EXISTENCE OF SOLUTIONS FOR DIFFERENTIAL INCLUSIONS WITHOUT CONVEXITY(*)

by FRANCESCA PAPALINI (in Perugia)(**)

SOMMARIO. - *In questo lavoro otteniamo due teoremi di esistenza per inclusioni differenziali. Nel primo teorema proviamo una condizione per l'esistenza di soluzioni del problema di Cauchy:* $\dot{x} \in F(x) + f(t,x)$, $x(0) = \xi$, *ove "F" è un operatore multivoco di* $\mathbb{R}^n$ *e "f" è una perturbazione monodroma. Questo risultato contiene i Teoremi di esistenza conseguiti in* [4] *e* [1]. *Nel secondo teorema studiamo l'esistenza di soluzioni per il problema più generale:* $\dot{x} \in F(x) + G(t,x)$, $x(0) = \xi$, *ove "G" è una perturbazione multivoca.*

SUMMARY. - *In this note we obtain two existence theorems for differential inclusions. In the first theorem we prove a condition for the existence of solutions to the Cauchy problem:* $\dot{x} \in F(x) + f(t,x)$, $x(0) = \xi$, *where "F" is a multivalued operator of* $\mathbb{R}^n$ *and "f" is a singlevalued perturbation. This result improves the existence Theorems obtained in* [4] *and* [1]. *In the second theorem we study the existence of solutions for the more general problem:* $\dot{x} \in F(x) + G(t,x)$, $x(0) = \xi$, *where "G" is a multivalued perturbation.*

1. Introduction.

In the study of the existence of solutions to the Cauchy problem

$$(*) \quad \begin{cases} \dot{x} \in F(x) \\ x(0) = \xi \,, \end{cases}$$

where "F" is a multivalued operator of $\mathbb{R}^n$, the conditions imposed on "F", in order to obtain existence Theorems, are of two kinds: regularity conditions on the multifunction "F" (such as continuity, upper semicontinuity, lower semicontinuity, ...) and conditions on

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(*) Pervenuto in Redazione il 20 aprile 1993.

(**) Indirizzo dell'Autore: Dipartimento di Matematica, Università degli Studi, Via Vanvitelli 1, 06100 Perugia (Italy).

the values of "F" (such as convexity, compactness, ...). It is known (cf. [4]) that the only conditions of upper semicontinuity on "F" and compactness (without convexity) on the values of "F" are not sufficient to have solutions for (∗). Therefore, in order to have an existence result in the context in which "F" is an upper semicontinuous multifunction with compact values, it is necessary to make additional assumptions on "F".

In 1989 A. Bressan, A. Cellina and G. Colombo [4] have obtained an existence Theorem for the problem (∗) in the case which $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is an upper semicontinuous an cyclically monotone multifunction with compact, not necessarily convex, and non empty values.

Later in 1990 F. Ancona and G. Colombo [1] have studied the following "perturbed" problem

$$(1) \quad \begin{cases} \dot{x} \in F(x) + f(t,x) \\ x(0) = \xi \,, \end{cases}$$

where $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is like in [4], and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a function that satisfies the conditions:

β) for every $x \in \mathbb{R}^n$, $t \mapsto f(t,x)$ is measurable;

$\beta\beta$) for a.e. $t \in \mathbb{R}$, $x \mapsto f(t,x)$ is continuous on $\mathbb{R}^n$;

$\beta\beta\beta$) $\exists m \in L^2(\mathbb{R},\mathbb{R})$ such that

$$\|f(t,x)\| \leq m(t), \quad \text{for a.e. } t \in \mathbb{R}, \text{ for all } x \in \mathbb{R}^n \,.$$

The result obtained in [1] contains the result of [4], as a particular case.

In the first part of this paper we consider the problem (1) to prove the existence of a solution under weaker assumptions than those which are assumed in [1]. In fact, for us, as for F. Ancona and G. Colombo, $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is an upper semicontinuous and cyclically monotone multifunction with compact and non empty values, while $f : [0,b] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the conditions β), $\beta\beta$) and

$\beta\beta\beta)_w$ $\exists p \in]1,2[$ and $\exists h \in L^p([0,b],\mathbb{R}) \cap L^2_{loc}([0,b],\mathbb{R})$, such that $\|f(t,x)\| \leq h(t)$, for a.e. $t \in [0,b]$, for all $x \in \mathbb{R}^n$.

