PERIODIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH A p-LAPLACIAN AND ASYMMETRIC NONLINEARITIES(*)

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SOMMARIO. - *In questa nota si ottengono risultati di esistenza per il problema con condizioni alla frontiera*

$$\begin{cases} (\Phi_p(x'))' + f(t,x) = 0 \ , \\ x(0) = x(T), \ x'(0) = x'(T) \ , \end{cases}$$

dove $\Phi_p(s) = |s|^{p-2}s$, *la funzione non lineare* f *essendo asimmetrica (una cosiddetta "jumping nonlinearity"). Il metodo di dimostrazione è basato su argomenti della teoria del grado topologico. Limiti a priori per possibili soluzioni sono ottenuti per mezzo del calcolo del numero di rivoluzioni nel piano delle fasi.*

SUMMARY. - *In this note we obtain existence results for the periodic boundary-value problem*

$$\begin{cases} (\Phi_p(x'))' + f(t,x) = 0 \ , \\ x(0) = x(T), \ x'(0) = x'(T) \ , \end{cases}$$

where $\Phi_p(s) = |s|^{p-2}s$, *the nonlinear function* f *being asymmetric (a so-called "jumping nonlinearity"). The method of proof is based on arguments of topological degree theory. A priori bounds for possible solutions are obtained by means of a count of the number of revolutions in the phase plane.*

(*) Pervenuto in Redazione il 15 giugno 1993.

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1. Introduction.

For $p > 1$, let us define $\Phi_p : \mathbb{R} \to \mathbb{R}$ by $\Phi_p(s) = |s|^{p-2}s$. We are interested in the periodic boundary-value problem

$$(\Phi_p(x'))' + f(t,x) = 0 \ , \tag{1}$$

$$x(0) = x(T),\ x'(0) = x'(T) \ . \tag{2}$$

If $p = 2$, the nonlinear operator $x \mapsto (\Phi_p(x'))'$ reduces to the Laplacian operator $x \mapsto x''$. The nonlinear operator is called a p-Laplacian.

The main objective of this paper is to show that techniques that have been used for the equation with the linear operator, like phase-plane analysis or the use of Prüfer variables, can be adapted to the general case, offering the possibility to generalize various results. As an illustration of the method, we will treat problems where the asymptotic behaviour of f, for x going to $\pm\infty$ is asymmetric, by which we mean that the ratio $f(t)/\Phi_p(x)$ has different limits, for x going to $-\infty$, and for x going to $+\infty$. Such a situation, sometimes referred to as a *jumping nonlinearity*, has been considered recently by M. Del Pino, R. Manasevich, A. Murua [1]. They assume that positive numbers a_+, a_-, b_+, b_- exist such that

$$a_- \leq \liminf_{x\to-\infty} \frac{f(t,x)}{\Phi_p(x)} \leq \limsup_{x\to-\infty} \frac{f(t,x)}{\Phi_p(x)} \leq b_- \ , \tag{3}$$

$$a_+ \leq \liminf_{x\to+\infty} \frac{f(t,x)}{\Phi_p(x)} \leq \limsup_{x\to+\infty} \frac{f(t,x)}{\Phi_p(x)} \leq b_+ \ , \tag{4}$$

the limits being uniform in t, and present conditions on a_+, a_-, b_+, b_- under which problem (1), (2) has at least one solution. Results along the same lines have also been obtained by Y. Huang and G. Metzen [6]. The conditions on a_+, a_-, b_+, b_- are related to the so-called Fučík spectrum and generalize conditions obtained by P. Drabek and S. Invernizzi [3] for $p = 2$, in which case they write

$$\frac{1}{\sqrt{a_+}} + \frac{1}{\sqrt{a_-}} < \frac{T}{n\pi} \ , \tag{5}$$

$$\frac{1}{\sqrt{b_+}} + \frac{1}{\sqrt{b_-}} > \frac{T}{(n+1)\pi} \ , \tag{6}$$

n being a positive integer.

The generalization of the above conditions for problems with a p-Laplacian is

$$\frac{1}{a_+^{1/p}} + \frac{1}{a_-^{1/p}} < \frac{T}{n\pi_p}, \tag{7}$$

$$\frac{1}{b_+^{1/p}} + \frac{1}{b_-^{1/p}} > \frac{T}{(n+1)\pi_p}, \tag{8}$$

where π_p is a number to be defined in Section 2.

In this paper, we will study modifications of the conditions (7), (8) in two different directions.

First, if the conditions (7), (8) are replaced by the equalities

$$\frac{1}{a_+^{1/p}} + \frac{1}{a_-^{1/p}} = \frac{T}{n\pi_p}, \tag{9}$$

$$\frac{1}{b_+^{1/p}} + \frac{1}{b_-^{1/p}} = \frac{T}{(n+1)\pi_p}, \tag{10}$$

additional conditions of Landesman-Lazer type, have to be imposed on the function f, in order to be able to obtain existence results. Such a situation has been considered recently by C. Fabry [4] for $p = 2$ and his results find thus here a generalization to equations involving a p-Laplacian. The conditions (9), (10), mean that the rectangle $[a_+, b_+] \times [a_-, b_-]$ touches two successive Fučík curves. This can be considered as a case of double resonance (at least when $a_+ = a_-, b_+ = b_-$).

Another situation concerns the case where one of the numbers b_+, b_- in (3), (4) becomes infinite, allowing the function f to grow at a superlinear rate. For example, we will allow b_- to go to infinity and replace (7), (8) by the conditions

$$\frac{1}{a_+^{1/p}} + \frac{1}{a_-^{1/p}} < \frac{T}{n\pi_p}, \tag{11}$$

$$b_+^{1/p} < \frac{(n+1)\pi_p}{T} \tag{12}$$

which can be interpreted as forcing the infinite rectangle $[a_+, b_+] \times [a_-, +\infty)$ to lie between two successive Fučík curves. Problems of that type have been considered by C. Fabry and P. Habets [5] for $p=2$.

