ON THE PROBLEM OF DISCRETE EXTRAPOLATION OF A BAND-LIMITED SIGNAL(*)

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SOMMARIO. - *Si considera il sistema lineare equivalente al problema della estrapolazione discreta di un segnale a banda limitata. Si dimostra che la matrice di iterazione del metodo di Gerchberg-Papoulis, metodo iterativo applicato a questo sistema, è una matrice convergente. Si verifica inoltre che la convergenza di tale metodo è così lenta da rendere tale metodo praticamente inutilizzabile.*

SUMMARY. - *We consider the linear system equivalent to the problem of discrete extrapolation of a band-limited signal. We show that the iteration matrix of the Gerchberg-Papoulis method, iterative method applied to this system, is a convergent matrix. We verify, moreover, that the convergence of such method is so slow as to make this method practically inapplicable.*

1. Introduction.

Let $g : [-T,T] \to \mathbb{C}$ be a known segment of an ω_0-band-limited signal of finite energy, i.e. of a function $f : \mathbf{R} \to \mathbb{C}$, satisfying the two conditions:

i) $\int_{-\infty}^{+\infty} |f(t)|^2 dt < +\infty$,

ii) the Fourier transform of f is zero in $\mathbf{R} - [-\omega_0, \omega_0]$.

The "continuous extrapolation problem" is to determine f out of $[-T,T]$. The problem is solvable ([1], pp. 184-186). The most widely known resolutive method is the Gerchberg-Papoulis iterative procedure ([1], pp. 244-248, [2], [3]); it may be described as follows. Let

$$f_0(t) = \begin{cases} g(t), & |t| \le T \\ 0, & |t| > T \end{cases} ;$$

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we determine the FT (Fourier Transform) $F_{n-1}(t)$ of $f_{n-1}(t)$ and its truncation

$$\Phi_n(\omega) = F_{n-1}(\omega) \cdot p_{\omega_0}(\omega) ,$$

where

$$p_c(x) = \begin{cases} 1, & |x| \le c \\ 0, & |x| > c \end{cases}$$

is the unitary impulse of duration $2c$; we determine the IFT (Inverse Fourier Transform) $\varphi_n(t)$ of $F_n(t)$; let

$$f_n(t) = g(t) + (1 - p_T(t)) \cdot \varphi_n(t) .$$

The sequence f_n converges to f.

The "discrete extrapolation problem" is to determine an L-band-limited sequence of complex numbers $(f(j))_{-N \le j \le N}$ $(L < N)$, given the "segment" $(g(j))_{-L \le j \le L}$, i.e.:

$$a) \qquad f(j) = g(j), -L \le j \le L .$$

Because of band-limitation, discrete Fourier Transform (DFT) of f is zero out of $[-L, L]$; therefore we have:

$$b) \quad \sum_{j=-N}^{N} f(j) \cdot \exp(-\frac{2\pi i}{M} \cdot jk) = 0, L < |k| \le N \ (M = 2N + 1) .$$

If $z(k)$, $-N \le k \le N$, denote the DFT of $f(j)$, using the discrete inversion (DIFT) formula, the previous conditions a), b) become:

$$a') \qquad \sum_{k=-L}^{L} z(k) \cdot \exp(\frac{2\pi i}{M} \cdot kj) = g(j), \ -L \le j \le L ,$$

$$b') \qquad f(j) = \sum_{k=-L}^{L} z(k) \cdot \exp(\frac{2\pi i}{M} \cdot kj), -N \le j \le N .$$

The determinant of linear system a′) is the Vandermonde determinant of the numbers $\exp(\frac{2\pi i}{M} j)$, $-L \le j \le L$, all pairwise distinct; then a') determines the values $z(k)$, $-L \le k \le L$, that, substituted in b') give the problem's solution.

In 1983 Sanz and Huang [4] suggested the following conjecture: the solution of the discrete extrapolation problem concerning a band-limited continuous signal of finite energy, when samples $g(j)$ in $[-T, T]$ are given, tends to the solution of the continuous extrapolation problem, for the sampling period that tends to zero and DFT's length M that tends to $+\infty$. Such conjecture was proved in 1985 by Schlebush and Splettstösser [5] and by Xingwel and Xiangen [6].