It is obvious that every function "f" satisfying the assumption $\beta\beta\beta$) satisfies also the condition $\beta\beta\beta)_w$, but there exist (cf. here, Remark 3) functions "f" that verify the condition β), $\beta\beta$), $\beta\beta\beta)_w$ that do not satisfy the assumption $\beta\beta\beta$).

In the second part of this note we obtain an existence theorem for the Cauchy problem:

$$(1)' \quad \begin{cases} \dot{x} \in F(x) + G(t,x) \\ x(0) = \xi \ , \end{cases}$$

where $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is an upper semicontinuous and cyclically monotone multivalued operator with compact and non empty values and $G : [0,b] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a multifunction with the properties:

i) $G(t,x)$ is non empty, closed and convex, $\forall (t,x) \in [0,b] \times \mathbb{R}^n$;

ii) $\forall x \in \mathbb{R}^n$, $t \mapsto G(t,x)$ is measurable;

iii) $\forall t \in [0,b]$, $x \mapsto G(t,x)$ is lower semicontinuous and it has closed graph;

iv) $\exists p \in]1,2[$ and $\exists h \in L^p([0,b],\mathbb{R}) \cap L^2_{loc}([0,b],\mathbb{R})$, such that $\|y\| \leq h(t), \forall y \in G(t,x)$, for a.e. $t \in [0,b]$ and for all $x \in \mathbb{R}^n$.

In order to obtain this existence result we first prove a proposition that is a sufficient condition to get Caratheodory's selections for a multifunction. This proposition extends a Caratheodory's selection Theorem stated by G. Bonanno in 1989 (cf. [3, Theorem 3.1]), in the sense that there exist multifunctions that satisfy the conditions of our proposition that do not satisfy the assumptions of the Theorem of G. Bonanno (cf. here, Remark 4).

2. Let $[a,b]$ be an interval of the real line and μ the Lebesgue measure on $[a,b]$. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$ we set
$B(x,\varepsilon) = \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$ endowed by the scalar product $\langle \cdot, \cdot \rangle$ and given a subset A of $\mathbb{R}^n$, we put $B(A,\varepsilon) = \{x \in \mathbb{R}^n : \rho(x,A) < \varepsilon\}$, where

$\rho(x, A) = \inf\{\|y - x\| : y \in A\}$. If $1 \leq p < +\infty$, we put

$$L^p_{loc}([a,b],\mathbb{R}^n) = \{x : [a,b] \to \mathbb{R}^n : \text{“}x\text{” is measurable in } [a,b] \text{ and } \int_c^d \|x(t)\|^p dt < +\infty, \ \forall c,d \in]a,b[\}$$

$$W^{1,p}([a,b],\ \mathbb{R}^n) = \{x : [a,b] \to \mathbb{R}^n : \text{“}x\text{” is absolutely continuous on } [a,b] \text{ and such that } \dot{x} \in L^p([a,b],\mathbb{R}^n)\} \ (1) .$$

A function $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be “*proper*” if $D(V) \neq \emptyset$, where $D(V) = \{x \in \mathbb{R}^n : V(x) < +\infty\}$. If “$V$” is proper, convex and lower semicontinuous, the multifunction $\partial V : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, defined by

$$\partial V(x) = \{y \in \mathbb{R}^n : V(\xi) - V(x) \geq \langle y, \xi - x\rangle, \ \forall \xi \in \mathbb{R}^n\}, \ \forall x \in \mathbb{R}^n ,$$

is called “*sub-differential*” of “V”.

REMARK 1. It is known (cf. [5, Example 2.3.4]) that the sub-differential “∂V” of a proper, convex, lower semicontinuous function “V” is a monotone maximal operator and $D(\partial V) \subset D(V)$, where $D(\partial V) = \{x \in \mathbb{R}^n : \partial V(x) \neq \emptyset\}$.

A multifunction $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is called “*lower semicontinuous* (*upper semicontinuous*) if

$\forall x \in \mathbb{R}^n$ and $\forall \varepsilon > 0$ there exists $\delta > 0$ such that

$$F(x) \subset B(F(y), \varepsilon) \ \ (F(y) \subset B(F(x), \varepsilon)), \quad \forall y \in B(x, \delta) .$$

Moreover “F” is said to have “*closed graph*” if the set

$$GrF = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in F(x)\}$$

is closed in $\mathbb{R}^n \times \mathbb{R}^n$.

