ASYMPTOTIC BEHAVIOUR OF NONLINEAR SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAY (*)

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SOMMARIO. - *In questo lavoro si considerano equazioni differenziali non lineari con ritardo distribuito. Si studiano le proprietà asintotiche delle loro soluzioni non oscillanti.*

SUMMARY. - *In the paper nonlinear differential equations with distributed delay of neutral type are considered. The asymptotic properties of their non-oscillating solutions are investigated.*

1. Introduction.

In the recent years the qualitative properties of the solutions of neutral differential equations of second and higher order were considered in numerous papers [1]-[14]. In the present paper the asymptotic behaviour of the nonoscillating solutions of the neutral equation

$$\frac{d^2}{dt^2}[x(t) + \varepsilon.p(t) \cdot x(t - \tau(t))] - \int_0^{\sigma(t)} f(x(t-s))dr(t,s) = 0 \quad (1)$$

is investigated. The aim of this paper is to generalize and complement the results in [7], where the equation

$$\frac{d^2}{dt^2}[x(t) + p(t) \cdot x(t-\tau)] - q(t) \cdot f(x(t-\sigma)) = 0 \quad (2)$$

is investigated, for equations in which the delays are not constant. We shall note that the only papers in which equations of the type

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(2) are considered for $q(t) \geq 0$ are [2] and [7]. Both cases in which the function $\tau(t)$ in equation (1) is nonnegative and nonpositive are investigated. Some of the results obtained are new even for the equation

$$\frac{d^2}{dt^2}[x(t) + px(t-\tau)] - qx(t-\sigma) = 0 \tag{3}$$

2. Preliminary Notes.

We shall say that conditions (H) are met if the following conditions hold:

H1. $p(t) \in C([t_0, \infty), \mathbb{R}_+)$

H2. $\tau(t) \in C([t_0, \infty), \mathbb{R})$ and $\lim_{t\to\infty}(t - \tau(t)) = \infty$

H3. $\sigma(t) \in C([t_0, \infty), \mathbb{R}_+)$ and $\lim_{t\to\infty}(t - \sigma(t)) = \infty$

H4. $r(t, 0) = 0$ for $t \in [t_0, \infty)$

H5. $r(t, \sigma(t)) \in C([t_0, \infty), \mathbb{R}_+)$

H6. $r(t, s)$ is nondecreasing on s for $s \in [0, \sigma(t)]$

H7. $\int_{t_0}^{\infty} r(t, \sigma(t))dt = \infty$

H8. $f \in C(\mathbb{R}, \mathbb{R})$

H9. $u \cdot f(u) > 0$ for $u \in \mathbb{R}\backslash\{0\}$

H10.For any $\varepsilon > 0$ there exists $\delta > 0$ such that for $|u| > \delta$ the inequality $|f(u)| > \varepsilon$ be met.

Define the function $z(t)$ in the following way:

$$z(t) = x(t) + \varepsilon \cdot p(t) \cdot x(t - \tau(t)) \tag{4}$$

Then

$$\ddot{z}(t) = \int_0^{\sigma(t)} f(x(t-s))dr(t,s) \tag{5}$$

DEFINITION 1. The function f is said to enjoy the property $\mathcal{K}$ if there exists t_0 such that for $t \geq t_0$ the function enjoys the property $\mathcal{K}$.

DEFINITION 2. The function X defined for all sufficiently large values of t is said to be an *eventual solution* of (1) if for all t large enough x is a continuous function, z is twice differentiable and x eventually satisfies equation (1).

REMARK 1. In the paper no solutions are considered for which $x(t) \equiv 0$ eventually.

DEFINITION 3. The eventual solution $x(t)$ of (1) is said to *oscillate* if its set of zeros is unbounded above. Otherwise the solution is called *nonoscillating*.

By Definition 3 the nonoscillating solution of (1) are characterized as eventually positive or eventually negative.

LEMMA 1. *Let conditions* (H) *hold and* $\varepsilon = -1$. *Moreover, let the following condition hold*

$$0 \leq p(t) \leq p \tag{6}$$

Then, if $x(t)$ is an eventually positive solution of (1), then either

$$z(t) > 0, \dot{z} > 0, \ddot{z}(t) > 0 \text{ and } \lim_{t\to\infty} z(t) = \lim_{t\to\infty} \dot{z}(t) = \infty \tag{7}$$

or

$$z(t) > 0, \dot{z}(t) < 0, \ddot{z}(t) > 0 \text{ and } \lim_{t\to\infty} z(t) = \lim_{t\to\infty} \dot{z}(t) = 0 \tag{8}$$

Proof. From H6, H9, (5) and from the inequality $x(t) > 0$ it follows that $\ddot{z}(t) > 0$ eventually. Consequently, $\dot{z}(t)$ is an eventually increasing function and $z(t)$ is an eventually monotone function. Suppose that $\dot{z}(t) > 0$ eventually. Since $\dot{z}(t)$ is an eventually increasing function, we obtain that $\lim z(t)_{t\to\infty} = \infty$. Then from (4) and from H1 it follows that $\lim_{t\to\infty} x(t) = \infty$. Integrate (5) from t_1 to t, where t_1 is large enough and obtain

$$\dot{z}(t) \quad = \quad \dot{z}(t_1) + \int_{t_1}^{t} \int_{t_1}^{\sigma(v)} f(x(v-s))dr(v,s)dv \geq$$

