DOUBLY MULTIPLIABLE MATRICES(*)

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SOMMARIO. - *Si studiano i sottospazi nei quali le matrici doppiamente moltiplicabili (A, B) si scompongono secondo i loro divisori elementari, e l'effetto di A e B come applicazioni tra essi. Nasce il concetto di coppia tra matrici doppiamente moltiplicabili e, sulla base di questo, si studia quando due matrici P e Q possono scomporsi nella forma AB e BA, rispettivamente, con una costruzione esplicita delle matrici A e B. Infine, si studia l'equivalenza tra coppie di matrici doppiamente moltiplicabili.*

SUMMARY. - *We study the subspaces into which the doubly multipliable matrices (A, B) decompose according to their elementary divisors, and the effect of A and B as applications between them. There appears the concept of coupling between doubly multipliable matrices and on the basis of this we study when two matrices P and Q can decompose into the form AB and BA respectively, with an explicit construction of the matrices A and B. Finally, we study the equivalence between doubly multipliable pairs of matrices.*

Introduction.

Let A be a matrix $m \times n$ and B of the order $n \times m$ in $\mathbb{C}$. We can consider A as a morphism of $\mathbb{C}^n$ in $\mathbb{C}^m$ and B of $\mathbb{C}^m$ in $\mathbb{C}^n$. It is clear that the products AB and BA, which are endomorphisms of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, have meaning. The article by Harley Flanders [2] contains a description of the relation between these products for two arbitrary matrices A and B. In [9] we can find another demonstration of the theorem of Flanders, together with a necessary and sufficient condition for two matrices P and Q to be simultaneously decomposed as a product of two matrices A and B. [1] makes a study of the behaviour of the matrices AB and BA on $\mathbb{C}^m$ and $\mathbb{C}^n$.

(*) Pervenuto in Redazione il 22 ottobre 1992.

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Our aim is to further explore the results of [1], studying the effect of the matrices A and B as morphisms between $\mathbb{C}^n$ qand $\mathbb{C}^m$, finding the subspaces in which these spaces decompose, and the effective construction of matrices A and B for pairs of matrices P and Q, which meet the necessary conditions.

Some of the results which can already be found in the references quoted have been repeated following the line of work presented here, as we are interested in emphasizing from the outset the role of the matrices A and B.

THEOREM 1. *The elementary divisors $(\lambda - \lambda_0)^p$ with $\lambda_0 \neq 0$ are the same in AB as in BA. Moreover, A and B are isomorphisms between the subspaces generated by these divisors. (The first part can also be seen in* [1], [2] *and* [9]*).*

Proof. Let $x \neq 0$ be a vector with minimum polynomial $(\lambda-\lambda_0)^p$. We have: $0 = B(AB - \lambda_0)^p x = (BA - \lambda_0)^p Bx \Rightarrow (\lambda - \lambda_0)^p$ is a polynomial which cancels the vector Bx (with respect to BA).

Furthermore, if $(\lambda - \lambda_0)^{p-1}$ cancelled Bx, a parallel reasoning would indicate that $(\lambda - \lambda_0)^{p-1}$ cancels ABx, and therefore

$$\begin{aligned} 0 &= (AB-\lambda_0)^p x = (AB-\lambda_0)^{p-1}(AB-\lambda_0)x \\ &= (AB-\lambda_0)^{p-1}ABx - (AB-\lambda_0)^{p-1}\lambda_0 x \\ &= -\lambda_0(AB-\lambda_0)^{p-1}x \end{aligned}$$

and, as $\lambda_0 \neq 0$, we are left with $(AB - \lambda_0)^{p-1}x = 0$!!!

Thus, the minimum polynomial of Bx is $(\lambda - \lambda_0)^p$.

We construct the subspaces

$I_1 = \{x | \exists k \in \mathbb{N} \text{ con } (AB - \lambda_0)^k x = 0\} \subset \mathbb{C}^m$

$I_1' = \{x' | \exists \in \mathbb{N} \text{ con } (BA - \lambda_0)^k x' = 0\} \subset \mathbb{C}^n$

We have just seen that $B(I_1) \subset I_1'$. Repeating the process symmetrically, $A(I_1') \subset I_1$.

Let us now see that B is injective. For this, we take a basis of the subspace I_1, $\langle x_1, x_2, \ldots x_r\rangle = I_1$ and the vectors $y_i = Bx_i$ for $i = 1, \ldots, r$; it is a question of checking that the set of vectors $\{y_i\}$ is linearly independent. Indeed:

If $\Sigma\alpha_i y_i = 0$, then $B(\Sigma\alpha_i x_i) = 0$. Let us take $z = \Sigma\alpha_i x_i$ and let us suppose $z \neq 0$. It is obvious that $z \in I_1$, and therefore there exists $k \in \mathbb{N}$ such that $(AB - \lambda_0)^k z = 0$ and $(AB - \lambda_0)^{k-1} z \neq 0$. As we have seen above, this means that $(BA - \lambda_0)^k Bz = 0$ and $(BA - \lambda_0)^{k-1} Bz \neq 0$, which is impossible, since $Bz = 0$. Therefore, $z = 0 \Rightarrow \alpha_i = 0\ \forall i \Rightarrow \{y_i\}$ is a linearly independent set, as we wished to see. Thus, B is injective.

Because B is injective, dim $I_i \leq \dim I_1'$.

Of course, A is also injective: dim $I_1' \leq \dim I_1$. The conclusion is clear: A and B are isomorphisms between I_1 and I_1'.

Given another eigenvalue $\lambda_1 \neq 0$, we find that A and B are isomorphisms between the respective subspaces I_2 and I_2'. But as $I_1 \cap I_2 = I_1' \cap I_2' = \{0\}$, A and B are isomorphisms between $I_1 \oplus I_2$ and $I_1' \oplus I_2'$.

Repeating the procedure, it will result that A and B are isomorphisms between two subspaces I and I' with minimum polynomial

$$(\lambda - \lambda_0)^{p_0}(\lambda - \lambda_1)^{p_1} \dots (\lambda - \lambda_h)^{p_h}$$

the minimum common multiple of all the elementary divisors with a non-nil root.

We will thus have $\mathbb{C}^m = I \oplus F$, $\mathbb{C}^n = I' \oplus F'$ with $I \simeq I'$, F and F' being the subspaces corresponding to the vectors with minimum polynomial λ^k.

