ON SIMPLE PSEUDORADIAL TOPOLOGIES CONNECTED WITH INFINITE ADDITIVE MEASURES ON BOOLEAN RINGS (*)

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SOMMARIO. - *In questo lavoro mostriamo che se $\mathcal{A}$ è un anello booleano ed m è una misura additiva di $\mathcal{A}$ in un gruppo commutativo G, allora, in collegamento naturale con proprietà di infinita additività, si può costruire su G per mezzo di m una topologia pseudoradiale (chain-net) che conferisce al gruppo G una struttura di gruppo topologico. Poiché ogni anello booleano può essere immerso in maniera standard in un anello booleano di insiemi, per semplicità espositiva ci riferiremo ad anelli booleani di quest'ultimo tipo.*

SUMMARY. - *In this paper we show that if $\mathcal{A}$ is a boolean ring and m is an additive measure from $\mathcal{A}$ into a commutative group G, then a simple topology on G connected in a natural way with usual properties of infinite additivity can be constructed by means of m. This topology is pseudo-radial (chain-net) and give the group G a structure of topological group. By virtue of well known embedding theorems we deal, without loss of generality, with boolean rings of subsets of a set S.*

Let $\mathcal{A}$ be a ring of subsets of a given set S (1) and let G be an (additive) abelian group. A function m from $\mathcal{A}$ into G is said to be a finite additive measure if the following property holds:

(1) $\quad \forall X, Y \in \mathcal{A} : X \cap Y = \emptyset \Rightarrow m(X \cup Y) = m(X) + m(Y)$

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(1) That is $\mathcal{A}$ is a subset of the power set $\mathcal{P}(S)$ of S and for any $XY \in \mathcal{A}$, $X \cup Y \in \mathcal{A}$ and $X - Y \in \mathcal{A}$. Consequently also $X \cap Y \in \mathcal{A}$, since $X \cap Y = = X \cup Y - ((X - Y) \cup (Y - X))$.

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Now let δ be an infinite set and let us assume that for any family $(X_i)_{i\in\delta}$ of elements of $\mathcal{A}$, the following property holds:

$$\cup\{X_i | i \in \delta\} \in \mathcal{A} \tag{2}$$

If δ is the set $\mathbb{N}$ of the natural numbers and $(R, +, \mathcal{T})$ is the additive topological group of the real numbers (where $\mathcal{T}$ is the set of the usual open sets of R) then a finite additive measure m from $\mathcal{A}$ into the set of the non negative real numbers is said a (countably additive) measure if for every disjoint sequence $(A_i)_{i\in N}$ of elements of $\mathcal{A}$ we have (cfr. [4] p, 30) $m(\cup\{A_i | i \in \mathbb{N}\}) = \sum_{i\in\mathbb{N}} m(A_i)$.

It is obvious that this latter definition depends on the natural topology $\mathcal{T}$ of $\mathbb{R}$.

We shall give a definition of "infinite" additivity of an additive measure m from our ring $\mathcal{A}$ into a group G (without topology) and aftwerwards we shall define a natural topology on G in such a manner that it shall become a chain-net (see [9], p. 40) topological group.

Thus we say that an additive measure m is a (weak) δ-measure if for arbitrary disjoint families $(A_i)_{i\in\delta}$ and $(B_i)_{i\in\delta}$ of elements of $\mathcal{A}$ ($A_h \cap A_k = \emptyset = B_h \cap B_k$ for any $h, k \in \delta$ with $h \neq k$) we have (cf. [10], p. 73):

(3) $\forall i \in \delta : m(A_i) = m(B_i) \Rightarrow m(\cup\{A_i | i \in \delta\}) = m(\cup\{B_i | i \in \delta\})$;

(4) $\forall i \in \delta : m(A_i) = m(B_i) \Rightarrow m(\cup\{A_i | i \in \delta\}) = -m(\cup\{B_i | i \in \delta\})$.