In section 2, we introduce and recall some properties of the functions $\sin_p$ and $\cos_p$, whose definitions can be found in [1],[2]. The function $\cos_p$ is used in section 3 for defining a change of variable that plays a key role in our study of equations with a p-Laplacian. In section 4, we express the number of revolutions in the phase-plane of solutions of (1), (2) by means of integrals. Section 5 is devoted to existence results concerning the case where a_+, a_-, b_+, b_- verify the equalities (9), (10); conditions of Landesman-Lazer type are introduced there. The proofs are based on the invariance by homotopy of the topological degree and make use of a count of the number of revolutions, of possible solutions, in the phase-plane. In section 6 , we establish some preliminary results for the superlinear case. Roughly speaking we prove that, if $xf(t,x)$ is positive and bounded away from 0 for large $|x|$, then a solution of (1) cannot escape to infinity without having an infinite number of zeros. This result is then used in section 7 in order to get an existence theorem when a_+, a_-, b_+ verify (11), (12).

2. The function $\sin_p$.

The solution of the homogeneous differential equation

$$(\Phi_p(u'))' + \Phi_p(u) = 0 \tag{13}$$

will play a major role in the sequel. Equation (13) can be solved by direct integration; one of the solutions is the function $\sin_p$ whose definition, given in [1],[2], is recalled below. Define the number π_p by

$$\pi_p = 2\int_0^{(p-1)^{1/p}} \frac{ds}{[1-s^p/(p-1)]^{1/p}}\,.$$

Let the function $w:[0,\pi_p/2]\to[0(p-1)^{1/p}]$ be defined implicitly by

$$\int_0^{w(t)} \frac{ds}{[1-s^p/(p-1)]^{1/p}} = t\,. \tag{14}$$

The function w will be extended to $\mathbb{R}$ as explained below, and the extension will be denoted by $\sin_p$. First, we define $\sin_p$ on $[\pi_p/2,\pi_p]$

by $\sin_p(t) = w(\pi_p - t)$, then we define $\sin_p$ on $[-\pi_p, 0]$ by assuming that the function is odd. Finally, we extend $\sin_p$ to $\mathbb{R}$ by $2\pi_p$-periodicity. From (14), it is easy to deduce the following relation:

$$(p-1)|\sin_p'(\theta)|^p + |\sin_p(\theta)|^p = p-1\ ; \tag{15}$$

we also observe that $\sin_p(0) = 0$, $\sin_p'(0) = 1$, $\sin_p(\pi_p/2) = (p-1)^{1/p}$, $\sin_p'(\pi_p/2) = 0$. We find it convenient to introduce a function $\cos_p$ defined by $\cos_p(t) = \sin_p(\pi_p/2 - t)$. Obviously, $\cos_p$ is also a solution of (13) and verifies

$$(p-1)|\cos_p'(\theta)|^p + |\cos_p(\theta)|^p = p-1\ . \tag{16}$$

We note that Huang and Metzen [6] give slightly different definitions of π_p, $\sin_p$, $\cos_p$, the differences corresponding to changes of scales.

3. A change of variables.

Equation (1) can be rewritten as a system:

$$x' = \Phi_p^{-1}(y) \tag{17}$$

$$y' = -f(t,x)\ . \tag{18}$$

In order to study such a system, we introduce a change of variables, analogous to the transformation to polar coordinates: for $\mu \in \mathbb{R}^+\backslash\{0\}$, let

$$\mu x = \rho \cos_p(\theta)\ , \tag{19}$$

$$y = -\rho^{p-1}\Phi_p(\cos_p'(\theta))\ . \tag{20}$$

The transformation is a local homeomorphism at each point (ρ, θ) of the set $\mathbb{R}^+\backslash\{0\} \times [0, 2\pi_p)$, since the Jacobian of the transformation is equal to ρ^{p-1}. It can be shown that the change of variables $H : (\rho, \theta) \mapsto (x, y)$ is a global homeomorphism from $\mathbb{R}^+\backslash\{0\} \times [0, 2\pi_p)$ onto $\mathbb{R}^2\backslash\{(0,0)\}$, taking into account the fact that

$$\lim_{\rho \to +\infty} \|H(\rho, \theta)\| = +\infty\ , \text{uniformly in } \theta\ .$$

Differentiating (19), (20), we obtain, since $\cos_p$ is a solution of (13),

$$\mu x' = \rho' \cos_p(\theta) + \rho \cos_p'(\theta)\theta'\ , \tag{21}$$

$$(22)\quad y' = -(p-1)\rho^{p-1}\rho'\Phi_p(\cos_p'(\theta)) + \rho^{p-1}\Phi_p(\cos_p(\theta))\theta' .$$

Because Φ_p is homogeneous of degree $(p-1)$, the above relations can be rewritten as

$$(23)\quad x' = \frac{\rho'}{\rho}x - \frac{1}{\mu}\Phi_p^{-1}(y)\theta' ,$$

$$(24)\quad y' = (p-1)\frac{\rho'}{\rho}y + \mu^{p-1}\Phi_p(x)\theta' .$$

Solving with respect to ρ', θ', we get

$$(25)\quad \theta' = \mu\frac{-(p-1)x'y + xy'}{(p-1)y\Phi_p^{-1}(y) + \mu^p x\Phi_p(x)} ,$$

$$(26)\quad \rho' = \rho\frac{\mu^p x'\Phi_p(x) + y'\Phi_p^{-1}(y)}{(p-1)y\Phi_p^{-1}(y) + \mu^p x\Phi_p(x)} .$$

Since $t \mapsto (x,(t),y(t))$ is a solution of (17), (18) this leads to

$$(27)\quad \theta' = \mu\frac{-(p-1)y\Phi_p^{-1}(y) - xf(t,x)}{(p-1)y\Phi_p^{-1}(y) + \mu^p x\Phi_p(x)} ,$$

$$(28)\quad \rho' = \rho\frac{\mu^p\Phi_p^{-1}(y)\Phi_p(x) - \Phi_p^{-1}(y)f(t,x)}{(p-1)y\Phi_p^{-1}(y) + \mu^p x\Phi_p(x)} .$$

But, from (15), (20) we see that

$$\begin{aligned}\mu^p x\Phi_p(x) + (p-1)y\Phi_p^{-1}(y) &= \rho^p|\cos_p(\theta)|^p + \rho^p(p-1)|\cos_p'(\theta)|^p \\ &= (p-1)\rho^p ,\end{aligned}$$

so that we obtain

$$(29)\quad \theta' = \mu\frac{-(p-1)y\Phi_p^{-1}(y) - xf(t,x)}{(p-1)\rho^p} ,$$

$$(30)\quad \rho' = \frac{\Phi_p^{-1}(y)[\mu^p\Phi_p(x) - f(t,x)]}{(p-1)\rho^{p-1}} .$$

The change of variables described above will be used later in the proof of the existence results.