By restriction of A and B to the subspaces I and I' ($A : I' \to I$, $B : I \to I'$), we construct the automorphisms $AB : I \to I$ and $BA : I' \to I'$ which are similar: $AB = B^{-1}(BA)B$, and which therefore have the same elementary divisors. ◇

PROPOSITION 1. If $E_k \subset \mathbb{C}^m$ is a cyclical subspace with minimum polynomial λ^k, then $B(E_k)$ is cyclical with minimum polynomial λ^k or λ^{k-1}.

(The proof may also be found in [1]. We present it here because it is performed with a different approach in the line of this work).

Proof. Let e, $(AB)e, \dots, (AB)^{k-1}e$ be a basis of E_k. Its image by B generates $B(E_k)$: $\langle Be, (BA)Be, \dots, (BA)^{k-1}Be\rangle = B(E_k)$. The image of these vectors by A is $(AB)e$, $(AB)^2e, \dots, (AB)^{k-1}e$, $(AB)^k e = 0$. It is thus deduced that at least the vectors Be, $(BA)Be, \dots, (BA)^{k-2}Be$ are independent.

As for the last vector, $(BA)^{k-1}Be$, two things may happen:

1. $(BA)^{k-1}Be = 0$

In this case $B(E_k)$ is cyclical, generated by Be and with minimum polynomial λ^{k-1}.

2. $(BA)^{k-1}Be \neq 0$

Let us see, by reductio ad absurdum, that it cannot be a linear combination of the above.

If it were $(BA)^{k-1}Be = \alpha_1 Be + \alpha_2(BA)Be + \ldots + \alpha_{k-1}(BA)^{k-2}Be$, we would be left with:

$$B[\alpha_1 e + \alpha_2(AB)e + \ldots + \alpha_{k-1}(AB)^{k-2}e - (AB)^{k-1}e] = 0$$

Applying A to this equality,

$$\alpha_1(AB)e + \alpha_2(AB)^2 e + \ldots + \alpha_{k-1}(AB)^{k-1}e - 0 = 0 \Rightarrow$$
$$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_{k-1} = 0$$

remembering the linear independence of these vectors.
Thus, $(BA)^{k-1}Be = 0$!!!

We thus have that $Be, (BA)Be, \ldots, (BA)^{k-1}Be$ form a basis of $B(E_k)$. Moreover, $(BA)^k Be = B(AB)^k e = 0$, which shows that the minimum polynomial of $B(E_k)$ is λ^k. $\diamond$

LEMMA 1. *If $E_k \subset \mathbb{C}^m$ is a cyclical subspace of annihilating polynomial λ^k and E' is an AB-invariant subspace such that $E_k \cap E' = \{0\}$ and $B(E_k) \cap B(E') \neq \{0\}$, then the annihilating polynomial of $B(E_k)$ is λ^k.*

Proof. Let $x \in B(E_k) \cap B(E')$, $x \neq 0$. As $x \in B(E_k)$, there exists a vector $y \in E_k$ such that $x = By$.

Let $e, (AB)e, \ldots, (AB)^{k-1}e$ be a basis of E_k. We have that

$$y = \sum_{i=0}^{k-1} \alpha_i (AB)^i e$$

and therefore,

$$x = \sum_{i=0}^{k-1} \alpha_i (BA)^i Be \Rightarrow Ax = \sum_{i=0}^{k-1} \alpha_i (AB)^{i+1} e = \sum_{i=0}^{k-2} \alpha_i (AB)^{i+1} e$$

since $(AB)^k e = 0$. Furthermore, $x \in B(E') \Rightarrow Ax \in AB(E') \subset E'$, ad E' is AB-invariant.

We are left with $Ax \in E_k \cap E' \Rightarrow Ax = 0 \Rightarrow \alpha_i = 0$ for $i = 0, \ldots, k-2 \Rightarrow x = \alpha_{k-1}(BA)^{k-1}Be$. We know that $x \neq 0$, which gives us $(BA)^{k-1}Be \neq 0$. We are in the second case of the above proposition. $\diamondsuit$

PROPOSITION 2. Let E_0 be the subspace of the vectors annulled by some power of AB. We can obtain a decomposition of E_0 in a direct sum of irreducible subspaces $E_0 = \oplus E_i$, such that $B(E_0) = \oplus B(E_i)$.

Proof. Let $E_0 = E_1 \oplus E_2 \oplus \ldots \oplus E_p$, the E_i being the subspaces given by the elementary divisors λ^k in decreasing order of degrees. Let us take as seen that $B(E_1 \oplus \ldots \oplus E_i) = B(E_1) \oplus \ldots \oplus B(E_i)$ for $i \geq 1$. Let λ^k be the minimum polynomial of the subspace E_{i+1}. Let us suppose that $B(E_{i+1})$ does not form a direct sum with $B(E_1 \oplus \ldots \oplus E_i)$. There then exists $x \in B(E_{i+1}) \cap B(E_1 \oplus \ldots \oplus E_i)$, $x \neq 0$. Taking $E' = E_1 \oplus \ldots \oplus E_i$, and e the generating vector of E_{i+1}, we can to take, in Lemma 1, $x = (BA)^{k-1}Be$.

Furthermore, $x \in B(E') \Rightarrow x = By$ with $y = y_1 + \ldots + y_i$, where each $y_j \in E_j$.

It is obvious that $Ax = (AB)^k e = 0 \Rightarrow ABy = 0 \Rightarrow ABy_1 + \ldots + ABy_i = 0$, being $ABy_j \in E_j$ (as they are invariants) $\Rightarrow ABy_j = 0$.

Let λ^{k_j} and e_j be the minimum polynomial and the generating vector of the subspace E_j. As $y_j \in E_j$, it is clear that

$$y_j = \sum_{h=0}^{k_j-1} \alpha_h (AB)^h e_j$$

From $ABy_j = 0$ it is deduced that

$$\sum_{s=0}^{k_j-2} \alpha_s (AB)^{s+1} e_j = 0 \Rightarrow \alpha_0 = \ldots = \alpha_{k_j-2} = 0 \Rightarrow y_j =$$

$$\alpha_{k_j-1}(AB)^{k_j-1}e_j$$

Due to the ordering taken in the subspaces E_r, we know that $k_j \geq k$. We will put

$y_j = \alpha_{k_j-1}(AB)^{k-1}(AB)^{k_j-k}e_j$

$\bar{y}_j = \alpha_{k_j-1}(AB)^{k_j-k}e_j (\in E_j)$

With this, $y = y_1 + \ldots + y_i = (AB)^{k-1}(\bar{y}_1 + \ldots + \bar{y}_i)$. We define $\bar{y} = \bar{y}_1 + \ldots + \bar{y}_i \in E_1 \oplus \ldots \oplus E_i$ and $\bar{e} = e - \bar{y}$. We also take $\bar{E}_{i+1} = \langle \bar{e}, (AB)\bar{e}, \ldots, (AB)^{k-1}\bar{e} \rangle$.