$$\geq \int_{t_1}^{t} \min_{[v-\sigma(v),v]} f(x(s)) \cdot r(v,\sigma(v))dv$$

From H3 and H10 it follows that if we choose t_1 large enough, then $\min_{[v-\sigma(v),v]} f(x(s)) \geq k > 0$ for $v \in [t_1, t]$. Using condition H7, we obtain that $\lim_{t\to\infty} \dot{z}(t) = \infty$, hence $\lim_{t\to\infty} z(t) = \infty$. Thus we showed that if $\dot{z}(t) > 0$ eventually, then (7) is realized. Let $\dot{z}(t) < 0$ eventually. Suppose that $\lim_{t\to\infty} \dot{z}(t) = -c$ $(c > 0)$. Since $\dot{z}(t)$ is an eventually increasing function, then $\dot{z}(t) < -c$ eventually, hence $\lim_{t\to\infty} z(t) = -\infty$. From (4) it follows that $\lim_{t\to\infty} p(t)\cdot x(t-\tau(t)) = \infty$ and since $p(t)$ is a bounded function, then $\lim_{t\to\infty} x(t-\tau(t)) = \infty$. From H2 it follows that $\lim_{t\to\infty} x(t) = \infty$. Further on, as above, it is shown that $\lim_{t\to\infty} \dot{z}(t) = \infty$. From the contradiction obtained it follows that $\lim_{t\to\infty} \dot{z}(t) = 0$. From the fact that $\dot{z}(t)$ is an eventually increasing function it follows that $\dot{z}(t) < 0$ eventually, hence $z(t)$ is an eventually decreasing function. We shall prove that $\lim_{t\to\infty} z(t) = 0$. Suppose that $\lim_{t\to\infty} z(t) = L$ and let $L > 0$. Then $z(t) > L$ eventually. From (4) it follows that $x(t) > L$ eventually. Since $\min_{[v-\sigma(v),v]} f(x(s)) > \tilde{L}$ for $v \in [t_1, t]$, where t_1 is large enough, and $\tilde{L}$ is a positive constant (its existence follows from condition H10), as above we obtain that $\lim_{t\to\infty} \dot{z}(t) = \infty$. Hence $L \leq 0$. Suppose that $L < 0$. From (4) and from the fact that $z(t)$ is an eventually decreasing function it follows that

$$\frac{L}{2} > x(t) - p(t) \cdot x(t-\tau(t)) > -p(t) \cdot x(t-\tau(t)) > -px(t-\tau(t))$$

Hence $x(t) > \frac{-L}{2p}$ eventually. Just as in the case when $L > 0$, from the last inequality it follows that $\lim_{t\to\infty} \dot{z}(t) = \infty$. Hence $L = 0$, i.e. $\lim_{t\to\infty} z(t) = 0$ and since $z(t)$ is an eventually decreasing function, then $z(t) > 0$ eventually. Thus we showed that if $\dot{z}(t) < 0$ eventually, then (8) is realized. ◇

3. Main Results.

We shall first consider the case when $\varepsilon = -1$.

THEOREM 1. *Let conditions* (H) *hold,* $\varepsilon = -1$ *and*

$$0 \le p(t) \le p < 1 \tag{9}$$

Then, if $x(t)$ *is a nonoscillating eventual solution of (1), then*

$$\lim_{t\to\infty} x(t) = 0 \text{ or } \lim_{t\to\infty} |x(t)| = \infty$$

Proof. Let $x(t) > 0$ eventually. By Lemma 1 either (7) or (8) holds. suppose that assertion (7) is valid. Then $\lim_{t\to\infty} z(t) = \infty$ and since $x(t) > z(t)$, we obtain that $\lim_{t\to\infty} x(t) = \infty$. Let assertion (8) be valid. We shall prove that $\liminf_{t\to\infty} x(t) = 0$. Suppose that this is not true. Let $\liminf_{t\to\infty} x(t) = c > 0$. Then $x(t) > \frac{c}{2}$ eventually and as in the proof of Lemma 1 we obtain that $\lim_{t\to\infty} \dot{z}(t) = \infty$, which contradicts (8). Hence $\liminf_{t\to\infty} x(t) = 0$. Suppose that $x(t)$ is an unbounded function. We can choose a sequence $\{\gamma_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \gamma_n = \infty$, $\lim_{n\to\infty} x(\gamma_n) = 0$ and $\max_{[\gamma_1,\gamma_n]} x(s) = x(\gamma_n)$. Then $z(\gamma_n) = x(\gamma_n) - p(\gamma_n)x(\gamma_n - \tau(\gamma_n)) > x(\gamma_n)(1-p)$ and $\lim z(\gamma_n) = \infty$. From the contradiction obtained, it follows that $x(t)$ is a bounded function. Suppose that $\limsup_{t\to\infty} x(t) = d > 0$. There exist sequences $\{\alpha\}_1^{\infty}$ and $\{\beta_n\}_1^{\infty}$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \infty$, $\lim_{t\to\infty} x(\alpha_1) = 0$ and $\lim_{n\to\infty}(\beta_n) = d$. Then for any ε there exists $N \in \mathbb{N}$ such that for $n > N$ we have $x(\alpha_n) < \varepsilon$, $x(\beta_n) > d - \varepsilon$ and $x(t) < d + \varepsilon$ for t large enough. Choose the positive integers k and m so that $\alpha_k < \beta_m$ and estimate the difference $z(\beta_m) - z(\alpha_k)$

$$\begin{aligned} z(\beta_m) - z(\alpha_k) &= x(\beta_m) - x(\beta_m - \tau(\beta_m)) \cdot p(\beta_m) \\ &\quad -x(\alpha_k) + p(\alpha_k) \cdot x(\alpha_k - \tau(\alpha_k)) > \\ &> x(\beta_m) - x(\alpha_k) - p(\beta_m)x(\beta_m - \tau(\beta_m)) \\ &> d - \varepsilon - \varepsilon - p(d + \varepsilon) = \\ &= d(1-p) - \varepsilon(2+p) \end{aligned}$$

We obtained that for suitably chosen m, k the inequality

$$z(\beta_m) - z(\alpha_k) > d(1-p) - \varepsilon(2+p)$$

holds. From (9) it follows that for ε small enough $z(\beta_m) - z(\alpha_k) > 0$, but since $\beta_m > \alpha_k$, then the inequality $z(\beta_m) > z(\alpha_k)$ contradicts the fact that $z(t)$ is an eventually decreasing function. Consequently, $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$. The case when $x(t) < 0$ eventually is considered analogously. ◇

REMARK 2. Under the conditions of Theorem 1 no restraints were imposed on the sign of the function $\tau(t)$. Hence the conclusion of Theorem 1 is valid not only for delays with constant signs. but for oscillating ones too.