Let $\mathcal{A}$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$; the multifunction “F” is called “*measurable*” if for any closed subset $C \subset \mathbb{R}^n$, we have

$$\{x \in \mathbb{R}^n : F(x) \cap C \neq \emptyset\} \in \mathcal{A} .$$

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(1) $W^{1,1}([a,b],\mathbb{R}^n)$ is the space of absolutely continuous functions on $[a,b]$.

The multivalued operator "F" is said to be "*cyclically monotone*" if for every cyclical sequence

$$x_0, x_1, \ldots, x_N = x_0$$

and for every sequence $y_1, \ldots, y_N$ such that $y_i \in F(x_i)$, $i = 1, \ldots, N$, we have

$$\sum_{i=1}^{N} \langle y_i, x_i - x_{i-1} \rangle \geq 0 \, .$$

REMARK 2. We recall that (cf. [5, Theorem 2.5]) "F" is cyclically monotone if and only if there exists a proper, convex, lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R} \cup (+\infty)$ such that

$$F(x) \subset \partial V(x), \forall x \in \mathbb{R}^n \, .$$

3. We consider the Cauchy problem

$$(1) \quad \begin{cases} \dot{x} \in F(x) + f(t,x) \\ x(0) = \xi \in \mathbb{R}^n \, , \end{cases}$$

where $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a multifunction and $f : [0,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a function which verify respectively the properties:

α) $F(x)$ is non empty and compact, $\forall x \in \mathbb{R}^n$;

$\alpha\alpha$) "F" is upper semicontinuous;

$\alpha\alpha\alpha$) "F" is cyclically monotone;

β) $\forall x \in \mathbb{R}^n$, the function $t \mapsto f(t,x)$ is measurable;

$\beta\beta$) for a.e. $t \in [0,b]$, the function $x \mapsto f(t,x)$ is continuous on $\mathbb{R}^n$;

$\beta\beta\beta)_w$ $\exists p \in]1,2[$ and $\exists h \in L^p([0,b],\mathbb{R}) \cap L^2_{loc}([0,b],\mathbb{R})$ such that $\|f(t,x)\| \leq h(t)$, for a.e. $t \in [0,b]$ for all $x \in \mathbb{R}^n$.

By a *solution* of our Cauchy problem we mean an *absolutely continuous function "x" that satisfies* (1) a.e.

Our existence result is the following

THEOREM. *If "F" and "f" satisfy the conditions* α*),* $\alpha\alpha$*),* $\alpha\alpha\alpha$*) and* β*),* $\beta\beta$*),* $\beta\beta\beta)_w$ *respectively, then there exist* $T > 0$ *and a solution* $x : [0,T] \to \mathbb{R}^n$ *of the Cauchy problem* (1).

We start by observing that from α) and $\alpha\alpha$) it is possible to find two positive real numbers R and M such that

$$\|y\| < M, \ \forall y \in F(x) \text{ and } \forall x \in B(\xi, R) \ . \tag{3.1}$$

By $\beta\beta\beta)_w$ it is possible to find a positive number T, non greater than b, with the property:

$$\int_0^T (h(t) + M)dt < R \ . \tag{3.2}$$

Now, we are going to introduce a sequence of functions defined in [0,T] and we will prove that a subsequence converges to a solution of the Cauchy problem (1).

We consider the sequence $(x_m)_m$, $x_m : [0,T] \to \mathbb{R}^n$, putting

$$\begin{aligned} x_m(0) &= \xi \ , \\ x_m(t) &= x_m\left(i\frac{T}{m}\right) + \int_{iT/m}^t f\left(s, x_m\left(i\frac{T}{m}\right)\right) ds + \left(t - i\frac{T}{m}\right) y_{m,i} \ , \end{aligned}$$

$$t \in \left[i\frac{T}{m}, (i+1)\frac{T}{m}\right] \ ,$$

where $y_{m,i} \in F\left(x_m\left(i\frac{T}{m}\right)\right)$, $\forall i \in \{0, 1, \ldots, m-1\}$.