It is clear that $E_1 \oplus \ldots \oplus E_i \oplus E_{i+1} = E_1 \oplus \ldots \oplus E_i \oplus \bar{E}_{i+1}$, since the vectors $\bar{e}, (AB)\bar{e}, \ldots, (AB)^{k-1}\bar{e}$ are still independent (if $\Sigma\alpha_s(AB)^s\bar{e} = 0, \Rightarrow \Sigma\alpha_s\ (AB)^s e - \Sigma\alpha_s(AB)^s\bar{y} = 0$, the first addend being a vector of E_{i+1} and the second one of $E_1 \oplus \ldots \oplus E_i$; this means that each of them must be nil, and therefore $\alpha_s = 0\ \forall s$).

Thus $B(\bar{E}_{i+1}) = \langle B\bar{e}, (BA)B\bar{e}, \ldots, (BA)^{k-1}B\bar{e} \rangle$.

But $(BA)^{k-1}B\bar{e} = (BA)^{k-1}Be - (BA)^{k-1}B\bar{y} = x - B(AB)^{k-1}\bar{y} = x - By = x - x = 0$.

With this, the minimum polynomial of $B(\bar{E}_{i+1})$ is λ^{k-1}. According to Lemma 1, this tells us that $B(\bar{E}_{i+1}) \cap B(E_1 \oplus \ldots \oplus E_i) = \{0\}$, and therefore,

$$B(E_1 \oplus \ldots \oplus E_i \oplus \bar{E}_{i+1}) = B(E_1) \oplus \ldots \oplus B(E_i) \oplus B(\bar{E}_{i+1})$$

Repeating the process, we will obtain the stated decomposition. ◊

Consequences of these results.

With all these results, we can take

$$\mathbb{C}^m \simeq E \oplus E_0 = E \oplus E_1 \oplus \ldots \oplus E_p \text{ and } \mathbb{C}^n \simeq E' \oplus E'_0$$

such that if the minimum polynomial of $\mathbb{C}^m$ with respect to AB is $\lambda^k p(\lambda)$ and that of $\mathbb{C}^n$ is $\lambda^{k'} q(\lambda)$ with respect to BA, it will be $E = Ker\ p(\lambda)$, $E' = Ker\ q(\lambda)$, $p(\lambda) = q(\lambda)$ and A, B being isomorphisms between E and E', according to Theorem 1.

Proposition 1 assures us that k' has the same value as $k-1$, k or $k+1$. Indeed: If in the minimum polynomial there appears the factor $\lambda^{k'}$, it is because there is some vector which has $\lambda^{k'}$ as a minimum polynomial. Let I be the subspace generated by it. This means that (Prop. 1) the minimum polynomial of subspace $A(I) \subset \mathbb{C}^m$ is $\lambda^{k'}$ or $\lambda^{k'-1}$, and therefore $k' \leq k$ or $k' - 1 \leq k$. Symmetrically, it can be deduced that $k \leq k'$ or $k-1 \leq k'$. In any case, we can state that $k-1 \leq k' \leq k+1$.

The first part of the proof of Theorem 1 tells us, besides, that $B(E_0) \subset E_0'$.

PROPOSITION 3. Each subspace $B(E_i)$ is a maximal cyclic or is contained in a maximal cyclic of one degree more.

Proof. Let $E_i = \langle e, (AB)e, \ldots, (AB)^{k-1}e \rangle$. The vector Be is the generator of $B(E_i)$. Let us remember that it is possible that $(BA)^{k-1}Be = 0$. If $B(E_i)$ is maximal, there remains nothing to say. Let us suppose, then, that it is not: $\exists F$ cyclic such that $B(E_i) \subset F$. As we know, it is possible to find a generator vector in F, e', such that

$$F = \langle e', (BA)e', \ldots, (BA)^r e', Be, \ldots, (BA)^{k-1}Be \rangle$$

with of course $Be = (BA)^{r+1}e'$.

With this,

$$A(F) = \langle Ae', (AB)Ae', \ldots, (AB)^r Ae', (AB)e, \ldots, (AB)^{k-1}e \rangle$$

Let $\bar{e} = (BA)^r e'$. We know that $(BA)\bar{e} = Be$. Moreover,

of $A\bar{e} \in E_0 = E_1 \oplus \ldots \oplus E_i \oplus \ldots \oplus E_p$ is $A\bar{e} = z_1 + \ldots + z_i + \ldots + z_p$ and then

$(BA)\bar{e} = Be = Bz_1 + \ldots + Bz_i + \ldots + Bz_p \in B(E_1) \oplus \ldots \oplus B(E_p)$ and necessary $Bz_j = 0 \;\; \forall j = 1, \ldots, p, j \neq i, Be = Bz_i$.

Furthermore,

$$z_i \in E_i \Rightarrow z_i = \sum_{s=0}^{k-1} \gamma_s (AB)^s e \Rightarrow Bz_i = \sum_{s=0}^{k-1} \gamma_s (BA)^s Be \text{ and then}$$

$$\sum_{s=0}^{k-1} \gamma_s (BA)^s Be = Be \Rightarrow \gamma_0 = 1, \quad \gamma_1 = \ldots = \gamma_{k-2} = 0$$

We cannot state anything on the coefficient γ_{k-1} since it is possible that $(BA)^{k-1}Be = 0$. With this

$$z_i = e + \gamma (AB)^{k-1} e$$

Let $\bar{E}_i$ be the subspace engendered by $A\bar{e}$. We will see that E_i can be replaced with $\bar{E}_i$ for all purposes. Firstly, do we still find $E_0 = E_1 \oplus \ldots \oplus \bar{E}_i \oplus \ldots \oplus E_p$?

Let us take $y_j \in Ej$, $j = 1, \ldots, p, j \neq i$ and $\bar{y}_i \in \bar{E}_i$ such that $y_1 + \ldots + \bar{y}_i + \ldots + y_p = 0$. We must see that all are nil.