The case when $p(t) \equiv 1$ and $\tau(t) \leq 0$ eventually, will be considered for the equation

$$\frac{d^2}{dt^2}[x(t) - x(t+\tau)] - \int_0^\sigma \operatorname{sgn}(x(t-s)) \cdot |x(t-s)|^\alpha dr(s) = 0 \qquad (10)$$

Obviously equation (10) is a particular case of (1) an is obtained from it for $p(t) \equiv 1$, $\sigma(t) = \sigma$, $r(t,s) = r(s)$ and $f(u) = |u|^\alpha \cdot \operatorname{sgn} u$.

THEOREM 2. *Let the following conditions hold:*
1. $\tau > 0$
2. $r(0) = 0$ and $r(\sigma) > 0$
3. $r(s)$ *is a nondecreasing function for* $s \in [0, \sigma]$
4. $0 < \alpha \leq 1$

Then, if $x(t)$ *is a nonoscillating solution of* (10), *then*

$$\lim_{t\to\infty} x(t) = 0$$

Proof. Without loss of generality let $x(t)$ be an eventually positive solution of (10). By Lemma 1 $\lim_{t\to\infty} z(t) = 0$ or $\lim_{t\to\infty} z(t) = \infty$. If $\lim_{t\to\infty} z(t) = \infty$, then from the inequality $z(t) < x(t)$ it follows that $\lim_{t\to\infty} x(t) = \infty$. Then there exists a sequence $\{t_n\}_1^\infty$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} x(t_n) = \infty$ and $\max_{[t_1,t_n]} x(s) = x(t_n)$. From $z(t) > 0$ eventually it follows that $x(t) > x(t+\tau)$ eventually, which obviously contradicts the last equality. Hence $\lim_{t\to\infty} z(t) = 0$. We shall prove that $z(t) \in L_1[t_0, \infty)$. From the fact that $x(t)$ is an eventually positive solution of (10) it follows that (5) becomes

$$\ddot{z}(t) = \int_0^\sigma x^\alpha(t-s) dr(s)$$

We integrate this equality from $\bar{t}$ to t, where $\bar{t}$ is a sufficiently large number and obtain

$$\dot{z}(t) = \dot{z}(\bar{t}) + \int_{\bar{t}}^{t} \int_{0}^{\sigma} x^{\alpha}(v - s)dr(s)dv$$

From $\lim_{t\to\infty} \dot{z}(t) = 0$ (Lemma 1) it follows that $\int_0^{\sigma} x^{\alpha}(t-s)dr(s) \in L_1(\bar{t}, \infty)$.

From the inequality $z(t) < x(t)$ we obtain the estimate

$$\int_0^{\sigma} x^{\alpha}(t-s)dr(s) > \int_0^{\sigma} z^{\alpha}(t-s)dr(s) > z^{\alpha}(t)r(\sigma)$$

Hence $z^{\alpha}(t) \in L_1[\bar{t}, \infty)$, and from condition 4 and $\lim_{t\to\infty} z(t) = 0$ it follows that $z(t) \in L_1[\bar{t}, \infty)$. We sum up the equalities

$$z(t + k\tau) = x(t + k\tau) - x[t + (k+1)\tau], \ k = 0, 1, \ldots, n$$

and obtain

$$\sum_{k=0}^{n} z(t + k\tau) = x(t) - x(t + (n+1)\tau) \tag{11}$$

Let $w(t) = \sum_0^{\infty} z(t + k\tau)$. The function $w(t)$ is correctly defined since from $z(t) \in L_1[\bar{t}, \infty)$ and from the monotone decreasing of the function $z(t)$ it follows that the series $\sum_{k=0}^{\infty} z(t + k\tau)$ corresponding to the improper integral $\int_{\bar{t}}^{\infty} z(s)ds$ is convergent for $t \geq \bar{t}$. Then from (11) it follows that there exists the limit $\lim_{n\to\infty} x(t+n\tau)$. Let $\mathcal{J}(t) = \lim_{n\to\infty} x(t+n\tau)$. From the definition of $w(t)$ it follows that $\lim_{t\to\infty} w(t) = 0$. Hence for any $\varepsilon > 0$ there exists $\bar{\bar{t}} > \bar{t}$ such that for $t > \bar{\bar{t}}$ the following inequality is valid

$$x(t) - \mathcal{J}(t) < \varepsilon \tag{12}$$

From (11) it follows that for any fixed t large enough the sequence $\{x(t+n\tau)\}_1^{\infty}$ is monotonely decreasing, hence for any t large enough and any $n \in \mathbb{N}$ we have $x(t + n\tau) > \mathcal{J}(t)$. From this inequality and from (12) it follow that for any $t > \bar{\bar{t}}$ and any $n \in \mathbb{N}$ the following inequality holds