4. Number of revolutions.

In this section, we present an auxiliary result which is of independent interest. The proof is an immediate adaptation of the proof in [4] concerning the case $p = 2$.

LEMMA 1. *Let $x \in H^2(0,T)$ satisfy conditions (1), (2). Assume that $x^2(t) + +x'^2(t) > 0$ for $t \in [0,T]$. Then there exists an integer k such that for all $a_+, a_- > 0$, the following relations hold:*

$$\begin{aligned} k\pi_p &= a_+^{1/p} \int_{I+} \frac{(p-1)x'\Phi_p(x') + xf(t,x)}{(p-1)x'\Phi_p(x') + a_+ x\Phi_p(x)}\, dt\,, \\ &= a_-^{1/p} \int_{I-} \frac{(p-1)x'\Phi_p(x') + xf(t,x)}{(p-1)x'\Phi_p(x') + a_- x\Phi_p(x)}\, dt\,, \end{aligned}$$

where $I_+ = \{t \in [0,T] \mid x(t) \geq 0\}$ and $I_-\{t \in [0,T] \mid x(t) \leq 0\}$.

Proof. Taking $y = \Phi_p(x')$, we use a slight modification of the change of variables (19), (20) taking different transformations in the two half-planes $x \geq 0$ and $x \leq 0$. For $x \geq 0$, we take

$$a_+^{1/p} x = \rho \cos_p(\theta)\,, \tag{31}$$

$$y = -\rho^{p-1}\Phi_p(\cos_p'(\theta))\,, \tag{32}$$

whereas, for $x \leq 0$, we use

$$a_-^{1/p} x = \rho \cos_p(\theta)\,, \tag{33}$$

$$y = -\rho^{p-1}\Phi_p(\cos_p'(\theta))\,. \tag{34}$$

It is clear that, globally, these transformations still define a homeomorphism from $\mathbb{R}^+ \backslash \{0\} \times [0, 2\pi_p)$ onto $\mathbb{R}^2 \backslash \{(0,0)\}$. Adapting (27) in an obvious way to (31), (32) we get for $x \geq 0$

$$\theta' = a_+^{1/p} \frac{-(p-1)x'\Phi_p(x') - xf(t,x)}{(p-1)x'\Phi_p(x') + a_+ x\Phi_p(x)}\,. \tag{35}$$

It is clear from the properties of the function $\cos_p$ that, if the solution curve $t \mapsto (x(t), y(t))$ makes k revolutions in the (x,y)-plane, then

$$-k\pi_p = \int_{I+} \theta'(t)dt = \int_{I-} \theta'(t)dt\,.$$

Using (35), we then obtain

$$\text{(36)} \qquad k\pi_p = a_+^{1/p} \int_{I^+} \frac{(p-1)x'\Phi_p(x') + xf(t,x)}{(p-1)x'\Phi_p(x') + a_+ x\Phi_p(x)}\, dt\ ;$$

similarly, we have

$$\text{(37)} \qquad k\pi_p = a_-^{1/p} \int_{I^-} \frac{(p-1)x'\Phi_p(x') + xf(t,x)}{(p-1)x'\Phi_p(x') + a_- x\Phi_p(x)}\, dt\ .$$

We note that if $x^2(t) + x'^2(t)$ does not vanish, then the same holds for $(p-1)x'(t)\Phi_p(x'(t)) + \mu x(t)\Phi_p(x(t))$, for any $\mu > 0$.

5. Existence results and Landesman-Lazer conditions.

Throughout this section, we will assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, i.e. $f(\cdot, x)$ is measurable on $[0,T]$ for all $x \in \mathbb{R}$, $f(t,\cdot)$ is continuous on $\mathbb{R}$, for almost every $t \in [0,T]$. We also suppose that there exists positive numbers a_+, a_-, b_+, b_- and a function $h \in L^2(0,T)$ such that the following conditions hold:

$$\left.\begin{array}{ll}(38) & f(t,x) - a_+\Phi_p(x) \geq -h(t) \\ (39) & f(t,x) - b_+\Phi_p(x) \leq h(t)\end{array}\right\} \text{for all } x \geq 0, \text{ for a.e. } t \in [0,T],$$

$$\left.\begin{array}{ll}(40) & f(t,x) - a_-\Phi_p(x) \leq h(t) \\ (41) & f(t,x) - b_-\Phi_p(x) \geq -h(t)\end{array}\right\} \text{for all } x \leq 0, \text{ for a.e. } t \in [0,T].$$

As explained in the introduction, we will study here the case where the rectangle $[a_+, b_+] \times [a_-, b_-]$ touches two successive Fučík curves. The proof of Theorem 1 makes use of the following simple lemma.

LEMMA 2. *Let $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ satisfy Carathéodory conditions and the conditions (38)-(41), where $h \in L^2(0,T)$. Then, we can write f as*

$$\text{(42)} \qquad f(t,x) = \Phi_p(x)\gamma(t,x) + r(t,x)\ ,$$

where

$$\text{(43)} \qquad a_+ \leq \gamma(t,x) \leq b_+ \text{ for } x > 0, \text{ for a.e. } t \in [0,T]\ ,$$

(44) $\qquad a_- \le \gamma(t,x) \le b_-$ *for* $x < 0$, *for a.e.* $t \in [0,T]$,

r satisfying Carathéodory conditions with

(45) $\qquad |r(t,x)| \le h(t)$ *for all* x, *for a.e.* $t \in [0,T]$.

