MEASURES, OUTER MEASURES AND NORMAL LATTICES (*)

by ROBERT W. SCHUTZ (in New York)(**)

SOMMARIO. - *In questo lavoro si definiscono le funzioni insiemistiche μ', μ'', μ_i, μ^* e $\mu^\wedge$ collegate alle misure $\mu \in M(\mathbb{L})$, dove $M(\mathbb{L})$ indica lo spazio delle misure limitate, finitamente additive, non negative, non banali definite su $A(\mathbb{L})$, l'algebra generata dal reticolo $\mathbb{L}$ dei sottoinsiemi dello spazio X. Queste funzioni insiemistiche sono misure interne o esterne o hanno proprietà simili a tali misure. Si generalizzano risultati precedenti che trattavano il caso di misure a due valori. Si studiano tali funzioni insiemistiche in particolare sotto l'ipotesi di normalità del reticolo sottostante $\mathbb{L}$, e si ottengono condizioni che valgono per le funzioni studiate.*

SUMMARY. - *In this paper we define set functions μ', μ'', μ_i, μ^* and $\mu^\wedge$ related to measures $\mu \in M(\mathbb{L})$, where $M(\mathbb{L})$ stands for the space of bounded finitely additive non-negative non-trivial measures defined on $A(\mathbb{L})$ the algebra generated by the lattice $\mathbb{L}$ of subsets of space X. These set functions are inner or outer measures or have properties similar to such functions. We generalize previous work that dealt with two valued measures. We look at such set functions, in particular under the hypothesis of normality of the underlying lattice $\mathbb{L}$, and obtain conditions that will hold among such functions.*

1. Introduction.

Normal lattices of sets have been investigated by a number of authors, (see e.g.[1,4,5]). In these papers equivalent characterizations of normal lattices have been given in terms of two valued (0 or 1) measures on the algebra $A(\mathbb{L})$ generated by the lattice $\mathbb{L}$. In this paper, we show how a large number of these results extend to the more general case of bounded finitely additive measures on $A(\mathbb{L})$. We

---

(*) Pervenuto in Redazione il 29 ottobre 1992.

(**) Indirizzo dell'Autore: P.O. Box 1149, West Babylon, New York 11704 (U.S.A.).

also investigate certain "outer" measures associated with a measure on $A(\mathbb{L})$, and study their behaviour in the case where $\mathbb{L}$ is normal; this also generalizes the work of others (see [3]) to the case of more general measures.

Our notation and terminology is consistent with that of the previous referred to papers. We give a brief review of just those concepts which are relevant to this paper in section 2 for the reader's convenience. Related material can be found in [2,3,6,7].

2. Background and Notation.

We begin by reviewing some notation and terminology which is fairly standard. We supply background material for the reader's convenience.

Let X be an abstract set, and $\mathbb{L}$ a lattice of subsets of X containing X and $\emptyset$. The complement of a set E of X will be denoted by E'. A delta lattice is one that is closed under countable intersections, and the delta lattice generated by $\mathbb{L}$ is denoted by $\delta\mathbb{L}$. $\mathbb{L}$ is countably paracompact (cp) if for every sequence $L_n \varepsilon \mathbb{L}$ and $L_n \downarrow \emptyset$ there exists $A_n \sim' \downarrow \emptyset$ st $A_n \sim' \supseteq L_n$ and $A_n \sim \varepsilon \mathbb{L}$. By a lattice being complement generated (cg) means that if $L \varepsilon \mathbb{L}$ there exists $L_n \varepsilon \mathbb{L} st L = \cap L_{n'} n = 1, \ldots,$. Note that cg implies cp since if $A_n \downarrow \emptyset$, $A_n \varepsilon \mathbb{L}$ then $A_n = \cap L_{N_K} n = 1, \ldots$. Call $B_{N'} = \cap L_{N_K} n = 1, \ldots, N$, and $L_K \sim' = \cap B_{K'}$, $K = 1, \ldots, N$, $L_K \sim' \supseteq A_K$ $L_K \sim \varepsilon \mathbb{L}$ and $L_K \sim' \downarrow \emptyset$. $A(\mathbb{L})$ will denote the algebra generated by $\mathbb{L}$ and $\sigma(\mathbb{L})$ the smallest σ-algebra generated by $\mathbb{L}$.

Let $M(\mathbb{L})$ denote the set of non-trivial bounded non-negative real valued finitely additive measures on $A(\mathbb{L})$, and let $M(\sigma^*, \mathbb{L})$ denote those elements of $M(\mathbb{L})$ that are sigma-smooth on $(\mathbb{L})$, i.e. if $L_n \downarrow \emptyset L_n \varepsilon \mathbb{L}, \mu \varepsilon M(\sigma^*, \mathbb{L})$ then $\lim \mu(L_n) = 0$ as n goes to infinity. Let $M(\sigma, \mathbb{L})$ denote those elements of $M(\mathbb{L})$ that are sigma-smooth on $A(\mathbb{L})$, i.e. if $L_n \downarrow \emptyset$ $L_n \varepsilon A(\mathbb{L})$, $\mu \varepsilon M(\sigma, \mathbb{L}$ then $\lim \mu(L_n) = 0$ as n goes to infinity. This is equivalent to μ being σ-smooth on $A(\mathbb{L})$. $MR(\mathbb{L})$ will stand for those measures on $A(\mathbb{L})$ that are $\mathbb{L}$- regular on $A(\mathbb{L})$, i.e. if $\mu \varepsilon MR(\mathbb{L})$ then $\mu(A) = \text{Sup } \mu(L)$ where $A \varepsilon A(\mathbb{L})$, $L \varepsilon \mathbb{L}$ and $A \supseteq L$. This is equivalent to μ being $\mathbb{L}$-regular on $\mathbb{L}'$. $MR(\sigma^*, \mathbb{L})$ will denote those measures that are sigma-smooth and $\mathbb{L}$-regular on $A(\mathbb{L})$. $I(\mathbb{L})$ will refer to those measures of $M(\mathbb{L})$ that are two valued $\{0,1\}$ on $A(\mathbb{L})$. $I(\sigma^*\mathbb{L})$ will denote those two valued measures that are sigma-smooth on $\mathbb{L}$. $I\mathbb{R}(\mathbb{L})$ will denote those two valued measures that are $\mathbb{L}$-regular on $A(\mathbb{L})$. $I\mathbb{R}(\sigma^*,\mathbb{L})$ will denote those two valued measures that are sigma-smooth and $\mathbb{L}$-regular on $A(\mathbb{L})$.