For every $m \in \mathbb{N}$ let be $I_{m,0} = \left[0, \frac{T}{m}\right]$ and $I_{m,i} = \left]i\frac{T}{m}, (i+1)\frac{T}{m}\right]$. $\forall i \in \{1, \ldots, m-1\}$; now we set:

δ_m, $\gamma_m : [0,T] \to [0,T]$ and f_m, $g_m : [0,T] \to \mathbb{R}^n$, where

$$\begin{aligned} \delta_m(t) &= i\frac{T}{m}, \quad \gamma_m(t) = (i+1)\frac{T}{m} \quad \forall t \in I_{m,i}, \forall i \in \{0, 1, \ldots, m-1\}; \\ f_m(t) &= f\left(t, x_m\left(i\frac{T}{m}\right)\right), g_m(t) = y_{m,i} \ , \end{aligned}$$

$\forall\, t \in I_{m,i}, \forall i \in \{0,1,\ldots,m-1\}$.

Let $k : [0,T] \to [0,T]$ be the function defined putting $k(t) = t$, $\forall t \in [0,T]$. It is easy to prove that

$$(\delta_m)_m \quad \text{and} \quad (\gamma_m)_m \text{ converge uniformly to "}k\text{"} . \tag{3.3}$$

Moreover, by construction, we have

$$g_m(t) \in F\left(x_m\left(\delta_m(t)\right)\right), \forall t \in [0,T], \forall m \in \mathbb{N} , \tag{3.4}$$

$$x_m(t) = \xi + \int_0^t (f_m(s) + g_m(s))ds, \ \forall t \in [0,T], \ \forall m \in \mathbb{N} , \tag{3.5}$$

and, by using (3.1), (3.4), (3.2) and $\beta\beta\beta)_w$, it is trivial to prove that

$$\|g_m(t)\| < M, \ \forall t \in [0,T], \ \forall m \in \mathbb{N} . \tag{3.6}$$

Then, taking (3.5), $\beta\beta\beta)_w$, (3.6) and (3.2) into account it follows that

$$(x_m)_m \text{ is equibounded in } [0,T] .$$

Now, by (3.5), we obtain that

$$\dot{x}_m = f_m + g_m \text{ a.e. in } [0,T] \text{ and } \forall m \in \mathbb{N} , \tag{3.7}$$

hence, by $\beta\beta\beta)_w$ and (3.6), we get

$$\|\dot{x}_m\|_{L^p[0,T]} \leq H, \ \forall m \in \mathbb{N} , \tag{3.8}$$

where $H = \left(\int_0^T (h(t) + M)^p dt\right)^{1/p}$.

Moreover, by using Hölder's inequality and (3.8) it is easy to see that

$$(x_m)_m \text{ is equiuniformly continuous in } [0,T] .$$

Hence, by taking Arzelà-Ascoli Theorem and the Theorem 3.27 of [6] into account, it follows that there exist a subsequence of $(x_m)_m$, still denoted by $(x_m)_m$, and an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that

$$(x_m)_m \text{ converges uniformly to "}x\text{"} \tag{3.9}$$

and

$$(\dot{x}_m)_m \text{ converges weakly in} L^p([0,T],\mathbb{R}^n) \text{ to } \text{“}\dot{x}\text{”} \ . \tag{3.10}$$

Since (cf. here, (3.3) and $\beta\beta$))

$$(f_m)_m \text{ converges to } f(\cdot,x(\cdot)) \text{ a.e. in } [0,T] \ , \tag{3.11}$$

by taking $\beta\beta\beta)_w$ into account, we have that:

$$(f_m)_m \text{ converges in } L^p([0,T],\mathbb{R}^n) \text{ to } f(\cdot,x(\cdot)) \ . \tag{3.12}$$

On the other hand, from (3.7), (3.4), (3.3) and (3.9) we get

$$\begin{aligned} &\lim_{m\to+\infty} \rho((x_m(t),\dot{x}_m(t)-f_m(t)),GrF) \leq \\ &\leq \lim_{m\to+\infty} \|x_m(t)-x_m(\delta_m(t))\| = 0 \ , \quad \text{a.e. in } [0,T] \ . \end{aligned} \tag{3.13}$$