As $\bar{E}_i = \langle A\bar{e}, (AB)A\bar{e}, \ldots, (AB)^{k-1}A\bar{e}\rangle$, we have:

$$\bar{y}_i = \alpha A\bar{e} + y_i \text{ with } y_i = \sum_{s=1}^{k-1} \delta_s (AB)^s e$$

for $(AB)^s A\bar{e} = A(BA)^{s-1}BA\bar{e} = (AB)^s e \ s = 1, \ldots, k-1$
then $\bar{y}_i = \alpha(z_1 + \ldots + z_i + \ldots + z_p) + \sum_{s=1}^{k-1} \delta_s (AB)^s e =$
$= \alpha z_1 + \ldots + \alpha e + \alpha\gamma(AB)^{k-1}e + \sum_{s=1}^{k-1} \delta_s (AB)^s e + \ldots + \alpha z_p$ and is
$0 = y_1 + \ldots + \bar{y}_i + \ldots + y_p = (y_1 + \alpha z_1) + \ldots + (\alpha e + \alpha\gamma(AB)^{k-1}e +$
$+ \sum_{s=1}^{k-1} \delta_s (AB)^s e) + \ldots + y_p + \alpha z_p) \in E_1 \oplus \ldots \oplus E_i \oplus \ldots \oplus E_p$ of where
$y_j + \alpha z_j = 0, \ j = 1, \ldots, p \ j \neq i,$
$\alpha e + \alpha\gamma(AB)^{k-1}e + \sum_{s=1}^{k-1} \delta_s (AB)^s e = 0$.

From the second equality it is deduced that $\alpha = 0$ (it is the only coefficient of the vector e) and therefore $y_i = 0$ and $\bar{y}_i = 0$ and also $y_j = 0 \ \forall j \neq i$.

The sum is therefore still direct.

The vector $A\bar{e}$ has the same minimum polynomial as e, since all the vectors z_j, $j \neq i$ are cancelled with AB (let us remember that $Bz_j = 0$). Thus, the subspace $\bar{E}_i$ corresponds to the same elementary divisor as E_i. Moreover, $B(\bar{E}_i) = B(E_i)$.

Let us redefine $E_i = \bar{E}_i$ and $e = A\bar{e}$. With this,

$A(F) =$

$$\langle Ae', \ldots, (AB)^{r-1}Ae', e, (AB)e, \ldots, (AB)^{k-1}e\rangle \text{ and } E_i \subset A(F)$$

As it is easy to check, each subspace E_i is maximal within E_0. Therefore, the vectors $Ae', \ldots, (AB)^{r-1}Ae'$ cannot exist. In other words, the subspace F is necessarily of the form

$$F = \langle \bar{e}, Be, \ldots, (BA)^{k-1}Be\rangle$$

one degree superior to that of $B(E_i)$. Moreover, $A(F) = E_i$. ◇

THEOREM 2. *There are decompositions in the direct sum of irreducibles of the subspaces* $E_0 = E_1 \oplus E_2 \oplus \ldots \oplus E_p$ *and* $E'_0 = E'_1 \oplus E'_2 \oplus \ldots \oplus E'_q$ *such that* $B(E_j) \subset E'_j$, $A(E'_j) \subset E_j$ *for* $j = 1, \ldots r$ $(r \leq min(p,q))$, *A or B being a monomorphism, the other with a kernel of dimension 1 and the dimensions of* E_j *and* E'_j *equal or different by one; furthermore,* $\forall i > r$ *we have that* $\dim E_i = 1$, $B(E_i) = \{0\}$ *and dim* $E'_i = 1$, $A(E'_i) = \{0\}$.

Proof. Let $E_1, \ldots, E_{r'}$ be the irreducibles of the decomposition of E_0 with $B(E_j) \neq \{0\}$. According to proposition 3, the subspace $B(E_j)$ is maximal or is contained in a maximal cyclic of a dimension one more than $B(E_j)$. Let E'_j be the maximal subspaces such that $B(E_j) \subset E'_j$. For a given subspace E_j, let e be its generating vector:

$$E_j = \langle e, (AB)e, \ldots, (AB)^{k-1}e \rangle$$

Then,

$$B(E_j) = \langle Be(BA)Be, \ldots, (BA)^{k-2}Be,\ (BA)^{k-1}Be \rangle$$

where the last vector may be nil.

If $E'_j = B(E_j)$, then $A(E'_j) = \langle (AB)e, \ldots, (AB)^{k-1}e \rangle \subset E_j$. In this case, if $(BA)^{k-1}Be \neq 0$, B is a monomorphism (in fact it is an isomorphism) and we will have $Ker\ (A) = \langle (BA)^{k-1}Be \rangle$, and thus $\dim Ker\ (A) = 1$; furthermore, $\dim E_j = \dim E'_j$. If $(BA)^{k-1}Be = 0$, then A is a monomorphsim and $Ker\ (B) = \langle (AB)^{k-1}e \rangle$, since $\dim E_j = \dim E'_j + 1$.

If $E'_j \neq B(E_j)$, is $E'_j = \langle \bar{e}, Be, \ldots, (BA)^{k-2}Be,\ (BA)^{k-1}Be \rangle$, with $A\bar{e} = e$. Then $A(E'_j) = E_j$.

If $(BA)^{k-1}Be \neq 0$, B is a monomorphism and $Ker\ (A) = \langle (BA)^{k-1}Be \rangle$, since $\dim E'_j = \dim E_j + 1$.

If $(BA)^{k-1}Be = 0$, A is a monomorphism (in fact it is an isomorphism) and we have $Ker\ (B) = \langle (AB)^{k-1}e \rangle$, since $\dim E'_j = \dim E_j$.

We still have to see that the subspaces E'_j form a direct sum. Let us take $y'_j \in E'_j$ such that $\Sigma y'_j = 0$. From this, $\Sigma A y'_j = 0$. As $Ay'_j \in E_j\ \forall j$, $\Rightarrow Ay'_j = 0\ \forall j$, since the subspaces E_j, form a direct sum. It may happen that $y'_j \in B(E_j)$ or that $y'_j \notin B(E_j)$ (if

$(B(E_j) \neq E_j')$. Let us analyze this second case: $B(E_j) \neq E_j'$.

$$\begin{aligned} y_j' \in E_j' \quad &\Rightarrow \quad y_j' = \alpha_0 \bar{e} + \sum_{s=0}^{k-1} \alpha_{s+1} (BA)^s Be \Rightarrow \\ &\Rightarrow \quad Ay_j' = \alpha_0 A\bar{e} + \sum_{s=0}^{k-2} \alpha_{s+1} (AB)^{s+1} e = \sum_{s=0}^{k-1} \alpha_s (AB)^s e \end{aligned}$$

since $A\bar{e} = e$. We know that $Ay_j' = 0$ and this means that $\alpha_s = 0$ $\forall s \leq k-1$, so $y_j' = \alpha_k (BA)^{k-1} Be \in B(E_j)$.

In any case, then $y_j' \in B(E_j)$. But the subspaces $B(E_j)$ form a direct sum, and then the E_j' do too.