$$x(t + n\tau) > x(t) - \varepsilon \tag{13}$$

Suppose that $\limsup_{t\to\infty} x(t) = c > 0$. Then there exists a sequence $\{t_n\}_1^\infty$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n) = c$. This implies the existence of $N \in \mathbb{N}$ such that for $n > N$ we have $x(t_n) > \frac{c}{2}$. Choose the positive integer m so that $m > N$ and $t_m > \bar{\bar{t}}$. Since $x(t)$ is a continuous function, then there exists $\delta > 0$ such that $x(t) > \frac{c}{3}$ for $t \in [t_m - \delta, t_m + \delta]$ and from (13) there follows the inequality

$$x(t) > \frac{c}{3} - \varepsilon \tag{14}$$

for $t \in [t_m\tau + n\tau - \delta, t_m + n\tau + \delta]$. From condition 2 it follows that there exists σ_0 such that $0 \le \sigma_0 - \frac{\delta}{2} < \sigma_0 + \frac{\delta}{2} \le \sigma$ and $r(\sigma_0 + \frac{\delta}{2}) - r(\sigma_0 - \frac{\delta}{2}) > 0$. Set $r = r(\sigma_0 + \frac{\delta}{2}) - r(\sigma_0 - \delta_2)$. From the choice of σ_0 and from (14) there follows the estimate

$$\int_{\bar{\bar{t}}}^{\infty} \int_0^\sigma x^\alpha(t-s)dr(s)dt > \sum_{n=1}^{\infty} \left[\int_{t_m+n\tau+\sigma_0-\frac{\delta}{2}}^{t_m+n\tau+\sigma_0+\frac{\delta}{2}} \int_0^\sigma x^\alpha(t-s)dr(s)dt \right] >$$

$$> \sum_{n=1}^{\infty} \left[\int_{t_m+n\tau+\sigma_0-\frac{\delta}{2}}^{t_m+n\tau+\sigma_0+\frac{\delta}{2}} \int_{\sigma_0-\frac{\delta}{2}}^{\sigma_0+\frac{\delta}{2}} x^\alpha(t-s)dr(s)dt \right] > \sum_1^\infty \delta(\frac{c}{3} - \varepsilon)^\alpha \cdot r$$

Hence $\int_{\bar{\bar{t}}}^\infty \int_0^\sigma x^\alpha(t-s)dr(s)dt = \infty$ which contradicts what was proved above.

Therefore, $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$. ◊

THEOREM 3. *Let conditions* (H) *hold,* $\varepsilon = -1, \tau(t) \le 0$ *eventually and*

$$1 < p_1 \le p(t) \le p_2 \tag{15}$$

Then, if $x(t)$ *is a nonoscillating eventual solution of* (1), *then*

$$\lim_{t\to\infty} x(t) = 0$$

Proof. Let $x(t) > 0$ eventually. By Lemma 1 $\lim_{t\to\infty} z(t) = 0$ or $\lim_{t\to\infty} z(t) = \infty$. If $\lim_{t\to\infty} z(t) = \infty$, then from (4) and from (15) we obtain the inequality $x(t) > p(t).x(t-\tau(t)) > x(t-\tau(t))$. Set $\gamma(t) = -\tau(t)$. Then $\gamma(t) \ge 0$ eventually and the inequality $x(t) > x(t+\gamma(t))$ holds. From $\lim_{t\to\infty} z(t) = \infty$ it follows that $\lim_{t\to\infty} x(t) = \infty$ and then there exists a sequence $\{t_n\}^\infty$, such that

$\lim_{t\to\infty} t_n = \infty$ and $\max_{[t_1,t_n+\gamma(t_n)]} x(s) = x(t_n + \gamma(t_n))$, which obviously contradicts the last inequality. Hence $\lim_{t\to\infty} z(t) = 0$. As in the proof of Theorem 1 we obtain that $\liminf_{t\to\infty} x(t) = 0$. Let $\limsup_{t\to\infty} x(t) > 0$. Then there exist sequences $\{\alpha_n\}^\infty$ and $\{\beta_n\}_1^\infty$ such that $\lim_{t\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \infty$, $\lim_{n\to\infty} x(\alpha_n + \gamma(\alpha_n)) = c$ and $\lim_{n\to\infty} x(\beta_n + \gamma(\beta_n)) = 0$. for any $\varepsilon > 0$ there exists a positive integer N such that for $n > N$ we have $x(\alpha_n + \gamma(\alpha_n)) > c - \varepsilon$, $x(\beta_n + \gamma(\beta_n)) < \varepsilon$ and $x(t) < c + \varepsilon$ for t large enough. Choose the positive integers k and m so that $\alpha_k < \beta_m$, and estimate the difference $z(\beta_m) - z(\alpha_k)$:

$$z(\beta_m) - z(\alpha_k) =$$

$$x(\beta_m) - p(\beta_m)x(\beta_m + \gamma(\beta_m)) - x(\alpha_k) + p(\alpha_k) \cdot x(\alpha_k + \gamma(\alpha_k)) >$$

$$> p(\alpha_k)x(\alpha_k + \gamma(\alpha_k)) - p(\beta_m) \cdot x(\beta_m + \gamma(\beta_m)) - x(\alpha_k) >$$

$$> p_1(c - \varepsilon) - p_2\varepsilon - c - \varepsilon = (p_1 - 1)c - \varepsilon(1 + p_1 + p_2) .$$

We obtained that for suitably chosen k and m the following inequality holds

$$z(\beta_m) - z(\alpha_k) > (p_1 - 1)c - \varepsilon(1 + p_1 + p_2).$$

From (15) it follows that for ε small enough we have $z(\beta_m) - z(\alpha_k) > 0$ and since $\beta_m > \alpha_k$, then the inequality $z(\beta_m) > z(\alpha_k)$ contradicts the fact that $z(t)$ is an eventually decreasing function. Therefore, $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$. The case $x(t) < 0$ is considered analogously. ◇