REMARK 1. It is obvious that if m is a δ-measure, then (2), (3) and (4) hold even if we consider instead of δ a nonvoid subset ρ of δ, and hence also if we consider a set whose cardinality is less than or equal to the cardinality of δ. In fact every family $(X_i)_{i\in\rho}$ of elements of $\mathcal{A}$ can be extended by associating the void set $\emptyset$ to every $i \in \delta - \rho$.

In the following m will be always a δ-measure from a fixed boolean ring $\mathcal{A}$ of subsets of a set S into a commutative group G.

Now if $\leq$ is a linear order relation on δ and $(X_i)_{i\in\delta}$ is a net (2) of elements of $\mathcal{A}$ such that $X_h \subseteq X_k$ when $h \leq k$ (hence we shall

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(2) We recall that if an order relation on a set δ is given such that for all $x, y \in \delta$ there is $z \in \delta$ such that $x, y \leq z$ (in particular a linear order relation) then any family having δ as the set of the indexes is said to be a net (with respect to that relation).

say that $(X_i)_{i\in\delta}$ is an increasing net) it could seem natural to say that the net $(m(X_i))_{i\in\delta}$ converges to $m(\cup\{X_i|i \in \delta\})$. This will be allowed by Theorem 7. In the meantime let us define a simple topology on G by means of a natural convergence. Thus let ρ be a set whose cardinality is less than or equal to the cardinalty of δ and let $\leq$ be a fixed well order relation on ρ.

DEFINITION 2. We shall say that a net $(a_i)_{i\in\rho}$ of elements of G *converges in a natural way* (in symbols *nat-converges*) to $a \in G$ if there are $t \in G$ and an increasing net $(A_i)_{i\in\rho}$ of elements of $\mathcal{A}$ such that $t + m(A_i) = a_i$ for all $i \in \rho$ and $t + m(\cup\{A_i|i \in \rho\}) = a$, or $t + m(A_i) = -a_i$ for all $i \in \rho$ and $t + m(\cup\{A_i|i \in \rho\}) = -a$.

REMARK 3. Definition 2 is very natural when $t = 0$. In order to give G a structure of topological group (see Theorem 5), definition 2 is given in such a manner that all the nat-converging nets are of the type $(t + m(A_i))_{i\in\rho}$ (that nat-converges to $t + m(\cup\{A_i|i \in \rho\})$) or $(-t - m(A_i))_{i\in\rho}$ (that nat-converges to $-(t + m(\cup\{A_i|i \in \rho\}))$, where $t \in G$, ρ is a set whose cardinality is less than or equal to the cardinality of δ and $(A_i)_{i\in\rho}$ is an increasing net of elements of $\mathcal{A}$.

Consequently (since one can consider the net $(A_i)_{i\in\rho}$, with $A_i = \emptyset$ for all $i \in \rho$) if $a_i = t \in G$ for every $i \in \rho$, then $(a_i)_{i\in\rho}$ nat-converges to t, even if t do not belong to the range of m.

Moreover, if $(a_i)_{i\in\rho}$ is a net of elements of G nat-converging to $a \in G$, if λ is a cofinal subset of ρ (3) and $(b_i)_{i\in\lambda}$ is a net of elements of G (with respect to the well order relation induced by $\leq$ on λ) such that $b_i = a_i$ for every $i \in \lambda$, (4) then also $(b_i)_{i\in\lambda}$ nat-converges to $a \in G$.

Now let 0 be the minimum of ρ. Moreover for every increasing net $(X_i)_{i\in\rho}$ of elements of $\mathcal{A}$ let ${}^*X_0 = X_0$ and, for all $j \in \rho - \{0\}$, let ${}^*X_i = X_i - \cup\{X_j|j < i\}$. Thus ${}^*X_i \cap {}^*X_j = \emptyset$ for all $i, j \in \rho$ such that $i \neq j$. Furthermore, for every (initial) section (5) λ of ρ having more than one element we have:

(5) $\cup\{X_i|i \in \lambda\} = \cup\{{}^*X_i|i \in \lambda\} = X_0 \cup (\cup\{{}^*X_i|i \in \lambda - \{0\}\})$.