From $\alpha\alpha$), (3.9), (3.10), (3.12) and (3.13), it follows that the multifunction “coF” and the sequences $(x_m)_m$ and $(\dot{x}_m - f_m)_m$ satisfy the assumptions of the Convergence Theorem 1.4.1 of [2], and then

$$\dot{x}(t) \in coF(x(t)) + f(t,x(t)) \text{ a.e. in } [0,T] \ ;$$

hence, by using $\alpha\alpha\alpha$) there exists (cf. here, Remark 2) a proper, convex and lower semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ such that

$$\dot{x}(t) \in \partial V(x(t)) + f(t,x(t)) \ , \quad \text{a.e. in } [0,T] \ . \tag{3.14}$$

Now, we fix a closed interval $J = [c,d] \subset \ ]0,T[$; by using $\beta\beta\beta)_w$ it follows that

$$f_m, \ f(\cdot,x(\cdot)) \in L^2(J,\mathbb{R}^n) \ , \ \ \forall m \in \mathbb{N} \ . \tag{3.15}$$

Since (cf. here, (3.7), $\beta\beta\beta)_w$ and (3.6))

$$\|\dot{x}_m\|_{L^2(J)} \leq \left(\int_c^d (h(t)+M)^2 dt\right)^{1/2} \ , \ \forall m \in \mathbb{N} \ , \tag{3.16}$$

we have, from (3.10) and Theorem 2 of [8, p. 222], that

$$(\dot{x}_m)_m \text{ converges weakly in } L^2(J,\mathbb{R}^n) \text{ to } \text{“}\dot{x}\text{”} \ , \tag{3.17}$$

and so $x \in W^{1,2}(J, \mathbb{R}^n)$.

Therefore, by Lemma 3.3 of [5] (cf. here, (3.14) and (3.15)), it follows that the function $t \mapsto V(x(t))$ is absolutely continuous in J, and

$$\frac{d}{dt}V(x(t)) = \langle \dot{x}(t) - f(t, x(t)), \dot{x}(t) \rangle, \text{ a.e. in } J \ .$$

By integrating we obtain

$$V(x(d)) - V(x(c)) = \int_c^d \|\dot{x}(s)\|^2 ds - \int_c^d \langle f(s, x(s)), \dot{x}(s) \rangle ds \ . \quad (3.18)$$

On the other hand, by $\alpha\alpha\alpha$) and (3.4) we get

$$V(x_m((i+1)\frac{T}{m})) - V(x_m(i\frac{T}{m})) \geq \left\langle y_{m,i}, x_m((i+1)\frac{T}{m}) - x_m(i\frac{T}{m}) \right\rangle$$

$$\forall t \in I_{m,i}, \ \forall i \in \{0, 1, \ldots, m-1\} \text{ and } \forall m \in \mathbb{N} \ . \quad (3.19)$$

Since (cf. here, (3.7), (3.15) and (3.16))

$$\left\langle y_{m,i}, x_m((i+1)\frac{T}{m}) - x_m(i\frac{T}{m}) \right\rangle =$$

$$= \left\langle y_{m,i}, \int_{iT/m}^{(i+1)T/m} \dot{x}_m(s) ds \right\rangle =$$

$$= \int_{iT/m}^{(i+1)T/m} \langle \dot{x}_m(s) - f_m(s), \dot{x}_m(s) \rangle ds =$$

$$= \int_{iT/m}^{(i+1)T/m} \|\dot{x}_m(s)\|^2 ds - \int_{iT/m}^{(i+1)T/m} \langle f_m(s), \dot{x}_m(s) \rangle ds \ ,$$

$$\forall i \in \left\{ \gamma_m(c)\frac{m}{T}, \ldots, \delta_m(d)\frac{m}{T} - 1 \right\} \text{ and } \forall m \in \mathbb{N} \ ,$$

by adding in (3.19) with respect to i in $\{\gamma_m(c)\frac{m}{T}, \ldots, \delta_m(d)\frac{m}{T} - 1\}$ we obtain

$$V(x_m(\delta_m(d))) - V(x_m(\gamma_m(c))) \geq$$

$$\geq \int_{\gamma_m(c)}^{\delta_m(d)} \|\dot{x}_m(s)\|^2 ds - \int_{\gamma_m(c)}^{\delta_m(d)} \langle f_m(s), \dot{x}_m(s) \rangle ds, \forall m \in \mathbb{N} \ . \quad (3.20)$$

Now, by taking (3.9), (3.3) and $\alpha\alpha\alpha$) into account, from Proposition 2.12 of [5] we get

$$\lim_{m \to +\infty} V(x_m((\delta_m(d))) - V(x_m(\gamma_m(c))) = V(x(d)) - V(x(c)) \quad (3.21)$$