Let us now take the E_i such that $B(E_i) = \{0\}$. It is evident that $\dim E_i = 1$, according to proposition 1. Let E_0'' be an invariant subspace for BA, supplementary to $E_1' \oplus \ldots \oplus E_{r'}'$, within E_0'. For all $y_0' \in E_0''$ we have:

$$Ay_0' \in E_0 \Rightarrow Ay_0' = y_1 + \ldots + y_{r'} + y_{r'+1} + \ldots + y_p \text{ with } y_j \in E_j$$

As $B(y_{r'+1}) = \ldots = B(y_p) = 0$, since $B(E_j) = \{0\}$ for $j > r'$, we have

$$\begin{aligned} BAy_0' \quad &= \quad By_1 + \ldots + B(y_{r'}) \in (E_1' \oplus \ldots \oplus E_{r'}') \cap E_0'' \Rightarrow \\ &\Rightarrow \quad BAy_0' = 0, \quad By_j = 0 \quad j = 1, \ldots, r' \; . \end{aligned}$$

From the first equality we deduce that E_0'' has λ as its annihilating polynomial. From the others, we conclude the existence of $y_j' \in E_j'$ such that $Ay_j' = y_j$ (in the first part of the proof it has already been seen with the roles of A and B inverted). We can put

$$Ay_0' = A(y_1' + \ldots + y_{r'}') + y_{r'+1} + \ldots + y_p$$

It is clear that the subspace E_0'' decomposes into grade one cyclics:

$$E_0'' = \langle v_1 \rangle \oplus \ldots \oplus \langle v_q \rangle$$

For each vector v_i we obtain the vectors y_{ji}', as has been done for the vector y_0', and we define $w_i = v_i - (y_{1i}' + \ldots + y_{r'i}')$. The subspace $F = \langle w_i \rangle \oplus \ldots \oplus \langle w_q \rangle$ is still supplementary to $E_j' \oplus \ldots \oplus E_{r'}'$, and also $Aw_i \in E_{r'+1} \oplus \ldots \oplus E_p$, since

$$Aw_i = Av_i - A(y_{1i}' + \ldots + y_{r'i}') = y_{(r'+1)i} + \ldots + y_{pi}$$

It is clear that $A(F) = \langle Aw_1, \dots, Aw_q\rangle = \langle Aw_1, \dots, Aw_{r''}\rangle$, eliminating those that are nil and those that are a linear combination of the other ones (if there are any). We define $E_{r'+1} = \langle Aw_1\rangle, \dots, E_{r'+r''} = E_r = \langle Aw_{r''}\rangle$. All of them are grade 1 cyclic subspaces, since $(AB)Aw_i = A(BA)w_i = A(0) = 0$. We will put $e_{r'+i} = Aw_i$.

With this we have that $E_{r'+1} = \langle e_{r'+1}\rangle, E'_{r'+i} = \langle w_i\rangle$, all of dimension 1, with $B(E_i) = \{0\}$ and $A(E'_i) = E_i$. The subspaces $E_1, \dots E_r$ and $E'_1, \dots, E'_r$ fulfill the stated proposal.

We complete the decomposition into irreducibles of E_0:

$$E_0 = E_1 \oplus \dots \oplus E_r \oplus E_{r+1} \oplus \dots \oplus E_p$$

with $\dim E_{r+1} = \dots = \dim E_p = 1$ and $B(E_{r+1} = \dots = B(E_p) = \{0\}$.

In E''_0 we have $E''_0 = E'_{r'+1} \oplus \dots \oplus \langle w_{r+1}\rangle \oplus \dots \oplus E_r \oplus \langle w_q\rangle$, since

$$Aw_{r+i} = \sum_{j=1}^{r''} \alpha_{ji} Aw_j$$

We will take

$$e''_{r+i} = w_{r+i} - \sum_{j=1}^{r''} \alpha_{ji} w_j$$

The subspaces $\langle e''_{r+i}\rangle$ may replace the $\langle w_{r+i}\rangle$ in the direct sum, and of course $A(e''_{r+i}) = 0$.

We thus obtain the decomposition of the statement, taking

$$E'_{r+1} = \langle e''_{r+i}\rangle \ .$$

◇

Couplings.

Let P and Q be square matrices of the respective orders m and n. If there exist matrices A (of the order $m \times n$) and B (of the order $n \times m$) such that $P = AB$ and $Q = BA$, Theorem 2 ensures a decomposition

$$\mathbb{C}^m = E \oplus E_1 \oplus \dots \oplus E_p$$

$$\mathbb{C}^n = E' \oplus E'_1 \oplus \dots \oplus E'_q$$

where $E \simeq E'$ with minimum polynomial

$$(\lambda - \lambda_0)^{p_0}(\lambda - \lambda_1)^{p_1} \dots (\lambda - \lambda_h)^{p_h} \quad (\lambda_i \neq 0 \ \ \forall i)$$

and the subspaces E_i, E'_j are irreducible cyclics with minimum polynomial λ^k, giving strong relations between the two decompositions, characterized through A and B.

We will see that these relations are sufficient to ensure that, given P and Q, $\exists A$ and B such that $P = AB$ and $Q = BA$.

We will say that two square matrices P and Q of the respective orders m and n are *couplable* if there exist decompositions of $\mathbb{C}^m$ and $\mathbb{C}^n$,

$$\mathbb{C}^m = E \oplus E_1 \oplus \dots \oplus E_p$$
$$\mathbb{C}^n = E' \oplus E'_1 \oplus \dots \oplus E'_q$$

such that

a) in E and E' the matrices P and Q have the same elementary divisors

$$(\lambda - \lambda_i)^{p_i} \quad \lambda_i \neq 0$$

b) the miniumum polynomials of the subspaces E_i and E'_j are λ^k

c) $\exists r \leq min(p, q)$ such that:

- $|\dim E_i - \dim E'_i| \leq 1 \ \ \forall i \leq r$.
- $\dim E_j = \dim E'_k = 1 \ \forall j > r,\ k > r$.

Given two couplable matrices P and Q, we will call any arrangement of the subspaces E_i and E'_j which fulfills the above conditions, indicating one of the subspaces E_j or E'_j as principal if $\dim E_j = \dim E'_j$ (this will later involve choosing the meaning of a certain monomorphism), a (P, Q)-*coupling*.

We will call each of the pairs of subspaces which are related within a (P, Q)-coupling a *coupled pair*. We will also consider as coupled pairs those formed by $E_{r+1} \oplus \dots \oplus E_p$ with $\{0\}$ and $\{0\}$ with $E'_{r+1} \oplus \dots \oplus E'_q$, if they exist, as well as the pair (E, E').