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(3) That is for every $x \in \rho$ there is $y \in \lambda$ such that $x \leq y$.

(4) In this case we shall say that $(b_i)_{i\in\lambda}$ is a cofinal subnet of $(a_i)_{i\in\rho}$.

(5) That is, for all $x, y \in \rho$ such that $y \in \lambda$ and $x < y$ one has $x \in \lambda$.

THEOREM 4. *Let $(a_i)_{i\in\rho}$ be a net of elements of G that nat-converges to a and to b. Then $a = b$.*

Proof. The assertion is obvious when ρ is finite. Thus let us assume that ρ is infinite and let $0'$ be the minimum of $\rho - \{0\}$.

If $(A_i)_{i\in\rho}$ and $(B_i)_{i\in\rho}$ are two increasing nets of elements of $\mathcal{A}$ and $t, s \in G$, then we can consider three different cases:

$$t + m(A_i) = a_i = s + m(B_i) \text{ for all } i \in \rho\,, \tag{6}$$

$$t + m(\cup\{A_i|i \in \rho\}) = a \text{ and } s + m(\cup\{B_i|i \in \rho\}) = b\,;$$

$$t + m(A_i) = -a_i = s + m(B_i) \text{ for all } i \in \rho\,, \tag{7}$$

$$t + m(\cup\{A_i|i \in \rho\}) = -a \text{ and } s + m(\cup\{B_i|i \in \rho\}) = -b\,;$$

$$t + m(A_i) = a_i = -s - m(B_i) \text{ for all } i \in \rho\,, \tag{8}$$

$$t + m(\cup\{A_i|i \in \rho\}) = -a \text{ and } s + m(\cup\{B_i|i \in \rho\}) = -b\,.$$

Case (6) For all $j \in \rho - \{0\}$ we have $m(^*A_i) = m(^*B_i)$. Indeed $a_{0'} - t = m(A_{0'}) = m(A_0) + m(^*A_{0'}) = a_0 - t + m(^*A_0)$ and $a_{0'} - s = m(B_{0'}) = m(B_0) + m(^*B_{0'}) = a_0 - s + m(^*B_{0'})$ thus $m(^*A_{0'}) = a_{0'} - a_0 = m(^*B_{0'})$. By Remark 1 referred to (3), the demonstration of the assertion can be easily completed by transfinite induction.

As a consequence by (5) (with ρ instead of λ) we have $a = t + m(\cup\{A_i|i \in \rho\}) = t + m(A_0) + m(\cup\{^*A_i|i \in \rho - \{0\}\}) = s + m(B_0) + m(\cup\{^*A_i|i \in \rho - \{0\}\}) = s + m(\cup\{B_i|i \in \rho\}) = b$.

Case (7) is analogous to case (6).

Case (8) For all $j \in \rho - \{0\}$ we have $-m(^*A_i) = m(^*B_i)$. Indeed $a_{0'} - t = m(A_{0'}) = m(A_0) + m(^*A_{0'}) = a_0 - t + m(^*A_{0'})$ and $a_{0'} + s = -m(B_{0'}) = -m(B_0) - m(^*B_{0'}) = a_0 + s - m(^*B_{0'})$, hence $-m(^*A_{0'}) = a_0 - a_{0'} = m(^*B_{0'})$. By Remark 1 referred to (4), the demonstration of the assertion can be easily completed by transfinite induction. Hence one can end the proof, by means of (5), as in case (6). ◇

As ρ varies on the class of the sets whose cardinality is less than or equal to the cardinality of δ and $\leq$ varies on the well order relations of ρ, one has a "convergence" $\mathcal{C}$ [(6)] on G (cfr. [10]). As a consequence one can associate to $\mathcal{C}$ the greatest topology $\mathcal{T}_\delta$ in which all the nets that determine $\mathcal{C}$ converge.