THEOREM 4. *Let conditions* (H) *hold,* $\varepsilon = -1$, $\tau(t) \geq 0$ *eventually and*

$$1 \leq p(t) \leq p \tag{16}$$

Then, if $x(t)$ *is a nonoscillating eventual solution of* (1), *then*

$$\lim_{t\to\infty} |x(t)| = \infty$$

Proof. Let $x(t) > 0$ eventually. By virtue of Lemma 1 either assertion (7), or assertion (8) is valid. If (7) is valid, as in the proof of Theorem 1 it follows that $\lim_{t\to\infty} x(t) = \infty$. Let assertion (8) be valid. As in the proof of Theorem 1 it is shown that $\liminf_{t\to\infty} x(t) = 0$. Then there exists a sequence $\{t_n\}^\infty$, such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} x(t_n) = 0$ and

$$\min_{[t_1,t_n]} x(s) = x(t_n) \tag{17}$$

Since (8) is valid, then $z(t) > 0$ eventually and from (16) we obtain the inequality $x(t) > p(t) \cdot x(t - \tau(t)) \geq x(t - \tau(t))$. Consequently, the inequality $x(t_n) > x(t_n - \tau(t_n))$ holds for sufficiently large n, which contradicts (17). Hence under the condition of theorem 4 of the assertions (7) and (8) only (7) is valid and $\lim_{t\to\infty} x(t) = \infty$. Analogously the case is considered when $x(t) < 0$ eventually. ◊

COROLLARY 1. *Let the conditions* (H), (16) *hold,* $\varepsilon = -1$ *and* $\tau(t) \geq 0$ *eventually. Then each bounded solution of (1) oscillates.*

Further on in the work we shall investigate the case when $\varepsilon = 1$.

THEOREM 5. *Let the conditions* (H), (9) *hold and* $\varepsilon = 1$. *Then each bounded nonoscillating solution of (1) tends to zero.*

Proof. Let $x(t) > 0$ eventually. From H6 and H9 it follows that $\ddot{z}(t) > 0$ eventually, hence $\dot{z}(t)$ is an eventually increasing function. If $\dot{z}(t) > 0$ eventually, then $\lim_{t\to\infty} z(t) = \infty$, which contradicts the fact that $x(t)$ is a bounded solution of (1). Hence $\dot{z}(t) < 0$ eventually and $z(t)$ is an eventually decreasing function. As in Theorem 1 it is shown that $\liminf_{t\to\infty} x(t) = 0$. Let $\limsup_{t\to\infty} x(t) = c$ and suppose that $c > 0$. Then there exist sequences $\{\alpha_n\}_1^\infty$ and $\{\beta_n\}_1^\infty$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \infty$, $\lim_{n\to\infty} x(\beta_n) = c$ and $\lim_{n\to\infty} x(\alpha_n) = 0$. For any $\varepsilon > 0$ for sufficiently large $n \in \mathbb{N}$ we have $x(\alpha_n) < \varepsilon$, $x(\beta_n) > c - \varepsilon$ and $x(t) < c + \varepsilon$ for sufficiently large t. Choose the positive integers k and m so that $\alpha_k < \beta_m$ and estimate the difference $z(\beta_m) - z(\alpha_k)$

$$z(\beta_m) - z(\alpha_k) =$$

$$x(\beta_m) + p(\beta_m)x(\beta_m - \tau(\beta_m)) - x(\alpha_k) + p(\alpha_k) \cdot x(\alpha_k - \tau(\alpha_k)) >$$

$$> x(\beta_m) - x(\alpha_k) - p(\alpha_k)x(\alpha_k - \tau(\alpha_k)) > c - \varepsilon - \varepsilon - p(c + \varepsilon) =$$

$$= (1 - p) \cdot c - \varepsilon(2 + p)$$

We obtained for suitably chosen k and m the inequality

$$z(\beta_m) - z(\alpha_k) > (1-p)\cdot c - \varepsilon(2+p)$$

From (9) it follows that for ε small enough $z(\beta_m) - z(\alpha_k) > 0$, which contradicts the fact that $z(t)$ is an eventually decreasing function. Hence $c = 0$, i.e. $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$. The case when $x(t) < 0$ eventually is considered analogously. ◊

REMARK 3. Under the conditions of Theorem 5 no constraints were imposed on the sign of the function $\tau(t)$, i.e. it may oscillate. A stronger result is obtained when $\tau(t) \geq 0$ eventually, i.e. for equations with retarded argument.

THEOREM 6. *Let the conditions* (H), (9) *hold,* $\varepsilon = 1$ *and* $z(t) \geq 0$ *eventually. Then, if* $x(t)$ *is a nonoscillating eventual solution of* (1), *then* $\lim_{t\to\infty} x(t) = 0$ or $\lim_{t\to\infty} |x(t)| = \infty$.

