ALMOST PERIODICITY IN OPERATOR ALGEBRAS (*)

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SOMMARIO. - *Operatori Quasi invarianti sul cerchio sono stati studiati da Karel DeLeeuw ([7], [8]). In questo lavoro proponiamo una generalizzazione agli operatori Quasi periodici su un gruppo abeliano localmente compatto qualsiasi. Consideriamo inoltre l'analisi armonica di questi operatori usando le tecniche dell'analisi di Fourier.*

SUMMARY. - *The proper replacement of the almost invariant operators studied by Karel DeLeeuw ([7],[8]) in the case of the circle group has been shown to be what we call as almost periodic operators for any locally compact abelian group. A reasonable harmonic analysis has been carried out on these operators with standard Fourier analytic techniques.*

1. Introduction.

Our interest in Harmonic Analysis for Operators was initiated by a study undertaken by Karel DeLeeuw ([7],[8]) and U.B. Tewari and S. Madan [10]. De Leeuw considered almost invariant operators on Homogeneous Banach spaces on the Circle group and developed a Harmonic Analysis for these operators. In the present work we have been able to develop a similar analysis when G is any locally compact abelian group. In the context of this work it can be seen that when G is non-compact, the proper replacement for almost invariant operators is what we call as almost periodic operators.

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In Section 2 we introduce almost periodic operators and study their basic properties. We prove the main result namely an Approximation Theorem in Section 3. Next we prove the existence of an invariant mean for almost periodic operators and derive certain fundamental properties of the invariant mean using the Approximation Theorem. Following DeLeeuw [7], we consider a Fourier series for almost periodic operators and present a convergence result in this direction.

2. Definitions and Basic Properties.

DEFINITION 2.1. By a Homogeneous Banach space B on a locally compact abelian group G, we mean a Banach space of functions or equivalence classes of functions on G satisfying the following:

i) $f \in D$ implies $R_x f \in B$ for all $x \in G$, where $(R_x f)(y) = f(y-x)$ for $y \in G$.

ii) $\|R_x f\| = \|f\|$ for all $x \in G$, $f \in B$.

iii) $\lim_{x \to 0} \|R_x f - f\| = 0$ for every $f \in B$.

iv) B is closed under multiplication by characters of G.

Our familiar examples are the classical function spaces $L^p(G)$; $1 \leq p < \infty$, $C_0(G)$ (continuous functions vanishing at ∞) and also the character invariant Segal algebras on G.

In general, G will always denote a locally compact abelian group and B, a Homogeneous Banach space on G. $\mathcal{L}(B)$ will denote the algebra of bounded linear operators on B. In the following definition, (i) and (ii) are from DeLeeuw [7] and (iii) is new [(1)]. In fact the class of operators defined in (iii) is what we are interested in.

DEFINITION 2.2. $T \in \mathcal{L}(B)$ is defined to be

i) invariant if $TR_x = R_x T$ for all $x \in G$.

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(1) Our notion of almost periodic operators is different from the one introduced by C. Chou in [5].

ii) almost invariant if $\lim_{x\to 0} \|TR_x - R_xT\| = 0$. That is, $x \to R_{-x}TR_x$ is norm-continuous.

iii) almost periodic if $\{R_{-x}TR_x : x \in G\}$ is relatively compact in $\mathcal{L}(B)$ in the operator norm topology.

Let $\mathcal{L}_0$, $\mathcal{L}_*$, $\mathcal{L}_A$ denote the classes of invariant, almost invariant and almost periodic operators respectively. Let us note the following simple fact which has been mentioned for $\mathcal{L}_0$ and $\mathcal{L}_*$ in [7].

PROPOSITION 2.3. $\mathcal{L}_0, \mathcal{L}_*$ and $\mathcal{L}_A$ are closed subalgebras of $\mathcal{L}(B)$.