A lattice is said to be normal if for $L_1, L_2\varepsilon\mathbb{L}$ and $L_1 \cap L_2 = \emptyset$ there exists L_3, $L_4\varepsilon\mathbb{L}$ st $L_{3'} \supseteq L_1 L_4' \supseteq L_2$ and $L_{3'} \cap L_{4'} = \emptyset$. A lattice is said to be countably compact (cc) if $\mu\varepsilon I(\mathbb{L})$ implies that $\mu\varepsilon I(\sigma^*,\mathbb{L})$. A lattice is almost countably compact (acc) if $\mu\varepsilon I\mathbb{R}(\mathbb{L}')$ implies that $\mu\varepsilon I(\sigma^*,\mathbb{L})$. We now prove a result needed in the sequel.

THEOREM 2.1. *$\mathbb{L}$ almost countably compact and $\mu\varepsilon MR(\mathbb{L}')$ implies $\mu\varepsilon M(\sigma^*,\mathbb{L})$.*

Proof. Look at $FB(\mu)|\mathbb{L}' = \{L'|L\varepsilon\mathbb{L}, \mu(L') = \mu(X)\}$. Then $FB(\mu)|\mathbb{L}'$ is a filter base and thus is contained in an ultrafilter $UF(\mu)|\mathbb{L}'$.

Associated with $UF(\mu)|\mathbb{L}'$ is a measure $\mu_1\varepsilon I\mathbb{R}(\mathbb{L}') st \mu_1(L') = 1$ if $L'\varepsilon UF(\mu)|\mathbb{L}'$. In particular $\mu_1(L') = 1$ if $L'\varepsilon FB(\mu)|\mathbb{L}'$. Assume that μ is not in $M(\sigma^*,\mathbb{L})$, then there exists $L_n\varepsilon\mathbb{L}L_n \downarrow \emptyset$ and lim $\mu(L_n) > \varepsilon > 0$ for some $\varepsilon > 0$. Further since $\mathbb{L}$ is acc $\mu_1\varepsilon I(\sigma^*,\mathbb{L})$ and lim $\mu_1(L_n') = 1$ or $\mu_1(L_n) = 0$ for $n > N$. Since μ is $\mathbb{L}'$ regular there exists $L_n \sim' \varepsilon\mathbb{L}' st L_n \supseteq L_n \sim'$ and $\mu(L_n) - \mu(L_n \sim') < \varepsilon/2$ thus $\mu(L_n \sim') > \varepsilon/2$ for $n > N$. But this implies that $L_n \sim' \cap L' \neq \emptyset$ for $L'\varepsilon FB(\mu)|\mathbb{L}'$ and thus $L_n \sim' \varepsilon UF(\mu)|\mathbb{L}'$.

Therefore $1 = \mu_1(L_n \sim') \leq \mu_1(L_n) = 0$ for $n > N$. A contradiction $\mu\varepsilon M(\sigma^*,\mathbb{L})$.

Note: For $\mu, \mu_1\varepsilon M(\mathbb{L})$ we write $\mu \leq \mu_1(\mathbb{L})$ if $\mu(L) \leq \mu_1(L)$ for all $L\varepsilon\mathbb{L}$ and $\mu(X) = \mu_1(X)$.

A fact used repeatedly throughout this paper is this: If $\mu\varepsilon M(\mathbb{L})$ then there exists a $\nu\varepsilon MR(\mathbb{L})$ st $\mu \leq \nu(\mathbb{L})$ [see, 6]. If $\mu\varepsilon I(\mathbb{L})$ then there exists a $\nu\varepsilon I\mathbb{R}(\mathbb{L})$ st $\mu \leq \nu(\mathbb{L})$.

Normality of a lattice is equivalent to the following measure theoretic condition: If $\mu\varepsilon I(\mathbb{L})$, $\mu_1, \mu_2\varepsilon I\mathbb{R}(\mathbb{L})$, $\mu \leq \mu_1(\mathbb{L})$ and $\mu \leq \mu_2(\mathbb{L})$ then $\mu_1 = \mu_2$. We then have the following theorem which holds for normal lattices.

THEOREM 2.2. *If $\mathbb{L}$ is a normal lattice, $\mu\varepsilon M(\mathbb{L})$, $\nu_1, \nu_2\varepsilon MR(\mathbb{L})$,*

