ON THE FUZZES WHICH ARE COMPLETE RINGS OF SETS (*)

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Sommario. - *Sia $\mathcal{F}r$ la categoria di fuzzes il cui insieme soggiacente è un anello completo di insiemi. Ogni fuzz non banale in $\mathcal{F}r$ si può rappresentare come un prodotto sottodiretto di copie di oggetti di una famiglia $\mathcal{G}$ se e solo se $\mathcal{G}$ contiene gli oggetti* **2**, **3**, **R**. *Inoltre, ogni fuzz è l'immagine di un oggetto di $\mathcal{F}r$ mediante un morfismo fuzz.*

Summary. - *Let $\mathcal{F}r$ be the category of fuzzes whose underlying set is a complete ring of sets. Every non-trivial fuzz in $\mathcal{F}r$ can be represented as a subdirect product of copies of objects taken in a family $\mathcal{G}$ if and only if $\mathcal{G}$ contains the objects* **2**, **3**, **R**. *Furthermore, every fuzz is the image of an object of $\mathcal{F}r$ via a fuzz morphism.*

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1. Definitions and Notations.

Let $\mathcal{D}c$ denote the category of completely distributive complete lattices and complete lattice morphisms. A complete ring of sets is a complete sublattice of a power set. $\mathcal{I}L$ denotes the set of all completely join irriducible elements of a lattice L; it is well-known that $L \in \mathcal{D}c$ is a complete ring of sets if and only if every element of L is the supremum of a subset of $\mathcal{I}L$. In any complete lattice, 0 and 1 denote the smallest and the greatest element respectively.

A fuzz is an object of $\mathcal{D}c$ equipped with an order reversing involution $'$; the category of fuzzes and fuzz morphisms is denoted by $\mathcal{F}$; the category

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of fuzzes whose underlying set is a complete ring of sets is denoted by $\mathcal{F}r$. Given any fuzz L, we denote by c_L [1] the element: $\bigwedge\{l \vee l' : l \in L\}$. It is obvious that $c_L = 1$ if and only if L is a power set.

We denote by $\mathbf{2}$ and $\mathbf{3}$ the fuzzes (or lattices) with two and three elements respectively; $\mathbf{R}$ denotes the fuzz with four elements $\{0, a, b, 1\}$ equipped with the order reversing involution given by $a' = a$. As usual, I is the closed real unit interval, equipped with the order reversing involution $t' = 1 - t$.

Observe that $\mathcal{F}r$ is closed under the formation of products and subobjects, hence under the formation of subdirect products, while it is not closed under the formation of epimorphic images. Indeed, denote by T the chain $I \times \mathbf{2}$ lexicographically ordered, equipped with the involution defined componentwise. Clearly, T belongs to $\mathcal{F}r$ since the elements $(t, 1)$ are join irreducible, while I does not belong to $\mathcal{F}r$ since $\mathcal{I}I$ is empty, and the map $(t, i) \longrightarrow t$ is a map of fuzzes from T onto I.

We briefly recall the definitions of the functors C (clepsydra) and R (rhomb) from $\mathcal{D}c$ into $\mathcal{F}$, which we have introduced in [2].

Given $L \in \mathcal{D}c$, denote by $-L$ the opposite lattice, that is the lattice whose underlying set is $\{-l : l \in L\}$ and whose order is defined by $-l \leq -m \iff l \geq m$.

Consider the set $L \cup (-L)$ and identify 0_L with 1_{-L}; equip this set with the order which extends the original ones in such a way that $-l \leq m$ for all l, m belonging to L. The assignation $l' = -l$ induces a unique order reversing involution in this lattice and we denote by CL the fuzz obtained in this way. Trivially, $\mathbf{3} = C\mathbf{2}$ and it is easy to check that a fuzz is a clepsydra if and only if $c_L = c'_L$ and every element is comparable with c_L.

Now consider the lattice $L \times (-L)$ equipped with the componentwise order and with the order reversing involution $(l, -m)' = (m, -l)$. We denote this fuzz by RL. Trivially, $\mathbf{R} = R\mathbf{2}$ and $c_{RL} = 0$.

Con (L) denotes the complete lattice of complete congruence relations of L (as a lattice) [4]; Con $'(L)$ denotes the sublattice consisting of the elements of Con (L) compatible with the involution [2].

2. Results.

1. Proposition.

i) *The category of complete rings of sets is a full reflective subcategory of* $\mathcal{D}c$*;*

ii) *$\mathcal{F}r$ is a full reflective subcategory of $\mathcal{F}$.*

Proof.

i) For every $L \in \mathcal{D}c$, denote by ϑ_L the smallest complete congruence relation ϑ for which L/ϑ is a complete ring of sets. We show that $L \longrightarrow L/\vartheta_L$ is a reflection.