Proof. We prove the result for $\mathcal{L}_A$. Take $S_T = \overline{\{R_{-x}TR_x : x \in G\}}$ for $T \in \mathcal{L}_A$. For $T_1, T_2 \in \mathcal{L}_A$, S_{T_1}, S_{T_2} are compact and so is the image of $S_{T_1} \times S_{T_2}$ under the continuous mapping $(U, V) \to U + V$. $S_{T_1+T_2}$ is contained in this image and hence is compact. Thus $\mathcal{L}_A$ is closed under addition and similarly under multiplication. To see that $\mathcal{L}_A$ is closed, let $T_n \in \mathcal{L}_A$ and $T_n \to T \in \mathcal{L}(B)$. For $\varepsilon > 0$, choose N such that $\|T_N - T\| < \varepsilon/3$. Let $\{R_{-x_i}T_NR_{x_i}\}_{i=1}^m$ be a $\varepsilon/3$ net in $\{R_{-x}T_NR_x : x \in G\}$. For $x \in G$, there is a $j, 1 \le j \le m$ such that $\|R_{-x}T_NR_x - R_{-x_j}T_NR_{x_j}\| < \varepsilon/3$. Then

$$\begin{aligned}&\|R_{-x}TR_x - R_{-x_j}TR_{x_j}\| \le \|R_{-x}TR_x - R_{-x}T_NR_x\| + \\&+\|R_{-x}T_NR_x - R_{-x_j}T_NR_{x_j}| + \|R_{-x_j}T_NR_{x_j} - R_{-x_j}TR_{x_j}\| \\&= \|T - T_N\| + \|R_{-x}T_NR_x - R_{-x_j}T_NR_{x_j}\| + \|T - T_N\| < \varepsilon .\end{aligned}$$

Therefore $T \in \mathcal{L}_A$ and $\mathcal{L}_A$ is closed. ◇

The following result is crucial in further investigations of $\mathcal{L}_A$.

PROPOSITION 2.4. $\mathcal{L}_A \subseteq \mathcal{L}_*$.

Proof. For $T \in \mathcal{L}_A$, let $\{R_{-x_j}TR_{x_j}\}_{j=1}^n$ be a $\varepsilon/2$ – net for $\{R_{-x}TR_x : x \in G\}$.

Let $A_j = \{x \in G : \|R_{-x}TR_x - R_{-x_j}TR_{x_j}\| \le \varepsilon/2\}$. First we shall see that A_j is closed. Suppose $x_\alpha \in A_j$ and $x_\alpha \to x$.

For $f \in B$, $\|R_{-x_\alpha}TR_{x_\alpha}f - R_{-x}TR_xf\| =$

$$\begin{aligned}&\|TR_{x_\alpha}f - R_{x_\alpha - x}TR_xf\| \le \|TR_{x_\alpha} - fTR_xf\| + \\&+\|TR_xf - R_{x_\alpha - x}TR_xf\| \to 0\end{aligned}$$

by (iii) of 2.1.

For $\delta > 0$ and $\|f\| \leq 1$, choose α such that

$$\|R_{-x_\alpha}TR_{x_\alpha}f - R_{-x}TR_xf\| < \delta$$

Now $\|R_{-x}TR_xf - R_{-x_j}TR_{x_j}f\|$

$\leq \|R_{-x}TR_xf - R_{-x_\alpha}TR_{x_\alpha}f\| + \|R_{-x_\alpha}TR_{x_\alpha}f - R_{-x_j}TR_{x_j}\| < \delta + \varepsilon/2$

δ being arbitrary, $\|R_{-x}TR_xf - R_{-x_j}TR_{x_j}f\| \leq \varepsilon/2$. Hence A_j is closed.