$\nu \leq \nu_1(\mathbb{L})$ and $\nu \leq \nu_2(\mathbb{L})$ then $\nu_1 = \nu_2(\mathbb{L})$.
Assume that $\nu_1 \neq \nu_2$, then $\exists L\varepsilon\mathcal{L}$ st $\nu_1(L) > \nu_2(L)$ or $\nu_2(L') > \nu_1(L')$. Since ν_1, ν_2 are regular $\exists L_2\varepsilon\mathbb{L}$ for $\varepsilon > 0$ st $L' \supseteq L^{\sim}$ and $\nu_1(L') - \nu_1(L^{\sim}) < \varepsilon$ $\nu_2(L') - \nu_2(L^{\sim}) < \varepsilon$. Since $L \cap L^{\sim} = \emptyset$ and $\mathcal{L}$ is normal $\exists L_1, L_2 \in \mathbb{L}$ st $L_1' \supset L$ and $L_2' \supset L^{\sim}$ and $L_1' \cap L_2' = \emptyset$ or $L_1 \cup L_2 = X$. We have then that $L' \supseteq L_1 \supseteq L_2' \supseteq L^{\sim}$ or $L^{\sim'} \supseteq L_2 \supseteq L_1' \supseteq L$. Since ν_2 is regular $\nu_2(L) + \varepsilon > \nu_2(L^{\sim'}) > \nu_2(L_2)$ thus $\mu(X) = \mu(L_1) + \mu(L_2) - \mu(L_1 \cap L_2) \leq \nu_1(L') + \nu_2(L) + \varepsilon - \mu(L_1 \cap L_2)$.
Letting $\varepsilon \downarrow 0$ we have that

$$\mu(X) \leq \nu_1(L') + \nu_2(L)$$

But $\nu_1(L') + \nu_2(L) < \nu_1(L') + \nu_1(L) = \nu_1(X) = \mu(X)$ a contradiction.

We next introduce some set functions that are inner or outer measures or that have some of the properties of such set functions.

Let $\mu\varepsilon M(\sigma, \mathbb{L})$, then define for E, $X \supseteq E_1 \mu^*(E) = \inf \Sigma\mu(A_i')$ where $i = 1, 2 \ldots$, $\cup A_i' \supseteq E_1 A_i \varepsilon A(\mathbb{L})$. Then μ^* is an outer measure whose restriction to $A(\mathbb{L})$ agrees with μ.

Let $\mu\varepsilon M(\sigma^*, \mathbb{L})$, then define for E, $X \supseteq E$, $\mu''(E) = \inf \Sigma\mu(L_{i'})$ where $i = 1, 2, \ldots, \cup L_{i'} \supseteq E$ $L_i\varepsilon\mathbb{L}$. Then μ'' is an outer measure.

Let $\mu\varepsilon M(\mathbb{L})$, then define for E, $X \supseteq E$ $\mu'(E) = \inf \mu(L')_1$ $L' \supseteq E$ $L\varepsilon\mathbb{L}$. Then μ' is a finitely subadditive "outer measure".

Let $\mu\varepsilon M(\mathbb{L})$ and $X \supseteq E$, then define $\mu_i(E) = \sup \mu(L)_1 E \supseteq L$ $L\varepsilon\mathbb{L}$. Then μ_i has the properties of a finitely superadditive inner measure.

For $\mu\varepsilon M(\mathbb{L})_1$ $L\varepsilon\mathbb{L}$ if $\mu(L') = \sup \mu'(L \sim) L' \supseteq L \sim \varepsilon\mathbb{L}$ then μ is said to be weakly regular and the set of such measure is denoted by $MW(\mathbb{L})$.

3. Measure Theoretic Consequences of Normality.

In this section we investigate measure theoretic properties of normal lattices.

THEOREM 3.1. *$\mathbb{L}$ normal $\mu\varepsilon M(\mathbb{L})$ $\nu\varepsilon MR(\mathbb{L})$ $\mu \leq \nu(\mathbb{L})$ then $\nu(L') = \sup \mu(L \sim) L' \supseteq L \sim L$, $L \sim \varepsilon\mathbb{L}$.*

Proof. We have the following $\nu(L') > \nu(L\sim) \geq \mu(L\sim)$ for any $L' \supseteq L\sim$ and $L, L\sim \varepsilon \mathbb{L}$. Since $\nu\varepsilon MR(\mathbb{L})$ for $\varepsilon > 0$ chooses $L\sim \varepsilon\mathbb{L}$, $L' \supseteq L\sim$ st $\nu(L') - \nu(L\sim) < \varepsilon$. $L \cap L\sim = \emptyset$ and since $\mathbb{L}$ is normal there exists $L_1, L_2 \varepsilon \mathbb{L}$ st $L_{1'} \supseteq L\sim L_{2'} \supseteq L\sim$ and $L_{1'} \cap L_{2'} = \emptyset$. But this implies that $L' \supseteq L_i \supseteq L_2 \supseteq L\sim$ and $\nu(L') \geq \nu(L_1) \geq \mu(L_1) \geq \nu(L_2) \geq \nu(L\sim)$. Therefore since $\nu(L') - v(L\sim) < \varepsilon$, $L' \supseteq L_1$ and since ε is arbitrary, $\nu(L') = \sup \mu(L\sim) L' \supseteq L\sim$, $L\sim \varepsilon\mathbb{L}$.

THEOREM 3.2. *If* $\mathbb{L}$ *is normal,* $\mu_1 \leq \mu_2(\mathbb{L})$, $\mu_1 \varepsilon M(\sigma^*, \mathbb{L})$ *and* $\mu_2 \varepsilon MR(\mathbb{L})$ *then* $\mu_2 \varepsilon M(\sigma^*, \mathbb{L}')$.