If M is a complete ring of sets and $f : L \longrightarrow M$, define

$$\tilde{f} : L/\vartheta_L \longrightarrow M \qquad \text{by} \qquad \tilde{f}([a]) = f(a).$$

Since ϑ_L is a complete congruence relation, it is enough to show that $\tilde{f}$ is well defined, that is that $f(a) \neq f(b)$ implies $(a, b) \notin \vartheta_L$.

Suppose $f(a) \not\geq f(b)$: since $f(L)$ is a complete ring of sets, there exists an element $m \in \mathcal{I}f(L)$ such that $m \leq f(b)$, $m \not\leq f(a)$. The element $l^* = \bigwedge\{l : m \leq f(l)\}$ belongs to $\mathcal{I}L$; since $l^* \leq b$ and $l^* \not\leq a$, the pair (a, b) does not belong to the congruence relation whose classes are the filter $[l^*)$ and its complement $L \setminus [l^*)$, hence (a, b) does not belong to ϑ_L.

ii) It is enough to observe that ϑ_L belongs to Con $'(L)$ and that, if f is a $\mathcal{F}$-morphism, then $\tilde{f}$ preserves the order reversing involution as well. $\diamondsuit$

It is well-known [9] that an object of $\mathcal{D}c$ is a subdirect product of copies of **2** and I and it is obvious that a complete ring of sets is a subdirect product of copies of **2**.

In [2] we have shown that every fuzz is a subdirect product of copies of the fuzzes **2**, **3**, **R**, I, RI; as a consequence, we have the following:

2. Proposition. *A non trivial fuzz belongs to $\mathcal{F}r$ if and only if it is a subdirect product of copies of* **2**, **3**, **R** *(that is* **2**, C**2**, R**2***).*

Proof. Necessity: by [3, 4.1], ϑ_L, which is discrete, is the infimum of a set of congruence relations ψ_a such that, by [2, 3.11, 3.12], L/ψ_a is one of the fuzzes **2**, **3**, **R**. $\diamondsuit$

More precisely, the above result can be refined as follows:

i) ($L \in \mathcal{F}r$ and) $c_L = 1$ if and only if L is a power set;

ii) $L \in \mathcal{F}r$ and $c_L = c'_L$ if and only if L is a subdirect product of copies of **3**;

iii) $L \in \mathcal{F}r$ and $c_L = 0$ if and only if L is a subdirect product of copies of $\mathbf{R}$.

Observe that every non trivial homomorphic image of $\mathbf{2}$, $\mathbf{3}$, $\mathbf{R}$ is $\mathbf{2}$, $\mathbf{3}$, $\mathbf{R}$ respectively; moreover, $\mathbf{2}$ and $\mathbf{3}$ are subfuzzes of $\mathbf{R}$. Hence every non trivial object of $\mathcal{F}r$ is subfuzz of a power of $\mathbf{R}$, that is $\mathbf{R}$ is a coseparator of $\mathcal{F}r$. Therefore we obtain the following:

3. PROPOSITION. *Let $\mathcal{G}$ be a subfamily of $\mathcal{F}r$; then every object belonging to $\mathcal{F}r$ is a subdirect product of objects of $\mathcal{G}$ if and only if $\mathcal{G}$ contains* $\mathbf{2}$, $\mathbf{3}$, $\mathbf{R}$. ♢

The following theorem is the analog in fuzzy set theory of the classical result of Raney [8] for completely distributive complete lattices.

4. THEOREM. *Every fuzz L is the image of an object of $\mathcal{F}r$ via a fuzz morphism.*

Proof. By [2, 5.13], L is a subfuzz of a power $\prod_{x \in X} RI_x$, for a certain set of indices X, where $I_x = I$ for every $x \in X$. For every x put $T_x = T$ and let $f_x : T_x \longrightarrow I_x$ be the $\mathcal{D}c$-morphism defined by $f_x(t, i) = t$. Since f_x is onto, the $\mathcal{F}r$-morphism

$$f = \prod_x Rf_x : \prod_x RT_x \longrightarrow \prod_x RI_x$$

is onto, too.

The inverse image M of L under f is a subfuzz of $\prod_x RT_x$, therefore M belongs to $\mathcal{F}r$, and L is the image of M through the restriction of f to M. ♢

REMARK. The construction which gives T by means of I can be mimicked by replacing I with any fuzz L. We still obtain a fuzz if L is a chain, while in the general case the resulting lattice need not even be distributive.

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