Moreover, the convergence of $(f_m)_m$ to $f(.,x(.))$ in $L^2(J,\mathbb{R}^n)$ (cf. here, $\beta\beta\beta)_w$ and (3.11)) and the convergence of $(\dot{x}_m)_m$ to "$\dot{x}$" in the weak topology of $L^2(J,\mathbb{R}^n)$ (cf. here (3.17)) imply

$$\lim_{m\to+\infty}\int_c^d \langle f_m(s),\dot{x}_m(s)\rangle ds = \int_c^d \langle f(s,x(s)),\dot{x}(s)\rangle ds\ . \qquad (3.22)$$

Then, by using (3.20), (3.22), (3.21) and (3.18) we obtain

$$\limsup_{m\to+\infty} \|\dot{x}_m\|_{L^2(J)} \le \|\dot{x}\|_{L^2(J)}\ ,$$

and so (cf. here, (3.17) and Proposition 3.30 of [6, p. 52])

$$(\dot{x}_m)_m \textit{ converges strongly in } L^2(J,\mathbb{R}^n) \text{ to "}\dot{x}\text{"}\ .$$

Hence (cf. [6, Theorem 4.9, p. 58]) there exists a subsequence of $(\dot{x}_m)_m$, still denoted $(\dot{x}_m)_m$, which converges pointwisely a.e. in J to "$\dot{x}$". Since "GrF" is closed (cf. here, α), $\alpha\alpha$) and Proposition 1.1.2 of [2, p. 41]) taking (3.13) and (3.11) into account, we have that

$$\dot{x}(t) \in F(x(t)) + f(t,x(t)) \text{ a.e. in } J\ ,$$

that is the function $x : [0,T] \to \mathbb{R}^n$ (cf. here, (3.9)) satisfies the differential inclusion of the Cauchy problem (1) almost everywhere in J.

From the arbitrary choice of J, it follows that

$$\forall s \in \mathbb{N}\ ,\exists \text{ a closed interval } J_s \subset\ ]0,T[, \mu([0,T]\backslash J_s) < \frac{1}{s}\ ,$$

such that $\dot{x}(t) \in F(x(t)) + f(t,x(t))$ a.e. in J_s.

Now putting $D = \bigcup_{s\in\mathbb{N}} J_s$, we obtain

$$\dot{x}(t) \in F(x(t)) + f(t,x(t)) \text{ a.e. in } D\ ,$$

and $\mu([0,T]\backslash D) = 0$. Therefore, since $x_m(0) = \xi$, $\forall m \in \mathbb{N}$, finally it follows that "x" is a solution of our Cauchy problem.

REMARK 3. We observe that our proposition contains the existence Theorem of F. Ancona and G. Colombo [1]. In fact, it is obvious that if $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is a function satisfying the condition iii) of [1] then it satisfies our assumptions $\beta)$, $\beta\beta)$, $\beta\beta\beta)_w$ (cf.

[8, Theorem 6, p. 101]). On the other hand, there exist functions "f" that satisfy the conditions β, $\beta\beta$), $\beta\beta\beta)_w$ that do not satisfy the hypothesis iii) of [1], as it is evident from the following

EXAMPLE 1. $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f(t,x) = \begin{cases} \frac{1}{\sqrt{t}} & (t,x) \in\ ]0,1] \times \mathbb{R} \\ 0 & (t,x) \in \{0\} \times \mathbb{R}\ . \end{cases}$$

4. In this section we consider the Cauchy problem

$$(1)' \quad \begin{cases} \dot{x} \in F(x) + G(t,x) \\ x(0) = \xi \in \mathbb{R}^n \end{cases}$$

where $G : [0,b] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a suitable multifunction.

First, we prove the following preliminary proposition that is a sufficient condition to get Caratheodory's selections for a given multifunction.