Since $G = \cup_{j=1}^n A_j$, it follows from Baire Category theorem that we have some A_{j_0}, $x_0 \in A_{j_0}$ and a neighbourhood U of 0 in G such that $x_0 + U \subseteq A_{j_0}$. Now for $x \in U$,

$$\begin{aligned} \|R_{-x}TR_x - T\| &= \|R_{-(x_0+x)}TR_{(x_0+x)} - R_{-x_0}TR_{x_0}\| \\ &\leq \|R_{-(x_0+x)}TR_{(x_0+x)} - R_{-x_{j_0}}TR_{x_{j_0}}\| \\ &\quad + \|R_{-x_{j_0}}TR_{x_{j_0}} - R_{-x_0}TR_{x_0}\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Hence $T \in \mathcal{L}_0$. ◊

COROLLARY 2.5. *If G is compact, then $\mathcal{L}_* = \mathcal{L}_A$.*

Proof. The reverse inclusion $\mathcal{L}_* \subseteq \mathcal{L}_A$ also holds in this case since continuity of $x \to R_{-x}TR_x$ and compactness of G implies the compactness of $\{R_{-x}TR_x : x \in G\}$.

REMARK. If G is non-compact, then $\mathcal{L}_*$ properly contains $\mathcal{L}_A$ at least for certain Homogeneous Banach spaces as will follow from the next proposition since for non-compact groups there exist bounded uniformly continuous functions which are not almost periodic.

For suitable functions φ, the symbol M_φ will denote the operator on B defined as multiplication by φ. The following proposition interrelates the almost periodic behaviour of φ and M_φ.

PROPOSITION 2.6. Let $B = L^p(G)$, $1 \leq p < \infty$ and $\varphi \in L^\infty(G)$

OR

$B = C_0(G)$ and φ be a bounded continuous function on G. Then

i) M_φ is almost invariant if and only if φ is equivalent in $L^\infty(G)$ to a uniformly continuous function on G.

ii) M_φ is almost periodic if and only if φ is equivalent to an almost periodic function.

Proof. By a straightforward computation we have

$$R_x M_\varphi R_{-x} = M_{R_x\varphi} .$$

Now (i) follows from the following observation:

$$\|R_x M_\varphi R_{-x} - M_\varphi\| = \|M_{R_x\varphi} - M_\varphi\| = \|M_{(R_x\varphi-\varphi)}\| = \|R_x\varphi - \varphi\|_\infty$$

(ii) follows from

$$\|R_x M_\varphi R_{-x} - R_y M_\varphi R_{-y}\| = \|M_{R_x\varphi} - M_{R_y\varphi}\| = \|R_x\varphi - R_y\varphi\|_\infty$$

COROLLARY 2.7. *If $B = L^p(G)$, $1 \le p < \infty$ or $B = C_0(G)$, then $\mathcal{L}_A = \mathcal{L}_*$ if and only if G is compact.*

Proof. Follows from 2.6 and the Remark following 2.5.

Finally we come to the relation between compactness and almost periodicity in the classical function spaces.

PROPOSITION 2.8. Let G be compact and $B = L^p(G)$, $1 < p < \infty$. Then any compact operator on B is almost periodic (which is the same as almost invariance).

Proof. It is enough to show that any rank one operator T is almost invariant.

For every $f \in L^p(G)$, let $Tf = (f,g)h$ for some $g \in L^q(G)$, $h \in L^p(G)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $(f,g) = \int f\bar{g}$.

$$\begin{aligned}\text{For } \|f\| \le 1, \|R_xTf - TR_xf\| &= \|(f,g)R_xh - (R_xf,g)h\| \\ &\le \|(f,g)R_xh - (f,g)h\| + \|(f,g)h - (f,R_{-x}g)h\| \\ &\le \|g\|\|R_xh - h\| + \|g - R_{-x}g\|\|h\|\end{aligned}$$

Hence $\lim_{x\to 0} \|R_x T - T R_x\| = 0.$ $\diamondsuit$

For non-compact G, it will be shown in 4.5 that there is no non-zero compact almost periodic operator.

3. An Approximation Theorem.

De Leeuw [7] has shown that when G is compact, the norm closure of the set of finite sums of the form $\sum_{i=1}^{n} M_{\gamma_i} U_{\gamma_i}$, where M_{γ_i} is multiplication by the character γ_i and U_{γ_i} is an invariant operator, is the class of all almost invariant operators. Now when G is non-compact, we show that this norm closure is precisely the class of almost periodic operators. This characterization of almost periodic operators plays the key role in the analysis of this class of operators.