Proof. We introduce the function δ, defined for $u \le v$, by

$$\delta(u,x,v) = \begin{cases} u & \text{if } x < u \\ x & \text{if } u \le x \le v \\ v & \text{if } x > v \end{cases}$$

Let us define $\gamma(t,x)$ and $r(t,x)$ by

$$\gamma(t,x) = \begin{cases} \delta\left(a^+, \frac{f(t,x)}{\Phi_p(x)}, b^+\right) & \text{for } x > 0 \text{ ,} \\ \delta\left(a^-, \frac{f(t,x)}{\Phi_p(x)}, b^-\right) & \text{for } x < 0 \text{ ,} \\ 0 & \text{for } x = 0 \text{ ,} \end{cases}$$

$$r(t,x) = f(t,x) - \Phi_p(x)\gamma(t,x) \text{ .}$$

It is clear from the definition of $\gamma(t,x)$ that (43), (44) hold. On the other hand, (45) results from the inequalities (38)-(41). ◇

THEOREM 1. *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *satisfy Carathéodory conditions. Assume that there exists positive numbers* a_+, a_-, b_+, b_- *and a function* $h \in L^2(0,T)$ *such that the conditions (38)-(41) are fulfilled and that there exists* $n \in \mathbb{N}$ *such that*

$$\frac{1}{a_+^{1/p}} + \frac{1}{a_-^{1/p}} = \frac{T}{n\pi_p} \text{ ,}$$

$$\frac{1}{b_+^{1/p}} + \frac{1}{b_-^{1/p}} = \frac{T}{(n+1)\pi_p} \text{ .}$$

Assume moreover that, for any non-trivial solution ϕ *of the problem*

(46) $\qquad (\Phi_p(x'))' + a_+\Phi_p(x^+) - a_-\Phi_p(x^-) = 0$,

(47) $\qquad x(0) = x(T),\ x'(0) = x'(T)$,

the inequality

$$(48) \qquad 0 < \int_{\phi>0} \left(\liminf_{x\to+\infty}[f(t,x) - a_+\Phi_p(x)]\right)\phi(t)dt + \int_{\phi<0} \left(\limsup_{x\to-\infty}[f(t,x) - a_-\Phi_p(x)]\right)\phi(t)\,dt$$

is satisfied. Similarly, assume that for any non-trivial solution ψ of the problem

$$(49) \qquad (\Phi_p(x'))' + b_+\Phi_p(x^+) - b_-\Phi_p(x^-) = 0\ ,$$

$$(50) \qquad x(0) = x(T),\ x'(0) = x'(T)\ ,$$

the inequality

$$(51) \qquad 0 > \int_{\psi>0} \left(\limsup_{x\to+\infty}[f(t,x) - b_+\Phi_p(x)]\right)\psi(t)dt + \int_{\psi<0} \left(\liminf_{x\to-\infty}[f(t,x) - b_-\Phi_p(x)]\right)\phi(t)\,dt$$

holds. Then problem (1), (2) *has a solution.*

Proof. Taking $p_+ = \frac{a_++b_+}{2}$, $p_- = \frac{a_-+b_-}{2}$, it is clear that

$$\frac{T}{(n+1)\pi_p} < \frac{1}{p_+^{1/p}} + \frac{1}{p_-^{1/p}} < \frac{T}{n\pi_p}\ .$$

Consider the family of problems

$$(52) \quad (\Phi_p(x'))' + \lambda p_+\Phi_p(x^+) - \lambda p_-\Phi_p(x^-) + (1-\lambda)f(t,x) = 0\ ,$$

$$(53) \qquad x(0) = x(T),\ x'(0) = x'(T)\ ,$$

where $\lambda \in [0,1]$. It is easy to show with the above definition of p^+ and p^- that, for $\lambda = 1$, the above system has only the trivial solution.

By classical arguments from the theory of the topological degree, the theorem will be proven if we can find a priori bounds in $H^1(0,T)$ for the solution of problem (52), (53), independently of $\lambda \in (0,1)$ (notice that the degree for $\lambda = 1$ is odd; the proof, based on Borsuk's theorem, can be found in Lemma 4.3 of [1]).

By contradiction,suppose that there exists sequences $\{x_j\}$, $\{\lambda_j\}$ with $\|x_j\|_{H^1(0,T)} \to \infty$, $\lambda_j \in (0,1)$, such that x_j is a solution of (52), (53) for $\lambda = \lambda_j$. This means that

$$(54)\quad (\Phi_p(x_j'))' + \lambda_j p_+ \Phi_p(x_j^+) - \lambda_j p_- \Phi_p(x_j^-) + (1-\lambda_j) f(t,x_j) = 0\,.$$

Let $u_j = x_j/\|x_j\|_{H^1}$; using the decomposition of Lemma 2, we obtain

$$(55)\quad \begin{aligned} &(\Phi_p(u_j'))' + \lambda_j p_+ \Phi_p(u_j^+) - \lambda_j p_- \Phi_p(u_j^-) \\ &\quad + (1-\lambda_j)\gamma(t,x_j)\Phi_p(u_j) + (1-\lambda_j)\frac{r(t,x_j)}{\|x_j\|_{H^1}^{p-1}} = 0\,. \end{aligned}$$

As observed in [1], the problem of searching for T-periodic solutions of (1), (2), is equivalent to finding solutions in $H^1(0,T)$ of the equation $x = R_p(\Phi_p(x) + f(\cdot,x))$, where R_p is a completely continuous operator from $L^2(0,T)$ into $H^1(0,T)$. Since the Nemytskii operator $F : C[0,T] \to L^2(0,T) : x(\cdot) \mapsto f(\cdot, x(\cdot))$ is continuous, solutions of (1), (2), satisfy $x = R_p(\Phi_p(x) + F(x))$, where the right-hand side of this equation defines a completely continuous operator from $H^1(0,T)$ into itself. Thus, solving (52) is equivalent to searching for $x \in H^1(0,T)$ such that

$$(56)\; x = R_p(\Phi_p(x) + \lambda p_+ \Phi_p(x^+) - \lambda p_- \Phi_p(x^-) + (1-\lambda) f(\cdot,x))$$

and, similarly, (55) is equivalent to

$$(57)\quad \begin{aligned} u_j \;=\; R_p\Big(&\Phi_p(u_j) + \lambda_j p_+ \Phi_p(u_j^+) - \lambda_j p_- \Phi_p(u_j^-) \\ &+ (1-\lambda_j)\gamma(t,x_j)\Phi_p(u_j) + (1-\lambda_j)\frac{r(t,x_j)}{\|x_j\|_{H^1}^{p-1}}\Big)\,. \end{aligned}$$