Proof. Let $x(t) > 0$ eventually. If $x(t)$ is a bounded solution of (1), then from Theorem 5 it follows that $\lim_{t\to\infty} x(t) = 0$. Let $x(t)$ be an unbounded solution of (1). Then the function $z(t)$ is unbounded too, i.e. $\limsup_{t\to\infty} z(t) = \infty$. From the fact that $\dot{z}(t)$ is an eventually increasing function it follows that either $\dot{z}(t) > 0$ eventually or $\dot{z}(t) < 0$ eventually. If $\dot{z}(t) < 0$ eventually, then the function $z(t)$ is eventually decreasing, which contradicts the fact that $z(t)$ is a positive, unbounded function. Hence $\dot{z}(t) > 0$ eventually and $\lim_{t\to\infty} z(t) = \infty$. Let $\liminf_{t\to\infty} x(t) = d$ and suppose that $d < \infty$. There exists a sequence $\{\alpha_n\}_1^\infty$ such that $\lim_{n\to\infty} = \infty$ and $\lim_{n\to\infty} x(\alpha_n) = d$. Hence for sufficiently large n the inequality $x(\alpha_n) < 2d$ holds. Define the sequence $\{\beta_n\}_1^\infty$, in the following way: $\beta_n = \alpha_n - \tau(\alpha_n)$. Obviously $\beta_n < \alpha_n$ and $\lim_{n\to\infty} \beta_n = \infty$. Consider the difference $z(\alpha_n) - p(\beta_n)z(\beta_n)$:

$$z(\alpha_n) - p(\beta_n)\cdot z(\beta_n) = x(\alpha_n) - p(\alpha_n)\cdot p(\beta_n)\cdot x(\beta_n - \tau(\beta_n)) < x(\alpha_n)$$

Then

$$x(\alpha_n) > z(\alpha_n) - p(\beta_n)\cdot z(\beta_n) > z(\alpha_n) - pz(\alpha_n) = (1-p)z(\alpha_n)$$

Thus we obtained the inequality

$$z(\alpha_n) < \frac{x(\alpha_n)}{1-p} < \frac{2d}{1-p}$$

The last inequality contradicts the fact that $\lim_{t\to\infty} z(t) = \infty$.

Hence $\liminf_{t\to\infty} x(t) = \infty$, i.e. $\lim_{t\to\infty} x(t) = \infty$. The case when $x(t) < 0$ is considered analogously. $\diamond$

THEOREM 7. *Let conditions* (H) *hold,* $\varepsilon = 1$, $p(t) \equiv 1$ *and* $\tau(t) \equiv \tau$. *Then each nonoscillating, unbounded solution of (1) tends to infinity.*

Proof. Let $x(t)$ be an eventually positive, unbounded solution of (1). Then the function $z(t)$ is unbounded too. from the fact that $\dot{z}(t)$ is an eventually increasing function it follows that either $\dot{z}(t) > 0$ eventually or $\dot{z}(t) < 0$ eventually. If $\dot{z}(t) < 0$ eventually, then as in Theorem 6 we obtain a contradiction. Hence $\dot{z}(t) > 0$ eventually and $z(t)$ is an eventually increasing function. Since the function $\dot{z}(t)$ is eventually increasing, then there exists a positive constant d such that $\dot{z}(t) \geq d > 0$ eventually. Define the function $w(t)$ in the following way $w(t) = z(t) - z(t-\tau)$. Then

$$w(t) = x(t) - x(t-2\tau) \tag{18}$$

We sum up the equalities $w(t+2k\tau) = x(t+2k\tau) - x(t+2(k-1)\tau)$, $k = 1, \ldots n$ and obtain

$$\sum_{k=1}^{n} w(t+2k\tau) = x(t+2n\tau) - x(t) \tag{19}$$

Let $\liminf_{t\to\infty} x(t) = c$ and $0 < c < \infty$. The eventually the inequality $x(t) > \frac{c}{2}$ holds. From the definition of $w(t)$ it follows that for any positive integer k there exists a point α_k such that $t + (2_k - 1)\tau \leq \alpha_k \leq t + 2k\tau$ and such that $w(t+2k\tau) = \tau \cdot \dot{z}(\alpha_k)$. Hence for $t \geq \bar{t}$, where $\bar{t}$ is a sufficiently large number, the inequality $w(t+2k\tau) \geq d\tau$ holds. From (19) and from the last inequality we obtain that for any $t \geq \bar{t}$ we have $x(t+2n\tau) \geq nd\tau$. Choose the sequence $\{t_k\}_1^\infty$, so that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = c$. Hence for sufficiently large k the inequality $x(t_k) < 2c$ eventually holds. Let n be a positive integer such that $nd\tau > 2c$. For that fixed n choose the positive integer m such that $t_m - 2n\tau > \bar{t}$. Set $\bar{\bar{t}} = t_m - 2n\tau$. Then $x(\bar{\bar{t}} + 2n\tau) = x(t_m) < 2c$. On the other hand, from the choice of n and $\bar{\bar{t}}$ it follows that $x(\bar{\bar{t}} + 2n\tau) > nd\tau >$

2c. The contradiction obtained shows that either $c = 0$ or $c = \infty$. Suppose that $c = 0$, i.e. $\liminf_{t\to\infty} x(t) = 0$. Then there exists a sequence $\{\beta_n\}_1^\infty$ such that $\lim_{n\to\infty} \beta_n = \infty$, $\lim_{n\to\infty} x(\beta_n) = 0$ and $\min_{[\beta_1,\beta_n]} x(s) = x(\beta_n)$. From the definition of $w(t)$ and from the fact that $z(t)$ is an eventually increasing function it follows that $w(t) > 0$ eventually. From (18) we obtain that $x(t) > x(t - 2\tau)$ eventually. This inequality, however, contradicts the relation $\min_{[\beta_1,\beta_n]} x(s) = x(\beta_n)$. Hence $\liminf_{t\to\infty} x(t) = \infty$ and $\lim_{t\to\infty} x(t) = \infty$. The case when $x(t)$ is an eventually negative solution of (1) is considered analogously. ◇

THEOREM 8. *Let the conditions* (H) and (15) *hold,* $\varepsilon = 1$ *and* $\tau(t) \le 0$ *eventually. Then if* $x(t)$ *is a nonoscillating eventual solution of (1), then* $\lim_{t\to\infty} x(t) = 0$ *or* $\lim_{t\to\infty} |x(t)| = \infty$.