Before getting into the theorem itself let us note down the following equation. The equation is well known. Still we prove it for completeness.

LEMMA 3.1. *$M_\gamma R_x = (x,\gamma) R_x M_\gamma$ for $x \in G$, $\gamma \in \hat{G}$.*

Proof. For $f \in B$, $y \in G$

$$[M_\gamma R_x f](y) = (y,\gamma) f(y-x) = (x,\gamma)(y-x,\gamma) f(y-x)$$
$$= (x,\gamma)[R_x M_\gamma f](y), \text{ as required .}$$

$\diamondsuit$

Although the following lemma is not used in the proof of the Approximation Theorem, this is the right place to prove it for future use.

LEMMA 3.2. *$M_\gamma \cup M_{-\gamma} \in \mathcal{L}_0$ for any $U \in \mathcal{L}_0$ and $\gamma \in G$.*

Proof. For $x \in G$,

$$\begin{aligned} R_x M_\gamma U M_{-\gamma} R_{-x} &= (-x,\gamma) M_\gamma R_x U R_{-x} M_{-\gamma}(x,y), \text{ by 3.1} \\ &= M_\gamma U M_{-\gamma} \end{aligned}$$

Hence $M_\gamma U M_{-\gamma} \in \mathcal{L}_0$. $\diamondsuit$

Now we come to the main theorem.

THEOREM 3.3. *Let G be a locally compact abelian group and B, a Homogeneous Banach space on G. If $T \in \mathcal{L}_A$, then T is the norm limit of finite sums of the form $\sum_{i=1}^{n} M_{\gamma_i} U_{\gamma_i}$ where $\gamma_i \in \hat{G}$ and U_{γ_i} are invariant operators on B.*

Proof.

Step 1: Let $F_T = \overline{\{R_{-x}TR_x : x \in G\}}$. F_T is a compact metric space in the metric given by the operator norm. G acts on F_T as $g(S) = R_{-g}SR_g$ for $g \in G$, $S \in F_T$. Thus every $g \in G$ corresponds to an isometry on F_T. Let G_T be the closure of G in the group of isometries on F_T. The group of isometries on a compact metric space is compact in the topology given by the metric $\rho(S_1 S_2) = \sup_{x \in X} d(S_1(x), S_2(x))$ [See [9], p. 172, for instance].. So G_T is a compact group with composition of operators as the group operation and the metric d defined by $d(g_1, g_2) = \sup_{S \in F_T} \|g_1(S) - g_2(S)\|$. The continuity of the map $g \to g$ from G to G_T follows from the following observation:

$$\begin{aligned} \|g(R_{-x}TR_x) - R_{-x}TR_x\| &= \|R_{-(g+x)}TR_{(g+x)} - R_{-x}TR_x\| \\ &= \|R_{-g}TR_g - T\| . \end{aligned}$$

By almost invariance of T (See 2.4), given $\varepsilon > 0$ there is a neighbourhood V of 0 in G such that $\|R_{-g}TR_g - T\| < \varepsilon$ for $g \in V$. So $\|g(R_{-x}TR_x) - R_{-x}TR_x\| < \varepsilon$ for all $g \in V$ and $x \in G$. By continuity of g on F_T, $\|g(S) - S\| < \varepsilon$ for $g \in V$ and all $S \in F_T$. This proves continuity of th map $g \to g$ from G to G_T at $g = 0$ and hence on G, since it is a homomorphism. Now we can embed G in $\prod_{T \in \mathcal{L}_A} G_T$ as follows: $g \in G$ corresponds to a point in $\pi_{T \in \mathcal{L}_A} G_T$ whose T^{th} Co-ordinate is $g \in G_T$. Let $\bar{G}$ be the closure of G in $\prod_{T \in \mathcal{L}_A} G_T$. Continuity of $g \to g$ from G to $\bar{G}$ follows from continuity of $g \to g$ from G to G_T for every $T \in \mathcal{L}_A$. Actually $\bar{G}$ turns out to be the Bohr Compactification of G.