Let us see that if P and Q are couplable, there exist A and B (of the respective orders $m \times n$ and $n \times m$) such that $P = AB$ and $Q = BA$. Furthermore, A ane B will be isomorphisms between E and E' and the conditions laid out in Theorem 2 will be fulfilled.

When $\dim E_j = \dim E'_j$ we will make the monomorphism be A if E'_j has been indicated as principal or B if the principal subspace is E_j. Of course, the possible plurality of (P,Q)-couplings will provide different pairs of matrices (A,B) such that $P = AB$ and $Q = BA$. The existence of the matrices A and B will be checked by means of their effective construction.

With P and Q reduced to the Jordan form, arranging the blocks according to Theorems 1 and 2, and considering the correspondence between divisors given by the (P,Q)-coupling chosen, we will have:

1) Coupled pairs between the subspaces of diagonal blocks in P and Q with annuler $(\lambda - \lambda_0)^p$ for $\lambda_0 \neq 0$:

$$\begin{pmatrix} \lambda_0 & 0 & \dots & 0 & 0 \\ 1 & \lambda_0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \lambda_0 & 0 \\ 0 & 0 & \dots & 1 & \lambda_0 \end{pmatrix} \leftrightarrow \begin{pmatrix} \lambda_0 & 0 & \dots & 0 & 0 \\ 1 & \lambda_0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \lambda_0 & 0 \\ 0 & 0 & \dots & 1 & \lambda_0 \end{pmatrix}$$

Let $e_1, \dots e_p$ be the basis of the first subspace and $v_1, \dots, v_p$ of the second.

We will take B defined by

$$B(e_1) = v_1, \dots, B(e_p) = v_p$$

As $AB = P$, $(AB)e_i = \lambda_0 e_i + e_{i+1}$ for $i = 1, \dots p-1$ and $(AB)e_p = \lambda_0 e_p$, as indicated by the matrix P, it is clear that it must be

$$\begin{aligned} A(v_1) &= ABe_1 = \lambda_0 e_1 + e_2 \\ A(v_2) &= \lambda_0 e_2 + e_3 \\ &\vdots \\ A(v_{p-1}) &= \lambda_0 e_{p-1} + e_p \\ A(v_p) &= \lambda_0 e_p \end{aligned}$$

Let us observe that $(BA)v_i = B(\lambda_0 e_i + e_{i+1} = \lambda_0 v_i + v_{i+1}$ for $i = 1, \dots, p-1$ and $(BA)v_p = B(\lambda_0 e_p) = \lambda_0 v_p$, fully in agreement with the corresponding form of the matrix Q.

2) Coupled pairs of the type

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

with the first matrix of the order p and the second of the order $p-1$.

In these coupled paris A is always a monomorphism.

The morphism B is defined according to the relations

$$B(e_1) = v_1, B(e_2) = v_2, \dots, B(e_{p-1}) = v_{p-1}, B(e_p) = 0$$

With this, it must be

$$A(v_1) = e_2, A(v_2) = e_3, \dots, A(v_{p-1}) = e_p$$

3) Coupled pairs of the type

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

with the two matrices of the order p.

If we choose A as a monomorphism:

$$\begin{aligned} B(e_i) &= v_{i+1} \text{ for } i = 1, \dots, p-1; \; B(e_p) = 0 \\ A(v_i) &= e_i \text{ for } i = 1, \dots, p \end{aligned}$$

If the monomorphism is B:

$$\begin{aligned} B(e_i) &= v_i \text{ for } i = 1, \dots, p \\ A(v_i) &= e_{i+1} \text{ for } i = 1, \dots, p-1; \; A(v_p) = 0 \end{aligned}$$

4) Coupled pairs of the type

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

with the first matrix of the order $p-1$ and the second of the order p.
In these, the monomorphism is B. The definition is:

$$\begin{aligned} B(e_i) &= v_{i+1} \text{ for } i = 1, \dots, p-1 \\ A(v_i) &= e_i \text{ for } i = 1, \dots, p-1; \;\; A(v_p) = 0 \end{aligned}$$

5) Coupled pairs of the type

$$\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \leftrightarrow \emptyset \quad B(e_i) = 0, \quad A(0) = 0, \quad A \text{ monomorphism} .$$

$$\emptyset \leftrightarrow \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \quad B(0) = 0, \quad A(v_i) = 0; \quad B \text{ monomorphism} .$$

Taking $\mathbb{C}^m$ and $\mathbb{C}^n$ the basis used in these coupled pairs, the matrices B and A will be composed of diagonal blocks of the following types:

Type 1) The matrix of B is the identity. That of A is the type

$$\begin{pmatrix} \lambda_0 & 0 & \dots & 0 & 0 \\ 1 & \lambda_0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \lambda_0 & 0 \\ 0 & 0 & \dots & 1 & \lambda_0 \end{pmatrix}$$

In other words, the diagonal block corresponding to the matrix B is I_p, and that corresponding to A, $\lambda_0 I_p + H_p$, I_p being the identity of order p and H_p the matrix, also of the order p,

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

Type 2) The block B corresponding to subspaces that form this type of coupled pair is of the order $(p-1) \times p$:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} I_{p-1} & \vdots & 0 \end{pmatrix}$$

That corresponding to the matrix A is of the order $p \times (p-1)$:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ I_{p-1} \end{pmatrix}$$

Type 3) When A is a monomorphism, the block of B is H_p, and that of A is I_p; if B is a monomorphism, that of B is I_p and that of A is H_p.

Type 4) The block of B is of the order $p \times (p-1)$ and that of A is $(p-1) \times p$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

respectively.

We thus have the following theorem:

THEOREM 3. *Given two square matrices P and Q with respective orders m and n, there exist matrices A and B of orders $m \times n$ and $n \times m$, respectively, such that $P = AB$ and $Q = BA$ if and only if P and Q are couplable.*

Succession of nil power.

For any square matrix P, let $k_1 \geq k_2 \geq \ldots \geq k_p > 0$ be the ordered succession of the exponents of its elementary divisors in the form λ^k. We will call this succession made infinite by adding zeroes, the *succession of nil power*:

$$k_1 \geq k_2 \geq \ldots \geq k_p > k_{p+1} = k_{p+2} = \ldots = 0$$

It is clear that the matrix P will be fully described if we know its elementary divisors $(\lambda - \lambda_0)^{p_0}$ with $\lambda_0 \neq 0$ and its *succession of nil power*.