Without loss of generality, we can assume that λ_j converges to some $\lambda \in [0,1]$. Since $\gamma(t,x)$ verifies the conditions (43), (44), the functions $\gamma(\cdot, x_j(\cdot))$ are essentially bounded in [0,T] with a common bound, so we can assume, passing if necessary to a subsequence, that they converge weakly in $L^2(0,T)$ to some function $\Gamma(\cdot)$. Moreover the argument of R_p in (57) is bounded in $L^2(0,T)$. Since R_p is a completely continuous operator, passing to a subsequence, we can assume that $\{u_j\}$ converges strongly in $H^1(0,T)$ to a certain map

u. Since $\|u_j\|_{H^1} = 1$, we have $\|u\|_{H^1} = 1$. On the other hand, the functions $u \to \Phi_p(u)$ and $u \to \Phi_p(u^{\pm})$ being continuous in $C[0,T]$, letting j go to infinity in (57) yields

$$(58)\quad u = R_p(\Phi_p(u) + \lambda p_+ \Phi_p(u^+) - \lambda p_- \Phi_p(u^-) + (1-\lambda)\Gamma(t)\Phi_p(u)).$$

Therefore u satisfies

$$(59)\qquad (\Phi_p(u'))' + \lambda p_+ \Phi_p(u^+) - \lambda p_- \Phi_p(u^-) + $$
$$+ (1-\lambda)\Gamma(t)\Phi_p(u) = 0\ ,$$
$$(60)\qquad u(0) = u(T), \quad u'(0) = u'(T)\ .$$

As $\|u\|_{H^1} = 1$, u cannot be the trivial solution of that homogeneous problem, and consequently we will have $u^2(t) + u'^2(t) \neq 0$, for all $t \in [0,T]$. By Lemma 1, we then have, for some integer k,

$$(61)\qquad k\pi_p =$$
$$a_+^{1/p} \int_{I+} \frac{(p-1)u'\Phi_p(u') + u[\lambda p_+ \Phi_p(u^+) + (1-\lambda)\Gamma(t)\Phi_p(u)]}{(p-1)u'\Phi_p(u') + a_+ u \Phi_p(u)} dt$$

and also

$$k\pi_p =$$
$$b_+^{1/p} \int_{I+} \frac{(p-1)u'\Phi_p(u') + u[\lambda p_+ \Phi_p(u^+) + (1-\lambda)\Gamma(t)\Phi_p(u)]}{(p-1)u'\Phi_p(u') + b_+ u \Phi_p(u)} dt\ .$$

Since $a_+ \leq p_+ \leq b_+$, and $a_+ \leq \Gamma(t) \leq b_+$ for a.e. $t \in I_+$, we deduce from the above relations that

$$a_+^{1/p} \text{mes}\ (I^+) \leq k\pi_p \leq b_+^{1/p} \text{mes}\ (I_+)\ .$$

A similar argument leads to

$$a_-^{1/p} \text{mes}\ (I_-) \leq k\pi_p \leq b_-^{1/p} \text{mes}\ (I_-)\ .$$

Combining the above inequalities and using the fact that mes $(I_-)+$ mes $(I_+) = T$, we obtain

$$\frac{1}{b_+^{1/p}} + \frac{1}{b_-^{1/p}} \leq \frac{T}{k\pi_p} \leq \frac{1}{a_+^{1/p}} + \frac{1}{a_-^{1/p}}\ .$$

The assumptions (9), (10) then imply that $k = n$ or $k = n+1$. Moreover, going back to (61) and a similar equation on I_-, we see that, if $k = n$, we must have

$$\lambda p_+ + (1-\lambda)\Gamma(t) = a_+, \text{ a.e. on } I_+$$

and

$$\lambda p_- + (1-\lambda)\Gamma(t) = a_-, \text{ a.e. on } I_- \, .$$

This means that u must be a solution of (46), (47). Similarly, if $k = n+1$, u must be a solution of (49), (50).

Let us assume for the sequel that $k = n$, the case where $k = n+1$ being treated in a similar way. Going back to the sequence $\{x_j\}$ we can assume, passing if necessary to a subsequence, that x_j makes n revolutions in the phase plane. Using Lemma 1 again, we have

$$n\pi_p = \tag{62}$$

$$a_+^{1/p} \int_{x_j>0} \frac{(p-1)x_j'\Phi_p(x_j') + x_j[\lambda_j p_+ \Phi_p(x_j^+) + (1-\lambda_j)f(t,x_j)]}{(p-1)x_j'\Phi_p(x_j') + a_+ x_j \Phi_p(x_j)} dt$$

or, using the fact that $p_+ \geq a_+$

$$n\pi_p \geq a_+^{1/p} \text{mes}\{t \in [0,T] \mid x_j(t) > 0\} \tag{63}$$

$$+(1-\lambda_j)a_+^{1/p} \int_{x_j>0} \frac{[f(t,x_j) - a_+\Phi_p(x_j^+))]x_j dt}{(p-1)x_j'\Phi_p(x_j') + a_+ x_j \Phi_p(x_j)} .$$

Combining the above relation with the corresponding result obtained by working on the set $\{t \in [0,T] \mid x_j(t) < 0\}$, we get

$$\left(\frac{1}{a_+^{1/p}} + \frac{1}{a_-^{1/p}}\right) n\pi_p \geq T \tag{64}$$

$$+(1-\lambda_j) \int_0^T \frac{[f(t,x_j) - a_+\Phi_p(x_j^+) - a_-\Phi_p(x_j^-)]x_j}{(p-1)x_j'\Phi_p(x_j') + a_+ x_j^+ \Phi_p(x_j^+) + a_- x_j^- \Phi_p(x_j^-)} dt \, .$$