Proof. Let $A_n \varepsilon \mathbb{L}$ st $A_{n'} \downarrow \emptyset$ but $\lim \mu_2(A_{n'}) \neq 0$. Then since μ_2 is regular for $\varepsilon > 0$ there exists $B_n \varepsilon \mathbb{L}$ st $\mu_2(A_{n'}) - \mu_2(B_n) < \varepsilon/2$ $A_{n'} \supseteq B_n$ and without the loss of generality we can assume that $B_n \downarrow \emptyset$. Then $A_n \cap B_n = \emptyset$ and since $\mathbb{L}$ is normal, there exists C_n, $D_n \varepsilon \mathbb{L}$ st $C_{n'} \supseteq B_n$, $D_{n'} \supseteq A_n$ and $C_{n'} \cap D_{n'} = \emptyset$. Then $C_n \supseteq D_{n'} \supseteq A_n$ or $A_{n'} \supseteq D_n \supseteq C_{n'}$. Since $A_{n'} \downarrow \emptyset$ can assume without loss of generality that $D_n \downarrow \emptyset$, $C_{n'} \downarrow \emptyset$, $\mu_1(D_n) \geq \mu_1(C_{n'}) \geq \mu_2(C_{n'})$ and since $\mu_1 \varepsilon M(\sigma^*, \mathbb{L})$ as n goes to infinity $\lim \mu_1(D_n) = 0$. Therefore $\lim \mu_1(C_{n'}) = \lim \mu_2(C_{n'}) = 0$, and since $C_{n'} \supseteq Bn$ $\lim \mu_2(B_n) = 0$. Thus for $n > N$ $\mu_2(B_n) < \varepsilon/2$, which implies that $\mu_2(A_{n'}) < \varepsilon$, and letting $\varepsilon \downarrow 0$ $\lim \mu_2(A_{n'}) = 0$ and $\mu_2 \varepsilon M(\sigma^*, \mathbb{L}')$.

COROLLARY 3.1. *If* $\mathbb{L}$ *is normal and countably paracompact,* $\mu_1 \varepsilon M(\sigma^*, \mathbb{L})$, $\mu_2 \varepsilon MR(\mathbb{L})$ *and* $\mu_1 \leq \mu_2(\mathbb{L})$ *then* $\mu_2 \varepsilon MR(\sigma^*, \mathbb{L})$.

Proof. From theorem 3.2 $\mu_2 \varepsilon M(\sigma^*, \mathbb{L}')$ but since $\mathbb{L}$ is cp $M(\sigma^*, \mathbb{L}) \supseteq M(\sigma^*, \mathbb{L}')$. Therefore $\mu_2 \varepsilon MR(\sigma^*, \mathbb{L})$.

COROLLARY 3.2. *If* $\mathbb{L}$ *is normal and countably paracompact,* $\mu_1 \leq \mu_2(\mathbb{L})$, $\mu_1 \varepsilon M(\sigma^*, \mathbb{L})$,$\mu_2 \varepsilon M(\mathbb{L})$ *then* $\mu_2 \varepsilon M(\sigma^*, \mathbb{L})$.

Proof. Let $\mu_3 \varepsilon MR(\mathbb{L})$ be such that $\mu_1 \leq \mu_2 \leq \mu_3(\mathbb{L})$ then $\mu_1 \leq \mu_3(\mathbb{L})$ and by corollary 3.1 $\mu_3 \varepsilon MR(\sigma^*, \mathbb{L})$, therefore $\mu_2 \varepsilon M(\sigma^*, \mathbb{L})$.

THEOREM 3.3. *Suppose for each* $\mu \varepsilon M(\sigma^*, \mathbb{L})$ *there exists a unique* $\nu \varepsilon MR(\mathbb{L})$ *st* $\mu \leq \nu(\mathbb{L})$. *Then this property together with almost countably compact of* $(\mathbb{L})$ *implies the normality of* $\mathbb{L}$.

Proof. Let $\mu\varepsilon M(\mathbb{L})$ then $\mu\varepsilon M(\mathbb{L}')$ and there exists $\nu\varepsilon MR(\mathbb{L}')$ st $\mu \leq \nu(\mathbb{L}')$. Then $\nu \leq \mu(\mathbb{L})$ and thus there exists a $\nu_1\varepsilon MR(\mathbb{L})$ st $\nu \leq \mu \leq \nu_1(\mathbb{L})$. Now since $\mathbb{L}$ is almost countably compact, by theorem 2.1 $\nu\varepsilon M(\sigma^*, \mathbb{L})$, and $\nu \leq \nu_1(\mathbb{L})$. By the hypothesis of the theorem ν_1 is the unique element of $MR(\mathcal{L})$ such that this holds. Therefore it is thus also the unique $\nu_1\varepsilon MR(\mathbb{L})$ st $\mu \leq \nu_1(\mathbb{L})$. Then for any $\mu\varepsilon M(\mathbb{L})$, there exists a unique $\nu_1\varepsilon MR(\mathbb{L})$ st $\mu \leq \nu_1(\mathbb{L})$. But this is equivalent measure theoretically to $\mathbb{L}$ being normal by theorem 2.2.

Note you can reduce the first assumption in theorem 3.3 to simply: For $\mu\varepsilon I(\sigma^*, \mathbb{L}')$ there exists a unique $\nu\varepsilon I\mathbb{R}(\mathbb{L})$ st $\mu \leq \nu(\mathbb{L})$. In addition if $\mathbb{L}$ is countably paracompact or complement generated (which implies countably paracompact see section 2), and if also $\mathbb{L}$ is normal and almost countably compact, then $\mathbb{L}$ is countably compact. First $\mathbb{L}$ being almost countably compact implies that $M(\sigma^*, \mathbb{L}) \supseteq MR(\mathbb{L}')$. Thus for any $\mu\varepsilon M(\mathbb{L})$, $\mu\varepsilon M(\mathbb{L}')$ there exists a $\nu\varepsilon MR(\mathbb{L}')$ st $\mu \leq \nu(\mathbb{L}')$ or $\nu \leq \mu(\mathbb{L})$ $\nu\varepsilon M(\sigma^*, \mathbb{L})$. Since $\mathbb{L}$ is normal and countably paracompact by corollary 3.1 there exists a unique $\nu_1\varepsilon MR(\sigma^*, \mathbb{L})$ st $\nu \leq \nu_1$ $\mathbb{L})$ and thus a unique $\nu_1\varepsilon MR(\sigma^*, \mathbb{L})$ st $\mu \leq \nu_1(\mathbb{L})$. Therefore $\mu\varepsilon M(\sigma^*, \mathbb{L})$ for all $\mu\varepsilon M(\mathbb{L})$. In particular for $\mu\varepsilon I(\mathbb{L})$ this implies that $\mu\varepsilon I(\sigma^*, \mathbb{L})$ and thus $\mathbb{L}$ is cc.

