Theorem 11.1.)
9. Suppose that every pseudo-monotone function $f \in C(X)$ is UA. Is the space X UA?
10. Define 2-UA similarly to UA but with the set K of cardinality at most 2. Is 2-UA equivalent to UA? Note that Lemma 3.1 and its corollary holds with 2-UA instead of UA.

Acknowledgments

We thank M. Burke for providing us unpublished material. We also thank an anonymous referee for his very careful work and for pointing out a simpler proof that the "ladder space" is not UA.

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