Step 2: For $\psi \in L^1(\bar{G})$ define $\psi * T = \int_{\bar{G}} \psi(-g) g(T) dg$, dg being the normalized Haar measure on $\bar{G}$. For $\gamma \in \bar{G}$, let us see that $\gamma * T = M_\gamma U_\gamma$ where U_γ is an invariant operator.

$$\begin{aligned}
R_{-x}(\gamma * T)R_x &= \int_{\bar{G}} (-g,\gamma) R_{-x} g(T) R_x dg \\
&= \int_{\bar{G}} (-g,\gamma)(x+g)(T)dg \\
&= (x,\gamma)\int_{\bar{G}} (-x-g,\gamma)(x+g)(T)dg \\
&= (x,\gamma)(\gamma * T) \text{ by invariance of dg .}
\end{aligned}$$

Now

$$\begin{aligned}
M_{-\gamma}(\gamma * T)R_x &= M_{-\gamma} R_x R_{-x}(\gamma * T) R_x \\
&= (x,\gamma) M_{-\gamma} R_x(\gamma * T) \\
&= R_x M_{-\gamma}(\gamma * T) \text{ by 3.1 .}
\end{aligned}$$

Therefore $U_\gamma = M_{-\gamma}(\gamma * T)$ is invariant and hence $\gamma * T = M_\gamma U_\gamma$ as required.

Step 3: It is enough to show that T can be approximated in norm by finite sums of the form $\sum_{i=1}^n a_i(\gamma_i * T)$.

Given $\varepsilon > 0$ choose a symmetric neighbourhood V of 0 in $\bar{G}$ such that $\|g(T) - T\| < \varepsilon/2$ for $g \in V$. Choose $\psi \in L^1(\bar{G})$ such that $\psi \geq 0, \psi$ is supported in V and $\int_V \psi = 1$.

Then $\|\psi * T - T\| < \varepsilon/2$.

Choose $h = \sum_{i=1}^n a_i\gamma_i$ such that $\|h - \psi\|_1 < \frac{\varepsilon}{2(\|T\|+1)}$.

Then $\|h * T - \psi * T\| \leq \|h - \psi\|_1 \|T\| < \varepsilon/2$.

Thus $\|h * T - T\| < \varepsilon$, where $h * T = \sum_{i=1}^n a_i(\gamma_i * T)$.

This completes the proof of the theorem. ◊

4. The Invariant Mean.

For $T \in \mathcal{L}_A$ the integral $\pi_\gamma(T) = \gamma * T = \int_G (-g,\gamma) g(T) dg$ was encountered in the proof of Theorem 3.3. The notation $\pi_\gamma(T)$ goes with the notation used in [7]. $\pi_0(T) \in \mathcal{L}_0$ is checked directly as

$$R_{-x}\pi_0(T)R_x = \int_{\bar{G}} (x+g)(T)dg = \int_{\bar{G}} g(T)dg = \pi_0(T)$$

We call $\pi_0(T)$, the *invariant mean* of T. We already have an Invariant mean $\hat{\varphi}(0)$ for any almost periodic functions φ (see [6], p. 253). Once we realize $\pi_0(T)$ as some average, which we are going to do presently, it can be easily checked that $\pi_0(M_\varphi) = \hat{\varphi}(0)I$, where I is the identity operator on the Homogeneous Banach space B.

Now we shall obtain $\pi_0(T)$ by a standard limiting process. We start with a series of lemmas. The first of these is a particular case of Lemma 18.12 from page 254 of [6] and so we state it without proof.