4. Consequences of normality and set functions μ', μ'', μ_i and $\mu^\wedge$.

In this section we discuss consequences of a lattice being normal in terms of the set functions μ', μ'', μ_i and $\mu^\wedge$ (to be defined later) where $\mu\varepsilon M(\mathbb{L})$ or $\mu\varepsilon M(\sigma^*, \mathbb{L})$ as the case may be.

THEOREM 4.1. *If $\mathbb{L}$ is normal, $\mu\varepsilon M(\mathbb{L})$, $\nu\varepsilon MR(\mathbb{L})$, $\mu \leq \nu(\mathbb{L})$ then $\mu \leq \nu = \nu' = \mu'(\mathbb{L})$.*

Proof. From theorem 3.1 we know that $\nu(L') = \sup \mu(L \sim)$, $L, L \sim \varepsilon \mathbb{L}$ and $L' \supseteq L \sim$. But this implies that $\nu(L) = \inf \mu(L \sim')$ $= \mu'(L) L$, $L \sim \varepsilon\mathbb{L}$, $L \sim' \supseteq L$. Also $\nu = \nu'$ since $\nu\varepsilon MR(\mathbb{L})$. Therefore $\mu \leq \nu = \nu' = \mu'(\mathbb{L})$.

THEOREM 4.2. *If* $\mathbb{L}$ *is normal and* $\mu\varepsilon MW(\mathbb{L})$ *then* $\mu\varepsilon MR(\mathbb{L})$.

Proof. Let $\mu\varepsilon MW(\mathbb{L})$, then there exists a $\nu\varepsilon MR(\mathbb{L})$ st $\mu \leq \nu(\mathbb{L})$. From theorem 4.1 $\nu = \mu'(\mathbb{L})$. Since $\mu\varepsilon MW(\mathbb{L})$ then $\mu(L') = \sup \mu'(L \sim) = \sup \nu(L \sim) = \nu(L')$, L, $L \sim \varepsilon\mathbb{L}$, $L' \supseteq L \sim$ since ν is regular. Therefore $\mu = \nu$ and $\mu\varepsilon MR(\mathbb{L})$.

THEOREM 4.3. *Let* $\mathbb{L}$ *be normal and countably paracompact. Let* $\mu\varepsilon M(\sigma^*, \mathbb{L})$ *then* $\mu' = \mu''(\mathbb{L})$.

Proof. Since $\mathbb{L}$ is normal and countably paracompact from corollary 3.1 there exists a $\nu\varepsilon MR(\sigma^*, \mathbb{L})$ st $\mu \leq \nu(\mathbb{L})$. Then $\mu \leq \nu \leq \nu'' = \nu' \leq \mu'' \leq \mu'(\mathbb{L})$ and by theorem 4.1 $\nu' = \mu'(\mathbb{L})$ and $\nu' = \nu''(\mathbb{L})$ since $\nu\varepsilon MR(\sigma^*, \mathbb{L})$ thus $\mu'' = \mu'$.

THEOREM 4.4. *Let* $\mathbb{L}$ *be a normal lattice of subsets of a set* X. *Let* $X \supseteq E_1$, E_2 *st there exists* L_1, $L_2\varepsilon\mathbb{L}$ $L_1 \supseteq E_1$ $L_2 \supseteq E_2$ *and* $L_1 \cap L_2 = \emptyset$ *then for* $\mu\varepsilon M(\mathbb{L})$ $\mu'(E_1 \cup E_2) = \mu'(E_1) + \mu'(E_2) + \mu'(E_2)$ *and for* $\mu\varepsilon M(\sigma^*, \mathbb{L})$ $\mu''(E_1 \cup E_2) = \mu''(E_1) + \mu''(E_2)$ *holds.*

Proof. First μ' is finitely additive outer measure, thus $\mu'(E_1 \cup E_2) \leq \mu'(E_1) + \mu'(E_2)$ holds. Since $\mathbb{L}$ is normal and $L_1 \cap L_2 = \emptyset$ there exists $L_1 \sim, L_2 \sim \varepsilon\mathbb{L}$ st $L_1 \sim' \supseteq L_1$, $L_2 \sim' \supseteq L_2$, $L_1 \sim' \cap L_2 \sim' = \emptyset$, $L_1 \sim' \supseteq L_1 \supseteq E_1$, $L_2 \sim' \supseteq L_2 \supseteq E_2$. Then $\mu'(L_1 \sim' \cup L_2 \sim') = \mu'(L_1 \sim') + \mu'(L_2 \sim') \geq \mu(E_1) + \mu(E_2)$. Now if $L' \supseteq E_1 \cup E_2$ then $L' \cap L_1 \sim' \supseteq E_1$, $L' \cap L_2 \sim' \supseteq E_2$ and $L' \supseteq (L' \cap L_1 \sim') \cup (L' \cap L_2 \sim')$ and $(L' \cap L_1 \sim') \cap (L' \cap L_2 \sim') = \emptyset$. Therefore we can restrict coverings of $E_1 \cup E_2$ to ones where $L_1 \sim' \supseteq E_1$, $L_2 \sim' \supseteq E_2$, and $L_1 \sim' \cap L_2 \sim' = \emptyset$. Therefore $\mu'(E_1 \cap E_2) = \mu'(E_1) + \mu'(E_2)$. Similarly if $\cup A_{n'} \supseteq E_1 \cup E_2$, $\mu\varepsilon M(\sigma^*, \mathbb{L})$, $A_n\varepsilon\mathbb{L}$, $n = 1, 2, \ldots, \infty$ then $\cup(A'_n \cap L_1 \sim') \supseteq E_1$, $\cup(A'_n \cap L_2 \sim') \supseteq E_2$,$((\cup(A'_n \cap L_1 \sim')) \cap (\cup(A_{n'} \cap L_2 \sim'))) = \emptyset$ and $\cup A_{n'} \supset ((\cup(A'_n \cap L_1 \sim')) \cup (\cup(A'_n \cap L_2 \sim')))$ $n = 1, 2, \ldots, \infty$. Thus $\Sigma\mu(A'_n) \geq \Sigma\mu(A_{n'} \cap L_1 \sim') + \Sigma\mu(A_{n'} \cap L_2 \sim')$. Therefore $\mu''(E_1 \cap E_2) = \mu''(E_1) + \mu''(E_2)$ holds