Proof. Let $x(t)$ be an eventually positive solution of (1). Then $\ddot{z}(t) > 0$ eventually and $\dot{z}(t)$ is an eventually increasing function. Hence either $\dot{z}(t) < 0$ eventually of $\dot{z}(t) > 0$ eventually. Let $\dot{z}(t) > 0$ eventually. Then $\lim_{n\to\infty} z(t) = \infty$. Let $\liminf_{t\to\infty} x(t) = d$ and suppose that $d < \infty$. There exists a sequence $\{\alpha_n\}_1^\infty$, such that $\lim_{n\to\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} x(\alpha_n - \tau(\alpha_n)) = d$. Hence for sufficiently large $n \in \mathbb{N}$ the inequality $x(\alpha_n - \tau(\alpha_n)) < 2d$ holds. Define the sequence $\{\beta_n\}_1^\infty$, in the following way: β_n is some or the solutions of the equation $\beta_n - \tau(\beta_n) = \alpha_n$ (the existence of a solution of this equation follows from condition H2). From $\tau(t) \le 0$ it follows that $\beta_n \le \alpha_n$. from H2 it follows that $\lim_{t\to\infty} \beta_n = \infty$. We estimate the difference $z(\alpha_n) \cdot p(\beta_n) - z(\beta_n)$:

$$z(\alpha_n) \cdot p(\beta_n) - z(\beta_n) =$$
$$p(\alpha_n) \cdot p(\beta_n) \cdot x(\alpha_n - \tau(\alpha_n)) - x(\beta_n) < p_2^2 x(\alpha_n - \tau(\alpha_n))$$

Then

$$p_2^2 x(\alpha_n - \tau(\alpha_n)) > z(\alpha_n) \cdot p(\beta_n) - z(\beta_n) > p_1 \cdot z(\beta_n) - z(\beta_n) =$$
$$(p_1 - 1) \cdot z(\beta_n)$$

Thus we obtained the inequality

$$z(\beta_n) < \frac{p_2^2}{p_1 - 1} \cdot x(\alpha_n - \tau(\alpha_n)) < \frac{2d \cdot p_2^2}{p_1 - 1}.$$

Since $\lim_{n\to\infty}\beta_n = \infty$, then the last inequality contradicts the fact that $\lim_{t\to\infty} z(t) = \infty$. Hence $\liminf_{t\to\infty} x(t) = \infty$ and $\lim_{t\to\infty} x(t) = \infty$. Let $\dot{z}(t) < 0$ eventually. Then $z(t)$ is an eventually decreasing function. As in Theorem 1 it is shown that $\liminf_{t\to\infty} x(t) = 0$. Let $\limsup_{t\to\infty} x(t) = c$ and suppose that $c > 0$. There exist sequences $\{\alpha_n\}_1^\infty$ and $\{\beta_n\}_1^\infty$ such that $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = \infty$, $\lim_{n\to\infty} x(\beta_n - \tau(\beta_n)) = c$ and $\lim_{n\to\infty} x(\alpha_n - \tau(\alpha_n)) = 0$. For any $\varepsilon > 0$ for sufficiently large $n \in \mathbb{N}$ we have $x(\alpha_n - \tau(\alpha_n)) < \varepsilon$, $x(\beta_n - \tau(\beta_n)) > c-\varepsilon$ and $x(t) < c+\varepsilon$ for sufficiently large t. Choose the positive integers k and m so that $\alpha_k < \beta_m$ and estimate the difference $z(\beta_m) - z(\alpha_k)$:

$$z(\beta_m) - z(\alpha_k) =$$

$$x(\beta_m) + p(\beta_m)x(\beta_m) - \tau(\beta_m)) - x(\alpha_k) - p(\alpha_k)x(\alpha_k - \tau(\alpha_k)) >$$
$$> p(\beta_m)\cdot x(\beta_m - \tau(\beta_m)) - x(\alpha_k) - p(\alpha_k)\cdot x(d_k - \tau(\alpha_k)) >$$
$$> p_1(c-\varepsilon) - c - \varepsilon - p_2\cdot\varepsilon = (p_1 - 1)\cdot c - (1 + p_1 + p_2)\cdot\varepsilon$$

We obtained for the chosen k and m the inequality

$$z(\beta_m) - z(\alpha) > (p_1 - 1)\cdot c - (1 + p_1 + p_2)\cdot\varepsilon$$

From (15) it follows that for ε small enough we have $z(\beta_m) - z(\alpha_k) > 0$ which contradicts the fact that $z(t)$ is an eventually decreasing function. Hence $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$. ◇

THEOREM 9. *Let the conditions* (H), *(15) hold,* $\varepsilon = 1$ *and* $\tau(t) \geq 0$ *eventually. Then each bounded nonoscillating solution of* (1) *tends to 0.*

The proof of Theorem 9 is analogous to the proof of Theorem 8.

4. Comments.

In Theorems 1, 6 and 8 we claim that if $X(t)$ is a nonoscillating solutions of (1), then $\lim_{t\to\infty} x(t) = 0$ or $\lim_{t\to\infty} |x(t)| = \infty$. We should emphasize that for each of these theorems examples can be given in which both possibilities are realized. For instance, let us dwell on Theorem 1.

EXAMPLE 1. Consider the equation

$$[x(t) - \frac{t-1}{2t}x(t-1)]'' - (1-\frac{1}{2})^3 x^3(t-1) = 0$$

The above equation is a particular case of (1) for $p(t) = \frac{t-1}{2t}$, $\tau(t) \equiv 1$, $\sigma(t) \equiv 2$, $\varepsilon = -1$, $f(u) = u^3$ and

$$r(t,s) = (1-\frac{1}{2})^3 \cdot e(s-1), \text{where } e(s) = \begin{cases} 0 & s \leq 0 \\ 1 & s > 0 \end{cases}$$

A straightforward verification yields that $x(t) = \frac{1}{t}$ is a solution of this equation and $\lim_{t\to\infty} x(t) = 0$.