2. Discrete extrapolation by the Gerchberg-Papoulis method.

Let

$$\begin{aligned} &x \in \mathbb{C}^M, \quad x = (f(j))_{-N\leq j\leq N} = (x_{-1}, x_0, x_1), \; x_{-1}, x_1 \in \mathbb{C}^{N-L}, \\ &g \in \mathbb{C}^{2L+1}, \quad g = (g(j))_{-L\leq j\leq L}, \\ &b \in \mathbb{C}^M, \quad b = (0, g, 0). \end{aligned}$$

Let $\omega = \exp\left(-\frac{2\pi i}{M}\right)$ and let

$$\Omega = (\omega_{jk}) \in \mathbb{C}^{M\times M}, \omega_{jk} = \frac{1}{\sqrt{M}}\omega^{jk}, \; j, k = -N, \dots, N \tag{1}$$

be the Fourier matrix, and

$$T = \begin{pmatrix} 0_{N-L} & 0 & 0_{N-L} \\ 0 & I_{2L+1} & 0 \\ 0_{N-L} & 0 & 0_{N-L} \end{pmatrix} \tag{2}$$

be the "truncation matrix".

Given vectors $x_{-1}^{(0)}, x_1^{(0)}$ arbitrarily, we determine the DFT $F^{(n-1)} = \frac{1}{\sqrt{M}} \cdot \Omega x^{(n-1)}$ of $x^{(n-1)}$, its truncation $F_T^{(n-1)} = TF^{(n-1)}$, the DIFT $\tilde{x}^{(n)} = \sqrt{M} \cdot \Omega^H F_T^{(n-1)}$ of $F_T^{(n-1)}$; then we set the central components of $\tilde{x}^{(n)}$, with indices in $[-L, L]$, equal to the corresponding components in g.

So we obtain the vector (n-th iteration):

$$x^{(n)} = (I - T)\Omega^H T\Omega x^{(n-1)} + b .$$

Assuming

$$C = \Omega^H T \Omega \,, \tag{3}$$

$$H = (I - T)C \,, \tag{4}$$

we have:

$$x^{(n)} = Hx^{(n-1)} + b \,. \tag{5}$$

It follows that the Gerchberg-Papoulis method appears as an iterative method, with iteration matrix H, applied to the linear system:

$$(I - H)x = b \,. \tag{6}$$

Now we shall show that the system (6) may be obtained directly from conditions a) and b), used in §1 to define the discrete extrapolation problem.

In fact, with notations previously introduced, conditions a) and b) become:

$$\begin{cases} (I - T)\Omega x = 0 \\ Tx = b \end{cases} ,$$

which is equivalent to:

$$\begin{cases} (I - C)x = 0 \\ Tx = b \end{cases} . \tag{7}$$

Multiplying the first equality of (7), on the left, by $I - T$, we obtain:

$$\begin{cases} (I - T)(I - C)x = 0 \\ Tx = b \end{cases} . \tag{8}$$

System (8) is equivalent to (7) if and only if, assuming that τ and γ are the endomorphisms of $\mathbb{C}^M$ associated to $I - T$, $I - C$, respectively, we have:

$$\text{Ker } \tau \cap \text{Im } \gamma = \{0\} \,. \tag{9}$$

We prove (9): if $[e_1, \ldots, e_M]$ is the canonical basis of $\mathbb{C}^M$, we have:

$$\text{Ker } \tau = \langle e_{N-L+1}, \ldots, e_{N+L+1} \rangle \,,$$

$$\text{Im } \tau = \langle e_1, \ldots, e_{N-L}, e_{N+L+2}, \ldots, e_M \rangle \,.$$

Assuming that $\tilde{\omega}$ is the automorphism of $\mathbb{C}^M$ associated to Ω, we consider:

$$\tilde{\omega}(\text{Ker } \tau \cap \text{Im } \gamma) = \tilde{\omega}(\text{Ker } \tau) \cap \text{Im } \tau \circ \tilde{\omega} = \tilde{\omega}(\text{Ker } \tau) \cap \text{Im } \tau \ .$$

To determine the elements of such intersection, we look for vectors $(0, x_0, 0)$, such that the central components of $\Omega \begin{pmatrix} 0 \\ x_0 \\ 0 \end{pmatrix}$ are 0, i.e.

$$\Omega_L x_0 = 0 \ ,$$

where $\Omega_L \in \mathbb{C}^{(2L+1)\times(2L+1)}$ is the central matrix of Ω. As observed before, $\det(\Omega_L) \neq 0$; condition (9) follows.