DEFINITION 4.1. Denote by $S\mu''$ the class of all subsets of a set X that are μ'' measurable, i.e. $E\varepsilon S\mu''$ if $\mu''(F) = \mu''(F \cap E) + \mu''(F \cap E')$ for all st $X \supseteq F$.

THEOREM 4.5. *Let X be a set and $\mathbb{L}$ a lattice of subsets of X that is normal, and $\mu\varepsilon I(\sigma^*, \mathbb{L})$. Also let $A\varepsilon\mathbb{L}$, $A = \cap B'_k B_k \varepsilon \mathbb{L}$, $k = 1, 2, \ldots, \infty$, then $A\varepsilon S\mu''$.*

Proof. We have that $A' = \cup B_{k'}$. Define $C_n = \cup B_{m'} = 1, 2, \ldots, n$, $A' \supseteq C_n$. Then $A \cap C_n = \emptyset$ and since $\mathbb{L}$ is normal there exists D_n, $E_n \varepsilon \mathbb{L}$ st $D_{n'} \supseteq A$, $E_{n'} \supseteq C_n$ and $D_{n'} \cap E_n = \emptyset$. Further we can arrange so that $D_{n'} \downarrow$ and $E_{n'} \uparrow$. Then $\cup D_{n'} \supseteq A$, $\cup E_{n'} \supseteq A'$ or $A' \supseteq \cap D_n$ and $A \supseteq \cap E_n n = 1, 2, \ldots, \infty D_n \uparrow$ and $E_n \downarrow$. If $\mu(E_n) \neq 1$ for all n then $\mu(E_n) = 0$ for $n \geq M$. Therefore $\mu(D_{n'}) = 0$ for $n \geq M$ or $\mu(D_n) = 1$ for $n \geq M$ and $A' \supseteq D_n$. Then $A' \supseteq \cap D_k$ for $k = M, M+1, \ldots, \infty$ thus $A\varepsilon S\mu''$ from a result that for $\mu\varepsilon I(\sigma^*, \mathbb{L}) S\mu'' = \{E | E \supseteq \cap L_n,\ n = 1, 2, \ldots, \infty, \mu(L_n) = 1$ all n or $E' \supseteq \cap L_n$, $n = 1, 2, \ldots, \infty$, $\mu(L_n) = 1$ all $n\}$.

We next introduce a new set function $\mu^\wedge$ and point out relations to previously introduced set functions such as μ_i, μ^* and μ''. We next investigate $\mu^\wedge$ in detail in case $\mathbb{L}$ is normal. We then prove that $\mu^\wedge$ is a finitely additive set function if the lattice $\mathbb{L}$ is normal. We also show that the restriction to $A(\mathbb{L})$ is a regular measure.

DEFINITION 4.2. Let $\mu\varepsilon M(L)$ then define for $X \supseteq E$ $\mu^\wedge(E) = \inf \mu_i(L')$, $L' \supseteq E$. $L\varepsilon\mathbb{L}$.

THEOREM 4.6. *For $\mu\varepsilon M(L)$ $\mu = \mu_i \leq \mu^\wedge \leq \mu'(\mathbb{L})$ and $\mu^\wedge = \mu_i \leq \mu = \mu'(\mathbb{L}')$.*

Proof. The proofs follow immediately from the definitions and will be omitted.

We now prove a property of μ_i.

THEOREM 4.7. *μ_i is finitely additive on $\mathbb{L}'$.*

Proof. Let $A', B'\varepsilon\mathbb{L}$, $A' \cap B' = \emptyset$ then from the definition of μ_i and for $\varepsilon > 0$ there exists $L_1, L_2 \varepsilon \mathbb{L}$ st $A' \supseteq L_1$, $B_1 \supseteq L_2$ and $\mu(L_1) + \mu(L_2) + \varepsilon \geq \mu_i(A') + \mu_i(B')$ or $\mu_i(A' \cup B') + \varepsilon \geq \mu_i(A') + \mu_i(B')$ and letting $\varepsilon \downarrow 0$ $\mu_i(A' \cup B') \geq \mu_i(A') + \mu_i(B')$. Conversely look at $\mu_i(A' \cup B')$. Then there exists a $L\varepsilon\mathbb{L}$ for $\varepsilon > 0$ st $\mu(L) + \varepsilon \geq \mu_i(A' \cup B')$ and $A' \cup B' \supseteq L$. Then since $A' \cap B' = \emptyset$, $A' \supseteq L \cap B$, $B' \supseteq L \cap A$ and

$((L\cap B)\cup(L\cap A)) = L$. Thus $\mu(L)+\varepsilon = \mu(A\cap L)+\mu(B\cap L)+\varepsilon \geq \mu_i(A'\cup B')$. Letting $\varepsilon \downarrow 0$ $\mu_i(A'\cup B') \leq \mu_i(A') + \mu_i(B')$ and μ_i is finitely additive on $\mathbb{L}'$.