LEMMA 4.1. *Let G be a locally compact abelian group. Let V be a relatively compact neighbourhood of 0 in G and let $\varepsilon > 0$. Then there is a relatively compact neighbourhood H of 0 such that $V \subseteq H$ and $\frac{m[(H+V)\cap H^c]}{m(H)} < \varepsilon$. (Here $m(E)$ denotes the Haar measure of E).*

LEMMA 4.2. *Let G be a non-compact locally compact abelian group. We can choose relatively compact neighbourhoods H_α of 0 such that (i) $\cup_\alpha H_\alpha = G$ (ii) The index set can be ordered so that*

$$\lim_\alpha \frac{m[(x+H_\alpha)\cap H_\alpha^c]}{m(H_\alpha)} = 0 \text{ for every } x \in G\,.$$

Proof. Let $\{V_\alpha\}_{\alpha\in I}$ be a collection of all neighbourhoods of 0 with compact closure such that $U_\alpha V_\alpha = G$. We use Lemma 4.1 to get open sets H_α with compact closure such that $H_\alpha \supseteq V_\alpha$ and

$\frac{m[(H_\alpha+V_\alpha)\cap H_\alpha^c]}{m(H_\alpha)} < \frac{1}{m(V_\alpha)}$. Define $\alpha \geq \beta$ if $V_\alpha \supseteq V_\beta$. To see that (ii)

holds, for $x \in G$ choose β such that $x \in V_\beta$.

For $\alpha \geq \beta$, $$\frac{m[(x+H_\alpha)\cap H_\alpha^c]}{m(H_\alpha)} \leq \frac{m[(H_\alpha+V_\alpha)\cap H_\alpha^c]}{m(H_\alpha)} < \frac{1}{m(V_\alpha)} \to 0\,.$$

◇

LEMMA 4.3. *For the above choice of $\{H_\alpha\}$,*

$$\lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} (x,\gamma)dx = \begin{cases} 0 & \text{if } \gamma \neq 0 \\ 1 & \text{if } \gamma = 0 \end{cases}$$

(Here dx denotes Haar measure on G).

Proof. The case $\gamma = 0$ is obvious. If $\gamma \neq 0$, choose x_0 such that $(x_0, \gamma) \neq 1$.

$$\begin{aligned} &\left| \frac{1}{m(H_\alpha)} \int_{H_\alpha} [(x + x_0, \gamma) - (x, \gamma)]\, dx \right| = \\ &= \frac{1}{m(H_\alpha)} \left| \int_{H_\alpha - x_0} (x, \gamma) dx - \int_{H_\alpha} (x, \gamma) dx \right| \\ &\leq \frac{1}{m(H_\alpha)} \int_{(H_\alpha - x_0) \Delta H_\alpha} |(x, \gamma)| dx \end{aligned}$$

where Δ denotes symmetric difference of sets.

$$= \frac{1}{m(H_\alpha)} \left[m \left\{ [H_\alpha - x_0] \cap H_\alpha^c \right\} + m \left\{ [H_\alpha - x_0]^c \cap H_\alpha \right\} \right]$$

Since $\{[H_\alpha - x_0]^c \cap H_\alpha\} + x_0 = H_\alpha^c \cap [H_\alpha + x_0]$, the last expression $= \frac{1}{m(H_\alpha)} [m\{[H_\alpha - x_0] \cap H_\alpha^c\} + m\{H_\alpha^c \cap [H_\alpha + x_0]\}] \to 0$ by Lemma 4.2.

Therefore $\lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} (x, \gamma) dx [(x_0, \gamma) - 1] = 0$. since $(x_0, \gamma) \neq 1$, the lemma is proved.

PROPOSITION 4.4. For any $T \in \mathcal{L}_A$, $\lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} R_{-x} T R_x dx$ exists and is equal to $\pi_0(T) = \int_{\bar{G}} g(T) dg$.

Proof. Suppose $T = M_\gamma U$, U being invariant. Then

$$\begin{aligned} &\lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} R_{-x}(M_\gamma U) R_x dx \\ &= \lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} (x, \gamma) dx M_\gamma U, \text{ using 3.1} \\ &= \begin{cases} U & \text{if } \gamma = 0 \\ 0 & \text{if } \gamma \neq 0 \end{cases} \end{aligned}$$

Therefore the limit exists for any operator of the form $P = \sum_{i=1}^n M_{\gamma_i} U_{\gamma_i}$. For any $T \in \mathcal{L}_A$, given $\varepsilon > 0$ we can choose such a finite sum P_ε by our Approximation Theorem such that $\|T - P_\varepsilon\| < \varepsilon$.