EXAMPLE 2. Consider the equation

$$[x(t) - \frac{\exp(1)}{4} \cdot x(t-1)]'' - \frac{3\exp(1)}{4} \cdot x(t-1) = 0$$

This equation is a particular case of (1) for $p(t) = \frac{1}{4}\exp(1)$,

$$\tau(t) \equiv 1,\ \sigma(t) \equiv 2,\ \varepsilon = -1,\ f(u) = u \text{ and } r(t,s) = \frac{3}{4}(1) \cdot e(s-1)\ ,$$

where $e(s)$ is the function defined above.

A straightforward verification yields that $x(t) = e^t$ is a solution of this equation and $\lim_{t\to\infty} x(t) = \infty$.

It is immediately seen that in both examples all conditions of Theorem 1 are met. One can easily give examples illustrating the other theorems.

In order to have a complete classification of the nonoscillating solutions of (1) there remain the following two unsolved questions:

1. Let conditions (H) hold $\varepsilon = 1$, $p(t) \equiv 1$ and $\tau(t) \equiv \tau$. Let $x(t)$ be a bounded nonoscillating solution of (1). Is the equality $\lim_{t\to\infty} x(t) = 0$ valid?

2. Let the conditions (H) and (15) hold, $\varepsilon = 1$ and $\tau(t) \geq 0$ eventually. Let $x(t)$ be an unbounded nonoscillating solution of (1). Is the equality $\lim_{t\to\infty} |x(t)| = \infty$ valid?

We shall note that the first problem is not solved even for the simple linear equation

$$[x(t) + x(t-\tau)]'' - q(t) \cdot x(t-\sigma) = 0 \qquad (20)$$

where $\int_{t_0}^{\infty} q(t)dt = \infty$.

We shall point out two principal cases when the question can be answered affirmatively.

1.1 Let $q(t) \geq q > 0$.

Since $\ddot{z}(t) > 0$ eventually, then $\dot{z}(t)$ is an eventually increasing function, hence either $\dot{z}(t) > 0$ eventually or $\dot{z}(t) < 0$ eventually. If we suppose that $\dot{z}(t) > 0$ eventually, then $\lim_{t\to\infty} z(t) = \infty$ which contradicts the fact that $x(t)$ is a bounded solution of (20). Hence $\dot{z}(t) < 0$ eventually and $z(t)$ is a positive eventually decreasing function. Since $\dot{z}(t)$ is an eventually increasing negative function, then there exists the finite limit $\lim_{t\to\infty} \dot{z}(t) = c$. Then, integrating the equality $\ddot{z}(t) = q(t)x(t-\sigma)$ from t_1 to t, and passing to the limit as t tends to infinity, we obtain that $q(t)\cdot x(t-\sigma) \in L_1[t_1,\infty)$. From the inequality $q(t) \geq q > 0$ it follows that $x(t) \in L_1[t_1,\infty)$, hence $z(t) \in L_1[t_1,\infty)$. Since $z(t)$ is an eventually decreasing function, then $\lim_{t\to\infty} z(t) = 0$ and therefore $\lim_{t\to\infty} x(t) = 0$. ◇

1.2 Let $q(t) = t^{\alpha}$, where $-1 \leq \alpha < 0$.

As in 1.1 it is established that $z(t)$ is an eventually decreasing positive function, hence there exists the finite limit $\lim_{t\to\infty} z(t) = c \geq 0$. Suppose that $c > 0$. Then $x(t) + x(t-\tau) > c$ eventually. Consider an arbitrary interval $[t_1, t_1+2\tau]$, where t_1 is a sufficiently large number such that for $t \geq t_1$ the inequality $z(t) > c$ should hold. Let $\beta \in [t_1, t_1+\tau]$. Hence for at least one of the two points β and $\beta+\tau$ the inequality $x(t) > \frac{c}{2}$ is valid. Let $E_0 = \{t | t \in [t_1, t_1+2\tau],\ x(t) > \frac{c}{2}\}$. Obviously, the set E_0 is measurable as mes $E_0 \geq \tau$. Divide the set $[t_1,\infty)$ into intervals $[t_1+2k\tau,\ t_1+2(k+1)\tau]$, $k = 0,1,2,\ldots$ and let $E_k = \{t | t \in [t_1+2k\tau,\ t_1+2(k+1)\tau],\ x(t) > \frac{c}{2}\}$. Each of the sets E_k is measurable and mes $E_k \geq \tau$. Let $E = \{t \in [t_1,\infty),\ x(t) > \frac{c}{2}\}$. Then mes $E = \sum_{k=0}^{\infty} mes E_k$. Let $E_\sigma = \{t | t-\sigma \in E\}$. From the form of $q(t)$ it follows that $\int_{t_1}^{\infty} q(t)\cdot x(t-\sigma)dt \geq \int_{E_\sigma} q(t)x(t-\sigma)dt = \infty$, which contradicts th existence of the finite limit $\lim_{t\to\infty} \dot{z}(t)$. Hence $c = 0$ and $\lim_{t\to\infty} x(t) = 0$. ◇

What concerns the second problem, we shall emphasize that it is not solved even for the simplest linear equation

$$[x(t) + p,\ x(t-\tau)]'' - qx(t-\sigma) = 0\ ,$$

where $p > 1,\ q,\ \tau, \sigma > 0$.

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