By (9), it then follows that

$$\liminf_{j\to\infty} \int_0^T \frac{[f(t,x_j) - a_+\Phi_p(x_j^+) - a_-\Phi_p(x_j^-)]u_j}{(p-1)u_j'\Phi_p(u_j') + a_+ u_j^+ \Phi_p(u_j^+) + a_- u_j^- \Phi_p(u_j^-)} dt \leq 0 \, .$$

Because of inequalities (38)-(41), we can apply Fatou's lemma, which gives

$$\int_0^T \liminf_{j\to\infty}([f(t,x_j) - a_+\Phi_p(x_j^+) - a_-\Phi_p(x_j^-)]uj)dt \le 0 ,$$

taking into account that the quantity

$$(p-1)u'(t)\Phi_p(u'(t)) + a_+u^+(t)\Phi_p(u^+(t)) + a_-u^-(t)\Phi_p(u^-(t))$$

is constant on [0,T], since u is a solution of

$$(\Phi_p(u'))' + a_+\Phi_p(u^+) - a_-\Phi_p(u^-) = 0 .$$

It then follows that

$$\int_{u>0} \left(\liminf_{x\to+\infty}[f(t,x) - a_+\Phi_p(x)]\right) u(t)dt + \int_{u<0} \left(\limsup_{x\to-\infty}[f(t,x) - a_-\Phi_p(x)]\right) u(t)dt \le 0 .$$

But the last inequality contradicts condition (48), ending the proof in the case $k = n$. ◇

6. A priori bounds in the superlinear case.

The study of the superlinear case is based on an auxiliary result for which we will not restrict ourselves to equation (1), but will consider a one-parameter family of equations

(65) $$(\Phi_p(x')) + F(t,x,\lambda) = 0 .$$

We will assume that F is defined on $\mathbb{R}\times\mathbb{R}\times[0,1]$ and is continuous in its first two variables.

LEMMA 3. *Assume that $F(\cdot,\cdot,\lambda)$ is continuous on $\mathbb{R}\times\mathbb{R}$, for all $\lambda\in[0,1]$. Assume that there exists a number $\eta > 0$ such that*

(66) $$\liminf_{|x|\to\infty}(sgn\ x)F(t,x,\lambda) \ge \eta, \text{ uniformly in } (t,\lambda) .$$

Then, for any $\rho > 0$ there exists $R > 0$ such that, for any solution $x : [t_0, w] \to \mathbb{R}$ of (65) with $w > t_0$, $|x(t_0)| \ge R$, $x'(t_0) = 0$, and $|x(w)|^p + (p-1)|x'(w)|^p < \rho^p$, there exists $t_1 \in (t_0, w)$ such that

a) *x has at least two zeros in $[t_0, t_1]$,*

b) *for all $t \in [t_0, t_1]$, $|x(t)|^p + (p-1)|x'(t)^p \geq \rho^p$,*

c) $|x(t_1)| \geq \rho$, $x'(t_1) = 0$.

Proof. Take $\varepsilon \in (0, \eta/2)$. As in [5], it is possible to build continuous non-decreasing functions g, h such that, for all $(t, x, \lambda) \in \mathbb{R} \times \mathbb{R} \times [0,1]$,

$$g(x) + \varepsilon \leq F(t, x, \lambda) \leq h(x) - \varepsilon ,$$

g and h being defined in such that a way that $g(x) = \eta/2$ for large positive values of x, and $h(x) = -\eta/2$ for large negative values of x (for instance, remember that $F(t, x, \lambda) - \varepsilon$ becomes larger than $\eta/2$ for large positive x). Define then the convex functions G, H by

$$G(x) = \int_0^x g(u)du \ ; \ H(x) = \int_0^x h(u)du \ .$$

It is clear that

$$G(x) < H(x) \text{ for } x > 0 \ , \ G(x) > H(x) \text{ for } x < 0 \ . \tag{67}$$

Moreover, since $g(x) = \eta/2$ for large positive values of x, we have

$$\lim_{x \to +\infty} G(x) = +\infty, \text{ and similarly } \lim_{x \to -\infty} H(x) = +\infty \ .$$

In the (x, y)-plane, let $B_\rho = \{(x, y) \mid |x|^p + (p-1)|y|^p \leq \rho^p\}$. Let q be defined by

$$\frac{1}{p} + \frac{1}{q} = 1 \ .$$

Choose $K > 0$ such that for all $(x, y) \in B_\rho$,

$$\frac{1}{q}|y|^q + H(x) < K, \ \frac{1}{q}|y|^q + G(x) < K \ .$$

Let $\alpha < 0$ be such that $H(\alpha) = K$; such a number does exist, since $H(0) = 0$, and $\lim_{x \to -\infty} H(x) = +\infty$. Define next curves Γ_1, Γ_2 in the (x, y)-plane by:

$$\Gamma_1 = \{(x, y) \mid \frac{1}{q}|y|^q + H(x) = H(\alpha), \ y \geq 0\}$$

$$\Gamma_2 = \{(x, y) \mid \frac{1}{q}|y|^q + G(x) = G(\alpha), \ y \leq 0\}$$

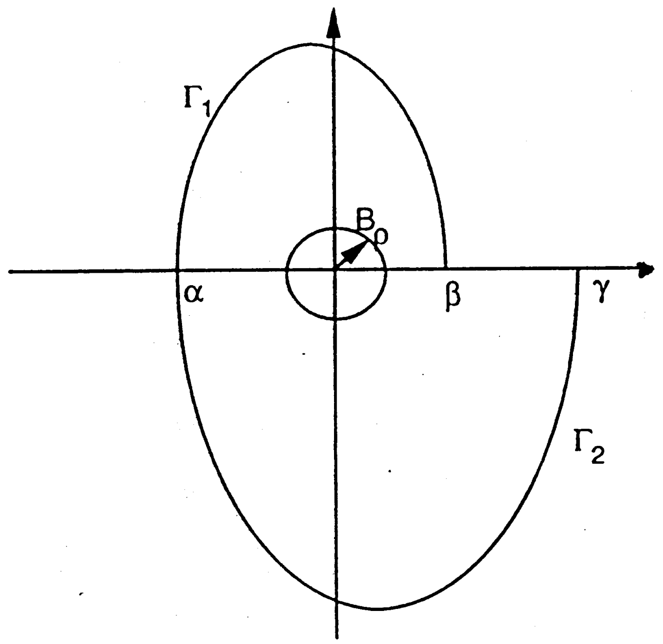

Figure 1: The curves Γ_1 and Γ_2

The curves Γ_1 and Γ_2 clearly lie outside the set B_ρ. Possible curves are shown in Fig.1. Since the function H is convex, for any $y \in \mathbb{R}$, these are at most 2 points x_1, x_2, such that $(x_1, y) \in \Gamma_1$, $(x_2, y) \in \Gamma_1$. The same holds true for Γ_2. Clearly, there exists $\beta > \rho$, such that $H(\beta) = H(\alpha)$. There also exists $\gamma > \rho$, such that $G(\gamma) = G(\alpha) > H(\alpha)$. since $G(x) < H(x)$, for $x > 0$, we will have $\gamma > \beta$. Now, let x be a solution of (65) defined on $[t_0, w]$ with $w > t_0$; that equation can be written as a system

$$x' = \Phi_p^{-1}(y) \tag{68}$$

$$y' = -F(t, x, \lambda)\ . \tag{69}$$

If the curve $t \mapsto (x(t), y(t))$ crosses Γ_1, the crossing must be from the "inside" towards the "outside". Indeed, along solutions of (68), (69), we have, for $y > 0$

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{q}|y|^q + H(x)\right) &= -y^{q-1}F(t,x,\lambda) + h(x)\Phi_p^{-1}(y) \\ &= y^{q-1}[h(x) - F(t,x,\lambda)] > 0\ , \end{aligned}$$

showing that, at points of Γ_1, the vector field associated to the differential system (68), (69) points outwards. A similar result holds for

Γ_2. It can also be shown (the details are left to the reader) that the vector field cannot "enter" at the point $(\alpha, 0)$. Moreover, the vector field points downwards along the half-line $\{(x, 0) \mid x \geq \gamma\}$. This results from the fact that, if a solution curve crosses that half-line at a point $x \geq \gamma$, we have

$$y' = -F(t, x, \lambda) \leq -g(x) \leq -g(\gamma) ,$$

where $g(\gamma) > 0$ (since otherwise, we would have $G(\gamma) \leq 0$). Consequently, if $x : [t_0, w] \to \mathbb{R}$ is a solution of (65) with $w > t_0$, $x(t_0) \geq \gamma$, $x'(t_0) = 0$, we see that the curve $t \mapsto (x(t), y(t))$ must circle at least once around B_ρ before crossing the segment $\{(x, 0) \mid \beta \leq x \leq \gamma\}$ and entering the set B_ρ.

A similar construction takes place for solutions x with $x(t_0) < 0$, $x'(t_0) = 0$. Hence, choosing R large enough, the conclusion follows. ◇

By iteration of Lemma 3, we can prove the next lemma.

LEMMA 4. *Let F be as in Lemma 3, Then for any $n \in \mathbb{N}$, and any $\rho_0 > 0$, there exists a number $R_0 > 0$ such that for any solution $x : [t_0, w] \to \mathbb{R}$ of (65) with $t_0 < w$, $|x(t_0)| \geq R_0$, $x'(t_0) = 0$, either $|x(t)|^p + (p-1)|x'(t)|^p \geq \rho_0^p$ for all $t \in [t_0, w]$, or x has at least $2n$ zeros on an interval $[t_0, t_n] \subset [t_0, w]$ and, for all $t \in [t_0, t_n]$, $|x(t)|^p + (p-1)|x'(t)|^p \geq \rho_0^p$.*

As observed in [5], the above lemma can also be rephrased as follows.

LEMMA 5. *Let F be as in Lemma 3. Then for any $n \in \mathbb{N}$, any $\rho_0 > 0$, there exists a number $R_0 > 0$ such that if $x : [t_0, w] \to \mathbb{R}$ is a solution of (65) with $|x(t_0)|^p + (p-1)|x'(t_0)|^p \leq \rho_0^p$, having at most $2n$ zeros, we have for all $t \in [t_0, w]$,*

$$|x(t)|^p + (p-1)|x'(t)|^p \leq R_0^p .$$

Lemma 5 shows that under condition (66), an a priori bound can be found for the solutions of equation (65) which enter the set B_ρ at some time t_0 and have less than a given number of zeros.

The lemmas 3, 4, 5 have been written using the norm $[|x|^p + (p-1)|y|^q]$ in the (x, y)-plane. Any other norm could obviously have been used in the statement of those lemmas.

7. Existence results for the superlinear case.

Our existence result for the superlinear case is as follows

THEOREM 2. *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be continuous and such that* $f(0,x) = f(T,x)$ *for all* $x \in \mathbb{R}$. *Assume that positive numbers* a_+, a_-, b_- *exist such that*

$$a_+ \le \liminf_{x\to+\infty} \frac{f(t,x)}{\Phi_p(x)} \le \limsup_{x\to+\infty} \frac{f(t,x)}{\Phi_p(x)} \le b_+ \ , \tag{70}$$

$$a_- \le \liminf_{x\to-\infty} \frac{f(t,x)}{\Phi_p(x)} \ , \tag{71}$$

the limits being uniform in t. *If, for some integer* $n \in \mathbb{N}$, *the inequalities*

$$\frac{1}{a_+^{1/p}} + \frac{1}{a_-^{1/p}} < \frac{T}{n\pi_p} \ , \tag{11}$$

$$b_+^{1/p} < (n+1)\frac{\pi_p}{T} \tag{12}$$

hold, then (1), (2) has a solution.