Let $\Phi(\alpha, T) = \frac{1}{m(H_\alpha)} \int_{H_\alpha} R_{-x} T R_x dx$.

Then $\|\Phi(\alpha, T) - \Phi(\beta, T)\| \leq \|\Phi(\alpha, T - P_\varepsilon)\| +$

$$\|\Phi(\alpha, P_\varepsilon) - \Phi(\beta, P_\varepsilon)\| + \|\Phi(\beta, P_\varepsilon - T)\|$$
$$< \|\Phi(\alpha, P_\varepsilon) - \Phi(\beta, P_\varepsilon)\| + 2\varepsilon \ .$$

Since $\lim_\alpha \Phi(\alpha, P_\varepsilon)$ exists, $\{\Phi(\alpha, T)\}$ is a Cauchy net and hence is convergent. To show that this limit is $\pi_0(T)$, let us first compute

$\pi_0(M_\gamma U)$. For $g \in G$, $g(M_\gamma U) = R_{-g} M_\gamma U R_g = (g, \gamma) M_\gamma U$

Hence $g(M_\gamma U) = (g, \gamma) M_\gamma U$ for all $g \in \bar{G}$.

Therefore $\pi_0(M_\gamma U) = \int_{\bar{G}} (g, \gamma) dg M_\gamma U = \begin{cases} U & \text{if } \gamma = 0 \\ 0 & \text{if } \gamma \neq 0 \end{cases}$.

This proves that $\pi_0(T) = \lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} (x, \gamma) R_{-x} T R_x dx$ for $T = M_\gamma U$ and therefore for $T = \sum_{i=1}^n M_{\gamma_i} U_{\gamma_i}$; it follows for any $T \in \mathcal{L}_A$ by our Approximation Theorem. [Note that $\|\pi_0(T)\| \leq \|T\|$ for any $T \in \mathcal{L}_A$].

This limit form of the invariant mean of an almost periodic operator is more handy in further analysis of these operators as illustrated by the following proposition which gives a simple result interesting in the context of 2.8.

PROPOSITION 4.5. Let G be a noncompact locally compact abelian group. Then there does not exist any non-zero compact almost periodic operator on $L^p(G)$, $1 \leq p < \infty$.

Proof.

For $T \in \mathcal{L}_A$, we have $\pi_\gamma(T) = \lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} (x, \gamma) R_{-x} T R_x dx$, where H_α's are as in 4.2. Then from the proof of 3.3, T lies in the closed linear span of $\{\pi_\gamma(T) = \gamma * T : \gamma \in \hat{G}\}$. Also $\pi_\gamma(T) = M_\gamma U_\gamma$ where $U_\gamma = \lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} R_{-x}(M_{-\gamma} T) R_x dx$, using Lemma 3.1.

Now if T is also compact, then for any $\gamma \in \hat{G}$, U_γ is a compact multiplier on $L^p(G)$ and hence is zero (See Akemann [1]).

Hence if $T \in \mathcal{L}_A$ is compact, then $T = 0$. ◇

Now we proceed to investigate certain intrinsic properties of the invariant mean $\pi_0(T)$ of an almost periodic operator T on $L^2(G)$, which play a key role in the next section when we study the Fourier series of these operators. The next proposition is a simple observation and its proof is omitted.

PROPOSITION 4.6. Let $B = L^2(G)$, Then $\mathcal{L}_0$, $\mathcal{L}_*$ and $\mathcal{L}_A$ are $*$-subalgebras of $\mathcal{L}(B)$.

Before we take up the next result we make the following definition.

DEFINITION 4.7. i) A linear map ψ from an algebra of operators on a Hilbert space into another such algebra is said to be positive if $T \geq 0$ implies $\psi(T) \geq 0$.

ii) A linear map ψ of a $*$-algebra B_1 into a $*$-algebra B_2 is said to be faithful if $\psi(T^*T) = 0$ implies $T = 0$ for $T \in B_1$.