THEOREM 4.8. *If $\mathbb{L}$ is normal, then $\mu^\wedge$ is a finitely subadditive outer measure for $\mu\varepsilon M(\mathbb{L})$.*

Proof. a) Let $F \supseteq E$ then $\mu^\wedge(E) \leq \mu^\wedge(F)$ if $F \supseteq E$. Since if $L_{1'} \supseteq E$, $L_{2'} \supseteq F$, $L_{2'}$ not containing $L_{1'}$, $L_{1'} - L_{2'} \neq \emptyset$, then $L_{1'}\cap L_{2'} \supseteq E$, $L_{1'}\cap L_{2'}\varepsilon\mathbb{L}'$ $L_{2'} \supseteq L_{1'}\cap L_{2'}$ and $L_{1'} \supseteq L_{1'}\cap L_{2'}$). Thus taking infs over all $L'\varepsilon\mathbb{L}'$, $\mu^\wedge(E) \leq \mu^\wedge(F)$ if $F \supseteq E$.

b) $\mu^\wedge(\emptyset) = 0$ $\mu^\wedge(X) = \mu(X)$ obviously.

c) Look at $\mu^\wedge(E\cup F)$, $X \supseteq E, F$. One need only look at $\mathbb{L}'$, A', $B' \in L$ coverings of $E\cup F$, where $A' \supseteq E$, $B' \supseteq F$, since if $L' \supseteq E\cup F$, $L'\varepsilon\mathbb{L}'$ then for any coverings of E and $F A' \supseteq E$, $B' \supseteq F$, $C' = (L'\cap A')\cup(L'\cap B') \supseteq E\cup F$, $C\varepsilon\mathbb{L}'$, $L' \supseteq C'$, $L'\cap B' \supseteq F$ and $L'\cap A' \supseteq E$. Next for any $L\varepsilon\mathbb{L}$ st $C' \supseteq L$ if we can prove that there exists $L_1, L_2\varepsilon\mathbb{L}$ st $A' \supseteq L'\cap A' \supseteq L_1$, $B' \supseteq L'\cap B' \supseteq L_2$ and $L = L_1\cup L_2$ then for $\varepsilon > 0$ there exists a $L\varepsilon\mathbb{L}$ st $A'\cup B' \supseteq E\cup F \supseteq L$. Thus we have that $\mu_i(E\cup F) \leq \mu(L)+\varepsilon \leq \mu(L_1)+\mu(L_2)+\varepsilon \leq \mu^\wedge(E)+\mu^\wedge(F)+\varepsilon$ and letting $\varepsilon \downarrow 0$ $\mu^\wedge(E\cup F) \leq \mu^\wedge(E)+\mu^\wedge(F)$ and $\mu^\wedge$ is a finite outer measure.

We now prove the lemma.

LEMMA 4.1. *If $A'\cup B' \supseteq L$ for $L, A, B\varepsilon\mathbb{L}$* and if the lattice $\mathbb{L}$ is normal then there exists $L_1, L_2\varepsilon\mathbb{L}$ st $L = L_1\cup L_2$.

Proof. Since $L' \supseteq A\cap B$, $(L\cap A)\cap(L\cap B) = \emptyset$ or $L\cap B = \emptyset$.If $L\cap B = \emptyset$ or $L\cap A = d$, then take $L_1 = L$, $L_2 = \emptyset B' \supseteq L_1, A' \supseteq L_2$ and $L \neq L_1\cup L_2$. Otherwise $L\cap A = \emptyset$ and $L\cap B \neq \emptyset$ and $(L\cap A)\cap(L\cap B) = \emptyset$. Then since $\mathbb{L}$ is normal there exists $L_1, L_2\varepsilon\mathbb{L}$ st $L_1' \supseteq L\cap A$, $L_{2'} \supseteq L\cap B$ or $L'\cup A' \supseteq L_1$, $L'\cup B' \supseteq L_2$, $L_{1'}\cap L_{2'} = \emptyset$ or $L_1\cup L_2 = X$, then $A' \supseteq A'\cap L \supseteq L\cap L_1$, $B' \supseteq B'\cap L \supseteq L\cap L_2$ and $((L\cap L_2)\cup(L\cap L_1)) = L$.

We first make a definition of $S\mu^\wedge$ measurable sets and then look at a criterion for a set to be $\mu^\wedge$ measurable. in the remaining theorems we will assume that $\mathbb{L}$ is normal.

DEFINITION 4.2. Denote by $S\mu^\wedge$ the class of all subsets of a set X

that are $\mu^\wedge$ measurable, i.e. $E\varepsilon S\mu''$ if $\mu^\wedge(F) = \mu^\wedge(F\cap E)+\mu^\wedge(F\cap E')$ for all sets st $X \supseteq F$.

THEOREM 4.9. *$E\varepsilon S\mu^\wedge$iff$\mu^\wedge(A'\cap E') + \mu^\wedge(A'\cap E) \le \mu^\wedge(A')$ for $A'\varepsilon\mathbb{L}'$.*

Proof. If $E\varepsilon S\mu^\wedge$ then for $X \supseteq F$ $\mu^\wedge(F) \ge \mu^\wedge(F\cap E)+\mu^\wedge(F\cap E')$ holds. Taking $F = A'\varepsilon\mathbb{L}'$ $\mu^\wedge(A') \ge \mu^\wedge(E'\cap A') + \mu^\wedge(E\cap A')$ holds for all $A'\varepsilon\mathbb{L}'$.