We have the following proposition:

PROPOSITION 4. *Let P and Q be two square matrices with respective successions of nil power*

$$k_1 \geq k_2 \geq \ldots \geq k_p > k_{p+1} = k_{p+2} = \ldots = 0$$

$$k'_1 \geq k'_2 \geq \ldots \geq k'_q > k'_{q+1} = k_{q+2} = \ldots = 0$$

We have that P and Q are couplable if and only if

a) They have the same elementary divisors $(\lambda - \lambda_0)^{p_0}$, $\lambda_0 \neq 0$.

b) $|k_h - k'_h| \leq 1 \ \forall h$.

Proof. ($\Leftarrow$) Let $E_i \subset \mathbb{C}^m$ be the cyclic subspaces corresponding to the elementary divisors λ^k, and $E'_i \subset \mathbb{C}^n$ those corresponding to λ^k.

The pairs (E_i, E'_i), together with $(E_{q+1} \oplus \ldots \oplus E_p, \{0\})$ (if $p > q$) or $(\{0\}, E'_{p+1} \oplus \ldots \oplus E'_q)$ (if $p < q$), are coupled pairs and define a (P, Q)-coupling.

(⇒) Section a) forms part of the definition of (P,Q)-coupling. Let us see section b).

Let

$$\mathbb{C}^m = E \oplus E_1 \oplus \ldots \oplus E_p$$

$$\mathbb{C}^n = E' \oplus E'_1 \oplus \ldots \oplus E'_q$$

be the decomposition given by a given (P,Q)-coupling.

We rearrange the subspaces E_i and E'_j by decreasing order of degrees of their minimal polynomial (their dimension):

$$E_{i_1}, E_{i_2}, \ldots, E_{i_p}$$

$$E'_{j_1}, E'_{j_2}, \ldots, E'_{j_q}$$

We define $k_h = \dim E_{j_h}$, $k'_h = \dim E_{j_h}$, constructing with them the successions of nil power of P and Q:

$$k_1 \geq k_2 \geq \ldots \geq k_p > k_{p+1} = k_{p+2} = \ldots = 0$$

$$k'_1 \geq k'_2 \geq \ldots \geq k'_q > k'_{q+1} = k'_{q+2} = \ldots = 0$$

Let us suppose that for some value of h it is $k_h - k'_h > 1$. According to the definition of (P,Q)-coupling, the subspaces of $\mathbb{C}^n$ which form couled pairs with $E_{i_1}, E_{i_2}, \ldots, E_{i_{h-1}}$ must be in front of E_{j_h}, and are therefore $E'_{j_1}, E'_{j_2}, \ldots, E'_{j_{h-1}}$ (not necessarily in this order). The subspace E_{i_h} must also couple with one of these, which is clearly impossible. Therefore $k_h - k'_h \leq 1 \;\; \forall h$.

A parallel reasoning leads us to $k'_h - k_h \leq 1 \;\; \forall h$. ◇

Representative of a (P,Q)-coupling.

We will say that the pair of matrices (A,B) is a *representative* of the (P,Q)-coupling

$$\Omega = \{(E,E'), (E_1,E'_1) \ldots (E_r,E'_r), (E_{r+1} \oplus \ldots \oplus E_p, \{0\}),$$

$$(\{0\}, E'_{r+1} \oplus \ldots \oplus E'_q)\}$$

if:

i) A and B are isomorphisms between E and E'.

ii) $A(E_i') \subset E_i$, $B(E_i) \subset E_i'$, for $i \leq r$, being one of the two a monomorphism and the other with the nucleus of dimensions 1. If $\dim E_i = \dim E_i'$, the monomorphism will be suitable as indicated by the (P, Q)-coupling.

iii) $A(E_i') = B(E_i) = \{0\}, \forall i > r$.

iv) $AB = P$, $BA = Q$.

Equivalent couplings.

Let

$$\Omega_1 = \{(E, E'), (E_1, E_1'), \ldots, (E_r, E_r'), (E_{r+1} \oplus \ldots \oplus E_p, \{0\}), \\ (\{0\}, E_{r+1}' \oplus \ldots \oplus E_q')\}$$

$$\Omega_2 = \{(F, F'), (F_1, F_1'), \ldots, (F_r, F_r'), (F_{r+1} \oplus \ldots \oplus F_p, \{0\}), \\ (\{0\}, F_{r+1}' \oplus \ldots \oplus F_q')\}$$

be a (P_1, Q_1)-coupling and a (P_2, Q_2)-coupling respectively, P_1 and P_2 being square matrices of the order m and Q_1 and Q_2 square matrices of the order n.

We say that Ω_1 is equivalent to $\Omega_2 (\Omega_1 \dot{\sim} \Omega_2)$ if:

i) The matrices P_1 and P_2 have the same elementary divisors $(\lambda - \lambda_0)^p$ with $\lambda_0 \neq 0$ (and consequently to Q_1 and Q_2 also).

ii) There is a rearrangement of the coupled pairs such that

$$\left.\begin{array}{l} \dim E_{i_k} = \dim F_{j_k} \\ \dim E_{i_k}' = \dim F_{j_k}' \end{array}\right\} \quad \text{for } 1 \leq k \leq r$$

and such that the pairs (E_{i_k}, E_{i_k}'), (F_{j_k}, F_{j_k}') have the same principal subspaces at the same side, if $\dim E_{i_k} = \dim E_{i_k}'$.

It is clear that in these conditions P_1 is similar to $P_2 (P_1 \dot{\sim} P_2)$ and Q_1 is similar to $Q_2 (Q_1 \dot{\sim} Q_2)$.

Equivalence of doubly multipliable pairs of matrices.

We say that two pairs of doubly multipliable matrices (A_1, B_1) and (A_2, B_2) are equivalent $((A_1, B_1) \dot{\sim} (A_2, B_2))$ if there exist basis changes which transform one pair into another.

LEMMA 2. *Given two pairs of multipliable square matrices* (A_1, B_1) *and* (A_2, B_2) *with* B_1 *and* B_2 *invertible and such that* $A_1B_1 \dot{\sim} B_2A_2$, *then the two pairs are equivalent.*

Proof. We have that

$$A_1B_1 = T^{-1}B_2A_2T \Rightarrow A_1 = T^{-1}B_2A_2TB_1^{-1} = (B_2^{-1}T)^{-1}A_2(TB_1^{-1})$$

Defining $S = B_2^{-1}T$ and $R = TB_1^{-1}$, we are left with

$$R^{-1}B_2S = (B_1T^{-1})B_2(B_2^{-1}T) = B_1T^{-1}T = B_1$$

and therefore

$$A_1 = S^{-1}A_2R$$
$$B_1 = R^{-1}B_2S$$

which shows that the pairs of matrices considered are equivalent.

If the invertible matrices are A_1 and A_2, maintaining the other hypotheses the result is still true.