We recall that an open set H of the topology associated to a "convergence" $\mathcal{C}$ (a $\mathcal{C}$-open set) on a set G is a subset of G such that every net that define $\mathcal{C}$ (a $\mathcal{C}$-net) $\mathcal{C}$-converging to an element of H is eventually in H. Consequently if a $\mathcal{C}$-net $\mathcal{C}$-converges to a, then it converges to a under the associated topology.

Since in our case the nets of $\mathcal{C}$ are determined by linearly ordered sets, then the topology $\mathcal{T}_\delta$ is a chain-net topology. Moreover, we have the following

THEOREM 5. *The group G is a topological group that is T_1-separated. That is for every $a \in G$ the set $\{a\}$ is closed under $\mathcal{T}_\delta$.*

Proof. It is easy to verify that if $0 \in H \in \mathcal{T}_\delta$, then $0 \in -H \in \mathcal{T}_\delta$. Furthermore for all $t \in G$ and $H \subseteq G$, $H \in \mathcal{T}_\delta$ if and only if $t+H \in \mathcal{T}_\delta$ (see the first part of Remark 3). Thus G is a topological group.

Moreover it is T_1-separated otherwise there are $a \in G$, a well ordered set $(\rho, \leq)$ (with the cardinality of ρ less than or equal to the cardinality of δ) and a $\mathcal{C}$-net $(a_i)_{i\in\rho}$ that nat-converges to an element b of $G-\{a\}$ and is frequently in $\{a\}$. Consequently, there is a cofinal subnet of $(a_i)_{i\in\rho}$ whose range is $\{a\}$ and converges to a and b (see the second part of Remark 3). This is absurd by Theorem 4. ◇

In order to give some meaningful examples, let $(\mathbb{R}, +, \mathcal{T})$ be the additive topological group of the real numbers $\mathbb{R}$ where $\mathcal{T}$ is the usual topology), let $S := \mathbb{R}$ and let δ be the set $\mathbb{N}$ of the natural numbers (0 excluded).

As a first example we can set $\mathcal{A} := \mathcal{P}(\mathbb{R})$ and define m by setting $m(H) = 1 \in \mathbb{R}$ if $1 \in H$ and $m(H) = 0 \in \mathbb{R}$ if $1 \notin H$ (i.e. m is the usual two-valued measure associated to the principal real ultrafilter generated by $\{1\}$). It is easy to verify that a sequence $(a_i)_{i\in\mathbb{N}}$ of real numbers nat-converges if and only if it is constant or there are $n \in \mathbb{N}-\{1\}$ and $t \in \mathbb{R}$ such that $a_i = t$ for all $i \geq n$ and $a_i = t-1$ for all $i < n$. Hence the associated topology is the indiscrete topology on $\mathbb{R}$.

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(6) This means that $\mathcal{C}$ is a function from the class $\mathcal{N}$ of nets of G into the power set of G. If $x \in \mathcal{N}$ and $a \in \mathcal{C}(x)$ one says that x $\mathcal{C}$-converges to a.

More generally, it is easy to verify that every (countably additive) measure m on a ring of subsets of $\mathbb{R}$ determines a topology $\mathcal{T}_{\mathbb{N}}$ such that $\mathcal{T} \subseteq \mathcal{T}_{\mathbb{N}}$. Indeed it is obvious that if a sequence of real numbers $(a_i)_{i\in\mathbb{N}}$ nat-converges to $a \in \mathbb{R}$ with respect to m, then it converges to a also with respect to the natural topology $\mathcal{T}$ on $\mathbb{R}$.