So (8) is equivalent to (7); we conclude easily that it is also equivalent to

$$\begin{cases} (I - H)x = b \\ Tx = b \end{cases} . \tag{10}$$

In what follows it is convenient to write matrix C in the form:

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} , \tag{11}$$

where:

$$C_{11}, C_{13}, C_{31}, C_{33} \in \mathbb{C}^{(N-L)\times(N-L)}, C_{22} \in \mathbb{C}^{(2L+1)\times(2L+1)} \ ,$$

$$C_{12}, C_{32} \in \mathbb{C}^{(N-L)\times(2L+1)}, C_{21}, C_{23} \in \mathbb{C}^{(2L+1)\times(N-L)} \ .$$

Then we have

$$H = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} \end{pmatrix} , \tag{12}$$

and, therefore, assuming

$$H_1 = \begin{pmatrix} C_{11} & C_{13} \\ C_{31} & C_{33} \end{pmatrix} , \tag{13}$$

we can write the discrete Gerchberg-Papoulis method, for determining unknown vectors x_{-1}, x_1 in the form:

$$\begin{pmatrix} x_{-1}^{(n)} \\ \\ x_1^{(n)} \end{pmatrix} = H_1 \begin{pmatrix} x_{-1}^{(n-1)} \\ \\ x_1^{(n-1)} \end{pmatrix} + \begin{pmatrix} C_{12} \\ \\ C_{32} \end{pmatrix} g \, . \tag{14}$$

3. On matrix H_1 eigenvalues.

Let $J = [e_M, \dots, e_1]$; because $\Omega^2 = J = J^H$, we have $\Omega^2 T(\Omega^2)^H = JTJ = T$, and, therefore: $\Omega T \Omega^H = \Omega^H T \Omega$. Assuming

$$\tilde{H} = \Omega H \Omega^H \, , \tag{15}$$

we obtain:

$$\tilde{H} = (I - \Omega T \Omega^H)T = (I - C)T = \begin{pmatrix} 0 & -C_{12} & 0 \\ 0 & I_{2L+1} - C_{22} & 0 \\ 0 & -C_{32} & 0 \end{pmatrix} .$$

Then the not zero eigenvalues set of $\tilde{H}$ coincides with the not zero eigenvalues set of the matrix:

$$I_{2L+1} - C_{22} \, . \tag{16}$$

On the other hand such a set coincides with the not zero eigenvalues set of H (cfr. (15)) and, therefore, of H_1.

We exhibit some properties of matrix C, which we shall use afterwards:

i) C has eigenvalues 0 and 1 with respective multiplicities $2(N-L)$ and $2L+1$, because it is similar to T;

ii) the elements of C are

$$c_{jk} = \frac{1}{M} \cdot \left\{ 1 + 2 \cdot \sum_{h=1}^{L} \cos\left[\frac{2\pi}{M} h(j-k)\right] \right\} , \tag{17}$$

$$j, k = -N, \dots, N \, ,$$

and, therefore, C is real and symmetric; in fact, from (1), (2), (3), and assuming $\omega = \exp(-\frac{2\pi}{M}i) = \cos(-2\pi/M) + i \cdot \sin(-2\pi/M)$, we have

$$Mc_{jk} = \sum_{h=-L}^{L} \omega^{hj}\omega^{-hk} = \sum_{h=-L}^{L} \omega^{h(j-k)} = 1+\sum_{h=1}^{L}(\omega^{h(j-k)}+\omega^{-h(j-k)});$$

iii) we have

$$\begin{cases} c_{jk} = c_{j+1,k+1} \\ c_{jN} = c_{j+1,-N} \end{cases}, \quad j,k = -N,\dots,N-1 ,$$

i.e. C is the circulating matrix of the numbers:

$$\alpha_s = \begin{cases} c_{1s}, & s = 1,\dots,N+1 \\ c_{1,M-s+2}, & s = N+2,\dots,M . \end{cases} \tag{18}$$

From these properties we obtain that C_{22} is real, symmetric, and it is equal to the head submatrix of C, of order $2L+1$.