PROPOSITION 5. *If* (A_1, B_1) *and* (A_2, B_2) *are representatives of equivalent couplings, then they are equivalent pairs.*

Proof. Let (A_1, B_1) be a representative of a (P_1, Q_1)-coupling Ω_1, and (A_2, B_2) a representative of a (P_2, Q_2)-coupling Ω_2 where $\Omega_1 \dot{\sim} \Omega_2$.

On the coupled pairs of type 1 we have that

$$A_1B_1 = P_1 \dot{\sim} Q_2 = B_2A_2$$

A_i and B_i being invertible square matrices, because the matrices P_i and Q_i are Lemma 2 already ensures the equivalence of the pairs of matrices on coupled pairs of this type.

For coupled pairs of type 2, the matrices A_1 and A_2, of the order $p \times (p-1)$ are monomorphisms. Their rank is thus $p-1$. Also, the matrices B_1 and B_2, of the order $(p-1) \times p$, have the nucleus of dimension 1; their rank is also $p-1$. We can state that A_1 is "left-invertible" and B_1 is "right-invertible", i.e.

$$\exists M | MA_1 = I_{p-1} \quad \exists N | B_1 N = I_{p-1}$$

The similarity between the matrices $P_1 = A_1 B_1$ and $P_2 = A_2 B_2$ ensures the existence of an invertible matrix V such that $VA_1B_1 = A_2B_2V$.

With all this,

$$VA_1B_1 = A_2B_2V \Rightarrow VA_1B_1N = A_2B_2VN \Rightarrow A_1 = V^{-1}A_2(B_2VN)$$

$$\begin{aligned} VA_1B_1 &= A_2B_2V \Rightarrow A_1B_1 = V^{-1}A_2B_2V \Rightarrow \\ &\Rightarrow MA_1B_1 = MV^{-1}A_2B_2V \Rightarrow B_1 = (MV^{-1}A_2)B_2V \end{aligned}$$

Let us see the matrices (B_2VN) and $(MV^{-1}A_2)$ are inverse to each other:

$$(MV^{-1}A_2)(B_2VN) = M(V^{-1}A_2B_2V)N = MA_1B_1N = I_{p-1}$$

Indeed, defining $R = B_2VN$ (by which $R^{-1} = MV^{-1}A_2$) are inverse to each other:

$$A_1 = V^{-1}A_2R, \quad B_1 = R^{-1}B_2V$$

which ensures that $(A_1, B_1) \dot{\sim} (A_2, B_2)$ in this type of coupled pairs.

In type 3 coupled pairs the monomorphisms are in reality isomorphisms, which makes it possible to apply Lemma 2 again.

Type 4 coupled pairs are dealt with in parallel to those of type 2, using the similarity $B_1A_1 \dot{\sim} B_2A_2$ working with the matrices B_2 and A_2.

Those of type 5 show no problems. ◊

PROPOSITION 6. *If (A_1, B_1), (A_2, B_2) are pairs of equivalent doubly multipliable matrices, then they are representatives of equivalent couplings.*

Proof. As (A_1, B_1), (A_2, B_2) are equivalent pairs, there exist the invertible matrices V and W such that

$$A_1 = V^{-1}A_2W, \quad B_1 = W^{-1}B_2V$$

Let be

$$P_1 = A_1B_1, Q_1 = B_1A_1, P_2 = A_2B_2, Q_2 = B_2A_2$$

It is clear that

$$P_1 = V^{-1}P_2V, \quad Q_1 = W^{-1}Q_2W$$

Therefore, the four matrices have the same elementary divisors $(\lambda - \lambda_0)^p$ with $\lambda_0 \neq 0$, and also the same ones as the form λ^k and Q_1, Q_2 too.

Let

$$\Omega_1 = \{(E, E'), (E_1, E_1'), \ldots, (E_r, E_r'), (E_{r+1} \oplus \ldots \oplus E_p, \{0\}),$$
$$(\{0\}, E_{r+1}' \oplus \ldots \oplus E_q')\}$$

be the (P_1, Q_1)-coupling of which (A_1, B_1) is the representative.

We build

$$\Omega_2 = \{(F, F'), (F_1, F_1'), \ldots, (F_r, F_r'), (F_{r+1} \oplus \ldots \oplus F_p, \{0\}),$$
$$(\{0\}, F_{r+1}' \oplus \ldots \oplus F_q')\}$$

since

$$F = V(E), F' = W(E'); \quad F_i = V(E_i), F_i' = W(E_i') \ \forall i$$

Each F_i has the same minimum polynomial as the corresponding E_i, and the same occurs for F_i' and E_i'. It is also obvious that $\Omega_1 \dot{\sim} \Omega_2$ and that the pair (A_2, B_2) is a representative of the coupling Ω_2. $\diamondsuit$

With the last two propositions, we have the following theorem:

THEOREM 4 (of Equivalence). *Two pairs (A_1, B_1) and (A_2, B_2) of doubly multipliable matrices are equivalent if and only if there are representatives of equivalent couplings.*

Rank of the matrices A and B.

Given a (P,Q)-coupling, Theorem 4 ensures that all its representatives are equivalent. Therefore, if (A_1,B_1) and (A_2,B_2) are representatives of the same (P,Q)-coupling, we can ensure that $rg(A_1) = rg(A_2)$ and $rg(B_1) = rg(B_2)$. The construction of the matrices A and B, performed for the countable pairs (P,Q), allows us to calculate their rank. Indeed:

Let n_i for $i = 2,3,4$ be the number of coupled pairs of type i existing in the (P,Q)-coupling with which the matrices A and B have been constructed.

Each coupled pair of type two will produce a nil row in A and a nil column in B. Those of type 4 will produce a transposed effect: a nil column in A and a nil row in B. The coupled pairs of type 3 will become a nil row and nil column in A if B is a monomorphism, or in B if A is one; let n_3' and n_3'' be the respective number of them of each form (of course $n_3' + n_3'' = n_3$).

The coupled pairs of type 5 produce nil $p-r$ rows and $q-r$ columns in A; in matrix B there will be nil $q-r$ columns and $p-r$ rows.

The rank of matrices A and B, calculated by rows, is

$$rg(A) = m - n_2 - n_3' - p + r$$

$$rg(B) = n - n_4 - n_3'' - q + r$$

If we calculate it by columns, we have

$$rg(A) = n - n_4 - n_3' - q + r$$

$$rg(B) = m - n_2 - n_3'' - p + r$$

From these equalities it is deduced that for any P,Q-coupling,

$$m - n_2 - p = n - n_4 - q\ .$$

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