Another example is given by setting $\mathcal{A}$ equal to the set of the subsets of the real open interval (0,1) that are measurable with respect to the measure m of Lebesgue. In this case $\mathcal{T} = \mathcal{T}_{\mathbb{N}}$. In fact we observed that $\mathcal{T} \subseteq \mathcal{T}_{\mathbb{N}}$. Moreover $\mathcal{T}_{\mathbb{N}} \subseteq \mathcal{T}$. In fact if H is a neighborhood of 1/2 with respect to $\mathcal{T}_{\mathbb{N}}$, then it is a neighborhood of 1/2 with respect to $\mathcal{T}$ (otherwise it is easy to determine a suitable sequence $(a_i)_{i\in\mathbb{N}}$ nat-converging to 1/2 with respect to m, such that $a_i \notin H$ for all $i \in \mathbb{N}$). Thus Theorem 5 ensures that $\mathcal{T}_{\mathbb{N}} \subseteq \mathcal{T}$.

REMARK 6. Let $\leq$ be a linear order relation on a set τ and let $\mathcal{W}$ be the set of the subsets λ of τ that are well ordered with respect to the relation induced by the given $\leq$. Then $\mathcal{W}$ is an inductive ordered set under the relation defined by "$\lambda_1 \leq \lambda_2$ if and only if λ_1 is an (initial) section of λ_2". Thus by Zorn's lemma, $\mathcal{W}$ has a maximal element ρ, and hence ρ is a cofinal subset of τ. As a consequence, if $(A_i)_{i\in\tau}$ is an increasing net of elements of $\mathcal{A}$ and $(A_i)_{i\in\rho}$ is the restriction of $(A_i)_{i\in\tau}$ to ρ, then $\cup\{A_i|i \in \rho\} = \cup\{A_i|i \in \tau\}$ and hence $m(\cup\{A_i|i \in \rho\}) = m(\{A_i|i \in \tau\})$.

We can conclude with the following

THEOREM 7. *Let $\leq$ be a given linear order relation on the non-void set γ whose cardinality is less than or equal to the cardinality of δ. Let $(A_i)_{i\in\gamma}$ be an increasing net of elements of $\mathcal{A}$ and let $a = m(\cup\{A_i|i \in \gamma\})$. Then the net $(m(A_i))_{i\in\gamma}$ converges to a under $\mathcal{T}_\delta$. Moreover if $(m(A_i))_{i\in\gamma}$ converges to b under $\mathcal{T}_\delta$, then $a = b$.*

Proof. Let us assume that $(A_i)_{i\in\gamma}$ do not converge to a. Then there is an open neighbourhood H of a such that for every $i \in \gamma$ there is $j \in \gamma$ for which $j \geq i$ and $m(A_i) \notin H$. Consequently, if τ is the set of the elements $j \in r$ such that $m(A_j) \notin H$, then $\cup\{A_i|i \in \tau\} = \cup\{A_i|i \in \gamma\}$ and r is a cofinal subset of γ. Now let ρ be a well ordered and cofinal (under $\leq$) subset of τ. Thus, by Remark 6, $a = m(\cup\{A_i|i \in \rho\})$ and hence $(m(A_i))_{i\in\rho}$ nat-converges to a. Thus $(m(A_i))_{i\in\rho}$ converges to a under $\mathcal{T}_\delta$. This is absurd, since $m(A_i) \notin H$ for all $i \in \rho$.

It is easy to prove that if $(m(A_i))_{i\in\gamma}$ converges to b under $\mathcal{T}_\delta$, then $a = b$. In fact by Remark 6 we can consider a well ordered and cofinal (under $\leq$) subset ρ of γ and the restriction $(m(A_i))_{i\in\rho}$ of $(m(A_i))_{i\in\gamma}$ to ρ. Thus, since $m(\cup\{A_i|i \in \rho\}) = m(\cup\{A_i|i \in \gamma\})$, $(m(A_i))_{i\in\rho}$ nat-converges to a and b, hence $a = b$ by theorem 4. $\diamond$

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