PROPOSITION. *Let $G : [0,b] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a multifunction with the properties:*

i) *$G(t,x)$ is non empty, closed and convex, $\forall (t,x) \in [0,b] \times \mathbb{R}^n$;*

ii) *$\forall x \in \mathbb{R}^n$, $t \mapsto G(t,x)$ is measurable;*

iii) *$\forall t \in [0,b]$, $x \mapsto G(t,x)$ is lower semicontinuous and it has closed graph;*

In these conditions, there exists a function $g : [0,b] \times \mathbb{R}^n \to \mathbb{R}^n$ such that

(ψ) $g(t,x) \in G(t,x)$, $\forall (t,x) \in [0,b] \times \mathbb{R}^n$;

($\psi\psi$) *$\forall x \in \mathbb{R}^n$, $t \mapsto g(t,x)$ is measurable;*

($\psi\psi\psi$) *$\forall t \in [0,b]$, $x \mapsto g(t,x)$ is continuous.*

We start by observing that (cf. [7, Theorem 4.1]) for every $m \in \mathbb{N}$ there exists a closed set $P_m \subset [0,b]$, $\mu([0,b]\backslash P_m) < \frac{1}{m}$, such that the multifunction $G_{P_m \times \mathbb{R}^n}$ is lower semicontinuous.

Since $P_m \times \mathbb{R}^n$ is a paracompact space (cf. [10, Theorem 4.3]), by using Theorem 3.2'' of [9], there exists a continuous function $g_m : P_m \times \mathbb{R}^n \to \mathbb{R}^n$ with the property

$$g_m(t,x) \in G(t,x), \quad \forall (t,x) \in P_m \times \mathbb{R}^n . \tag{4.1}$$

We put $P = [0,b]\backslash \bigcup_{m\in\mathbb{N}} P_m$. By using again Theorem 3.2'' of [9], it follows that
$\forall t \in P$ there exists a continuous function $\lambda_t : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\lambda_t(x) \in G(t,x), \forall x \in \mathbb{R}^n, \forall t \in P . \tag{4.2}$$

Then the function $g : [0,b] \times \mathbb{R}^n \to \mathbb{R}^n$, defined putting:

$$g(t,x) = \begin{cases} g_1(t,x), & t \in P_1 , \quad x \in \mathbb{R}^n , \\ g_2(t,x), & t \in P_2\backslash P_1 , \quad x \in \mathbb{R}^n , \\ \dots & \\ g_i(t,x), & t \in P_i\backslash \bigcup_{j=1}^{i-1} P_j , \quad x \in \mathbb{R}^n , \\ \dots & \\ \lambda_t(x), & t \in P , \quad x \in \mathbb{R}^n . \end{cases}$$

satisfies the conditions (ψ), $(\psi\psi)$, $(\psi\psi\psi)$ (cf. here, (4.1) and (4.2)).

REMARK 4. We observe that the above proposition extends a Caratheodory's selection Theorem due to G. Bonanno (cf. [3, Theorem 3.1]) in the sense that there exist multifunctions that satisfy the conditions of our proposition that do not satisfy the assumptions of the Theorem of G. Bonanno, as it is evident by taking the following example into account.

EXAMPLE 2. $G : [0,1]\times\mathbb{R}^2 \to 2^{\mathbb{R}^2}$ is the multifunction so defined:

$$G(t,x) = \begin{cases} \{y = (y_1,y_2) \in \mathbb{R}^2 : y_2 = \|x\|\}, & (t,x) \in (\mathbb{R}\backslash\mathbb{Q} \cap [0,1]) \times \mathbb{R}^2 \\ \{y = (y_1,y_2) \in \mathbb{R}^2 : y_2 = \|x\| + 1\}, & (t,x) \in (\mathbb{Q} \cap [0,1]) \times \mathbb{R}^2 . \end{cases}$$

Finally, by using the above Proposition and our existence Theorem, it is easy to deduce the following

COROLLARY. *Let* $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *be a multivalued operator satisfying the conditions* α*)* $\alpha\alpha$*),* $\alpha\alpha\alpha$*) and* $G : [0,b] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *be a multifunction with the properties*: i), ii), iii) *and moreover:*

iv) $\exists p \in]1,2[$ *and* $\exists h \in L^p([0,b],\mathbb{R}) \cap L^2_{loc}([0,b],\mathbb{R})$, *such that* $\|y\| \leq h(t)$, $\forall y \in G(t,x)$, *for* a.e. $t \in [0,b]$ *and for all* $x \in \mathbb{R}^n$.

In these conditions, there exist $T > 0$ *and an absolutely continuous function* $x : [0,T] \to \mathbb{R}^n$ *that is a solution of the Cauchy problem* $(1)'$.

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