Conversely let $L' \supseteq F$ then $\mu_i(L') = \mu^\wedge(L') \ge \mu^\wedge(L'\cap E) + \mu^\wedge(L'\cap E') \ge \mu^\wedge(F\cap E)+\mu^\wedge(F\cap E')$ for any $L' \supseteq F$ $L'\varepsilon\mathbb{L}'$. Therefore $\mu^\wedge(F) \ge \mu^\wedge(F\cap E) + \mu^\wedge(E'\cap F)$.

THEOREM 4.10. *$S\mu^\wedge \supseteq A(\mathbb{L})$.*

Proof. We will prove that $S\mu^\wedge \supseteq \mathbb{L}'$ and since $S\mu^\wedge$ is an algebra $S\mu^\wedge \supseteq A(\mathbb{L})$. Must now show $\mu^\wedge(A') \ge \mu^\wedge(A'\cap E)+\mu^\wedge(A'\cap E')$ for A' , $E'\varepsilon\mathbb{L}'$ by theorem 4.9. Let $A'\cap E' \supseteq D$ and $A'\cap D' \supseteq F$ for D, $F\varepsilon\mathbb{L}$, then $A'\cap E'$, $A'\cap D'\varepsilon\mathbb{L}'$. Now $D\cap F = \emptyset$ clearly and $A' \supseteq D\cup F$ therefore $\mu^\wedge(A') = \mu_i(A') \ge \mu(F\cup D) = \mu(D) + \mu^\wedge(D'\cap E) \ge \mu(D) + \mu_i(A'\cap D') = \mu(D) + \mu^\wedge(A'\cap D') \ge \mu^\wedge(D'\cap E) + \mu(D)$ taking sups over all possible F st $A'\cap D' \supseteq F$ $F\varepsilon\mathbb{L}$ and the last inequality is due to the fact that $D' \supseteq A\cup E$, therefore $D' \supseteq E$ and the monotonicity of $\mu^\wedge$. Taking sups over all possible D st $A'\cap E' \supseteq D$ we have that $\mu^\wedge(A') \ge \mu^\wedge(A'\cap E) + \mu^\wedge(A\cap E')$ and we are done.

We now prove that $\mu^\wedge$ restricted to $A(\mathbb{L})$ is a regular measure.

THEOREM 4.11. *$\nu = \mu^\wedge|A(\mathbb{L})\varepsilon MR(\mathbb{L})$ and $\mu^\wedge(X) = \nu(X)$.*

Proof. To prove that $\nu\varepsilon MR(\mathbb{L})$ it suffices to prove that ν is $\mathbb{L}$ regular on $\mathbb{L}'$ for $L'\varepsilon\mathbb{L}'$. Then $\nu(L') = \mu^\wedge(L') = \mu_i(L') = \sup\mu(L_1)L' \supseteq L_1L_1\varepsilon\mathbb{L}'$. But this implies that ν is $\mathbb{L}$ on result of $\mathbb{L}'$, therefore $\nu = \mu|A(\mathbb{L})\varepsilon MR(\mathbb{L})$. Also $\nu(X) = \mu^\wedge(X) = \inf\ \mu_i(L') = \mu_i(X) = \mathrm{Sup}\mu(L\sim) = \mu(X)$ where the inf is over all $L'\varepsilon\mathbb{L}'$ and $L' \supseteq X$ and the sup is over all $L\sim\varepsilon\mathbb{L}$ st $X \supseteq L\sim$. Thus $\mu^\wedge(X) = \nu(X)$.

THEOREM 4.12. *$\mu^\wedge = \mu'(\mathbb{L})$ for $\mu\varepsilon M(\mathbb{L})$.*

Proof. From theorem 4.11 $\nu = \mu|MR(\mathbb{L})$ and $\nu = \mu^{\wedge} \geq \mu$. We know from previous work that if $\mu \leq \nu$ $\nu\varepsilon MR(\mathbb{L})$ then since $\mathbb{L}$ is normal from theorem 4.1 then $\nu = \mu'(\mathbb{L})$, therefore $\mu' = \mu^{\wedge}$ on $\mathbb{L}$.

REMARK. Let $\mu\varepsilon M(\mathbb{L})$. Define $\mu_j(E) = \text{Sup } \mu'(L) E \supseteq L$, $L\varepsilon\mathbb{L}$. Then immediately from their definitions $\mu_i \leq \mu_j \leq \mu^{\wedge} \leq \mu'$ everywhere. In addition, also from their definitions, $\mu = \mu_j(\mathbb{L}')$ iff $\mu\varepsilon MW(\mathbb{L})$ and $\mu = \mu'(\mathbb{L})$ iff $\mu\varepsilon MR(\mathbb{L})$. Then by theorem 4.2, if $\mathbb{L}$ is normal and $\mu\varepsilon MW(\mathbb{L})$, then $\mu = \mu_j = \mu'\varepsilon MR(\mathbb{L})$.

REFERENCES

[1] ALEXANDROFF A.D., *Additive set functions in abstract set spaces*, Mat. Sb. (N.S.) **9** (51) (1941), 563–628.

[2] BACHMAN G. and SULTAN A., *On regular extensions of measures*, Pacific J. Math. **86** (2) (1980), 389–395.

[3] CAMACHO J., *Extensions of lattice regular measures with applications*, J. of the Indian Math. Soc. **54** (1989), 233–244.

[4] EID G., *On normal lattices and Wallman spaces*, Internat. J. Math. and Math. Sci. **13** (1) (1990) 31–38.

[5] FROLIK Z., *Prime filters with the c.i.p.*, Comment. Math. Univ. Carolina **13** (3) (1972), 553–573.

[6] SCHUTZ R.W., *On normal lattices and separation and semiseparation of lattices*, Internat. J. Math. and Math. Sci. **15** (1) (1992) 83–90.

[7] SZETO M., *On maximal measures with respect to a lattice*, Measure Theory and its Applications, Proceedings of the 1980 Conference, Northern Illinois University (1980).