Applying the eigenvalues separation theorem (cf.[7], pp. 326) to matrix C, we have

$$\lambda_{\min}^{(k+1)} \leq \lambda_{\min}^{(k)} \leq \lambda_{\max}^{(k)} \leq \lambda_{max}^{(k+1)}, k = 1,\dots,M-1 ,$$

where $\lambda_{\min}^{(j)}$, $\lambda_{\max}^{(j)}$ are, respectively, the least and the greatest eigenvalue of the head submatrix of C, of order j; because $\lambda_{\min}^{(M)} = 0$, $\lambda_{\max}^{(M)} = 1$, it follows:

$$0 \leq \lambda_{\min}^{(2L+1)} \leq \lambda_{\max}^{(2L+1)} \leq 1 .$$

Because, as we know, $\det(\Omega_L) \neq 0$ and $C_{22} = \Omega_L^H \cdot \Omega_L$, then $\det(C_{22}) \neq 0$. So every eigenvalue λ of C_{22} verifies the condition

$$0 < \lambda \leq 1 .$$

Therefore the results stated on matrix (16) allow to assert that, for every eigenvalue μ of H_1, we have

$$0 \leq \mu < 1 . \tag{19}$$

4. On matrix H_1 spectral radius.

Conditions (19) assure the convergence of the iterative method (14). In order to investigate the convergence's speed and, therefore, the practical effectiveness of this method, we look for a valuation of the spectral radius ρ of matrix H_1.

We assume $p = N - L$, and, at first, we study the problem for $p = 1$ and $p = 2$; for such values of p, we establish an inferior limitation of ρ, which, with L increasing, tends rapidly to 1.

a) $p = 1$.

Making use of (17), (18), we obtain $\rho = \alpha_1 + \alpha_2$, where $\alpha_1 = \frac{2L+1}{2L+3}$, $\alpha_2 = \frac{2}{2L+3} \cdot \cos \frac{\pi}{2L+3}$, and we can easily establish the following limitation:

$$\rho > 1 - \frac{\pi^2}{(2L+3)^3} \; .$$

b) $p = 2$.

As before, making use of (17), (18), we obtain

$$\rho = \alpha_1 + \frac{1}{2} \cdot \left[\alpha_2 + \alpha_4 + \sqrt{(\alpha_2 - \alpha_4)^2 + 4(\alpha_2 + \alpha_3)^2} \right] \; ,$$

where

$$\alpha_1 = \frac{2L+1}{2L+5}, \; \alpha_2 = \frac{2}{2L+5} \left(\cos \alpha + \cos 3\alpha \right) \; ,$$

$$\alpha_3 = -\frac{2}{2L+5} \cdot \left(\cos 2\alpha + \cos 6\alpha \right) \; ,$$

$$\alpha_4 = \frac{2}{2L+5} \cdot \left(\cos 3\alpha + \cos 9\alpha \right) \; , \text{ and } \alpha = \frac{\pi}{2L+5} \; .$$

Because $\alpha_2 - \alpha_4 > 0$ for every L, we can write:

$$\rho = \alpha_1 + \frac{1}{2} \cdot \left[\alpha_2 + \alpha_4 + (\alpha_2 - \alpha_4)\sqrt{1 + 4r^2} \right] \; ,$$

where

$$r = \frac{\alpha_2 + \alpha_3}{\alpha_2 - \alpha_4} = \frac{\cos \alpha + \cos 3\alpha - \cos 2\alpha - \cos 6\alpha}{\cos \alpha - \cos 9\alpha} \; ;$$

from $\inf_{L\in\mathbb{N}} r = \frac{3}{8}$, we have:

$$\rho \geq \alpha_1 + \frac{1}{8}(9\alpha_2 - \alpha_4) > 1 - \frac{123}{2} \cdot \frac{\pi^4}{(2L+5)^3} .$$

Furthermore, we make numerical valuations of the spectral radius of H_1, for some values of L and $p = N - L$. In the following table we write the values of $1-\rho$:

L	$p=1$	$p=2$	$p=3$	$p=10$
1	7.6393 (−2)	1.2525 (−2)	3.3246 (−3)	2.7463 (−5)
2	2.8295 (−2)	1.6975 (−3)	2.0956 (−4)	8.2906 (−8)
3	1.3402 (−2)	3.6714 (−4)	2.4598 (−5)	7.3084 (−10)
10	8.0992 (−4)	9.5301 (−7)	4.5728 (−9)	5.4706 (−19)
100	1.1798 (−6)	3.6422 (−13)	7.2916 (−19)	1.0240 (−48)

5. Conclusions.

Valuations of §4 show that, also for low values of L and N, the spectral radius ρ of the iteration matrix of the Gerchberg-Papoulis method is very close to 1. To reduce the error of 1/10, we need approximately $-1/Log\rho$ iterations; such a number is very large. So, because of the effects of rounding errors, we can assert that the convergence of such a method is, in general, a merely theoretical fact.

References

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