Proof. For $\lambda \in [0,1]$, define the function F by

$$F(t,x,\lambda) = \lambda f(t,x) + (1-\lambda)[a_+\Phi_p(x^+) - a_-\Phi_p(x^-)] \ ;$$

by T-periodicity in t, that function will be extended to $\mathbb{R}\times\mathbb{R}\times[0,1]$. As in Theorem 1, we use degree theoretic arguments to prove the existence of at least one solution. However, instead of working with the H^1-norm, we will search here for a priori bounds in the sup-norm for the solutions of

$$(\Phi_p(x'))' + F(t,x,\lambda) = 0 \ , \tag{72}$$

$$x(0) = x(T),\ x'(0) = x'(T) \ , \tag{73}$$

the a priori bounds being independent of $\lambda \in (0,1)$. Since the inequalities (11), (12) still hold when a small positive constant is added or subtracted to a_+, b_+, a_- we can, without loss of generality, replace (70), (71), by the stronger assumption that, for some contant $K > 0$

$$a_+|x|^p - K \le xf(t,x) \le b_+|x|^p + K , \quad \text{for all } x \ge 0, \text{ for all } t \in \mathbb{R} , \tag{74}$$

$$a_-|x|^p - K \le xf(t,x), \text{for all } x \le 0, \text{for all } t \in \mathbb{R} . \tag{75}$$

Let x be a solution (72), (73) such that for some $t_0 \in [0,T]$, $|x(t_0)|^p + (p-1)|x'(t_0)|^p \ge R_0^p$. We will show that such a solution cannot exist if R_0 is large enough. For that purpose, we will use the change of variables of section 3, or more precisely, use (31), (32) in the half-plane $x \ge 0$ and (33), (34) in the half plane $x \le 0$. Assume that x is a solution of (72), (73); letting $I_+ = \{t \in [0,T] \mid x(t) \ge 0\}$ and $I_- = \{t \in [0,T] \mid x(t) \le 0\}$, we have, by the results of section 3 (see (36), (37)),

$$k\pi_p = a_+^{1/p} \int_{I+} \frac{(p-1)x'\Phi_p(x') + xF(t,x,\lambda)}{(p-1)x'\Phi_p(x') + a_+ x\Phi_p(x)}\, dt , \tag{76}$$

$$k\pi_p = a_-^{1/p} \int_{I-} \frac{(p-1)x'\Phi_p(x') + xF(t,x,\lambda)}{(p-1)x'\Phi_p(x') + a_- x\Phi_p(x)}\, dt . \tag{77}$$

We will find a priori bounds for the solutions of (72), (73) distinguishing 2 cases, depending on the number of zeros of the possible solution in [0,T]. In the sequel, the number n is the integer appearing in hypotheses (11), (12).

1st case: The solution x has at most $2n$ zeros in [0,T). Take ρ_0 large enough so that

$$\frac{n\pi_p}{T}\left(\frac{1}{a_+^{1/p}} + \frac{1}{a_-^{1/p}}\right) < 1 - \frac{K}{\rho_0^p} . \tag{78}$$

Since a_+ and a_- are positive, its results from (70), (71) that (sgn $x)f(t,x)$ and consequently also (sgn x) $F(t,x,\lambda)$ will become positive for $|x|$ large and bounded away from 0. Hence we can apply Lemma 4. Since the solution x is assumed to have at most $2n$ zeros in $[0,T)$, a number R_0 can be found using Lemma 4, such that, if $|x(t_0)| \ge R_0$, $x'(t_0) = 0$ for some $t_0 \in [0,T]$ then

$$a_+|x(t)|^p + (p-1)|x'(t)|^p \ge \rho_0^p$$

and

$$a_-|x(t)|^p + (p-1)|x'(t)|^p \geq \rho_0^p ,$$

for all $t \in [0,T]$. If k is the number of revolutions of the curve $t \mapsto (x(t), y(t))$, in the phase plane, we have, by (76),

$$k\pi_p = a_+^{1/p}\text{mes}(I^+) + a_+^{1/p} \int_{I^+} \frac{xF(t,x,\lambda) - a_+ x\Phi_p(x)}{(p-1)x'\Phi_p(x') + a_+ x\Phi_p(x)}\, dt ,$$

from which follows, using (74) that

(79) $$k\pi_p \geq a_+^{1/p}\text{mes}(I^+) - a_+^{1/p}\frac{K}{\rho_0^p}\text{mes}(I^+) .$$

Similarly, by (75) and (77), we obtain

(80) $$k\pi_p \geq a_-^{1/p}\text{mes}(I^-) - a_-^{1/p}\frac{K}{\rho_0^p}\text{mes } (I^-) .$$

Combining (79) and (80), we get

$$\frac{k\pi_p}{T}\left(\frac{1}{a_+^{1/p}} + \frac{1}{a_-^{1/p}}\right) \geq 1 - \frac{K}{\rho_0^p} ;$$

confrontation with (78) shows that $k > n$, leading to a contradiction. Hence we conclude that $|x(t)| \leq R_0$ for all $t \in [0,T]$, if x has at most $2n$ zeros in $[0,T]$.

2nd case: The solution x has at least $(2n+2)$ zeros in [0,T). Take ρ_0 large enough so that

(81) $$\frac{(n+1)\pi_p}{T}\frac{1}{b_+^{1/p}} > 1 + \frac{K}{\rho_0^p} .$$

Since the solution is assumed to have at least $(2n+2)$ zeros in $[0,T)$, using Lemma 4 again, a number R_0 can be found such that if for some $t_0 \in [0,T]$, $|x(t_0)| > R_0$ and $x'(t_0) = 0$, then

$$b_+|x(t)|^p + (p-1)|x'(t)|^p \geq \rho_0^p \text{ for all } t \in [0,T] .$$

Using (74) and (76) (with a_+ replaced by b_+), this leads to

(82) $$k\pi_p \leq b_+^{1/p}\text{ mes } (I^+) + b_+^{1/p}\frac{K}{\rho_0^p}\text{mes } (I^+) ,$$

implying that

$$k\pi_p \frac{1}{b_+^{1/p}} \leq \text{mes }(I^+)\left(1+\frac{K}{\rho_0^p}\right) \leq T\left(1+\frac{K}{\rho_0^p}\right) . \tag{83}$$

But, because of (81), this would imply that $k < (n+1)$, leading to a contradiction. Hence we must have in this case also $|x(t)| \leq R_0$ for all $t \in [0, T]$. ◊

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