PROPOSITION 4.8. Let $B = L^2(G)$. The invariant mean $\pi_0 : \mathcal{L}_A \to \mathcal{L}_0$ is a positive, faithful $*$-map.

Proof. Positivity of π_0 folllows from

$$\begin{aligned}(\pi_0(T)f, f) &= \lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} (R_{-x}TR_xf, f)dx \\ &= \lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} (TR_xf, R_xf)dx\end{aligned}$$

That π_0 is $*$-preserving follows from

$$\begin{aligned}(\pi_0(T)f, g) &= \lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} (R_{-x}TR_xf, g)dx \\ &= \lim_\alpha \frac{1}{m(H_\alpha)} \int_{H_\alpha} (f, R_{-x}T^*R_xg)dx \\ &= (f, \pi_0(T^*)g)\end{aligned}$$

We use a technique used by Arveson [2] to prove faithfulness of π_0. It suffices to produce a c^*-algebra $\mathcal{C}$, a positive faithful linear map ω of $\mathcal{C}$ into $\mathcal{L}_0$ and a $*$-morphism π of $\mathcal{C}$ onto $\mathcal{L}_A$ such that $\pi_0 \circ \pi = \omega$. Because if $T \in \mathcal{L}_A$ and $\pi_0(T^*T) = 0$, choose $C \in \mathcal{C}$ such that $T = \pi(C)$. Then $\omega(C^*C) = (\pi_0 \circ \pi)(C^*C) = \pi_0(\pi(C^*)\pi(C) = \pi_0(T^*T) = 0$. This implies $C = 0$ and hence $T = \pi(C) = 0$. Now $\mathcal{C}$ is constructed as follows:

Let $\mathcal{C}_1$ be the set of almost periodic functions $F : G \to \mathcal{L}(L^2(G))$. It can be easily seen that with pointwise operations $(F_1 + F_2)(x) = F_1(x) + F_2(x)$, $F_1F_2(x) = F_1(x)F_2(x)$, $F^*(x) = F(x)^*$ for $x \in G$ and the norm $\|F\| = \sup_{x\in G} \|F(x)\|$, $\mathcal{C}_1$ becomes a c^*-algebra. It can also be verified that π defined by $\pi(F) = F(0)$ is a $*$-morphism of $\mathcal{C}_1$ into $\mathcal{L}(L^2(G))$. Define $\omega(F) = \int_{\bar{G}} F(x)dx$. Then

$$\begin{aligned}\int_{\bar{G}} \|F(x)f\|^2 dx &= \int_{\bar{G}} (F^*(x)F(x)f, f)dx \\ &= (\omega(F^*F)f, f) \text{ for } f \in L^2(G) \,.\end{aligned}$$

Therefore $\omega(F^*F)= 0$ implies $F(x)f = 0$ for all $f \in L^2(G), x \in G$. So $F = 0$ and hence ω is faithful.

Now let $\mathcal{C}$ be the closure in $\mathcal{C}_1$ of the set of functions of the form $F(x) = \sum_{i=1}^n M_{\gamma_i} U_{\gamma_i}(x, \gamma_i)$. We shall see that $\mathcal{C}$ is a c^*-subalgebra of $\mathcal{C}_1$ containing the identity. For this the only thing which presents some difficulty is to show that $\mathcal{C}$ is closed with respect to multiplication. For this we note that $M_{\gamma_1}U_{\gamma_1}M_{\gamma_2}U_{\gamma_2} = M_{\gamma_1+\gamma_2}U$ where $U = M_{-\gamma_2}U_{\gamma_1}M_{\gamma_2}U_{\gamma_2}$. Also $M_{-\gamma_2}U_{\gamma_1}M_{\gamma_2} \in \mathcal{L}_0$ by Lemma 3.2. Hence $U \in \mathcal{L}_0$. This now implies that $\mathcal{C}$ is a c^*-algebra.

If F is given by $F(x) = \sum_{i=1}^n M_{\gamma_i}U_{\gamma_i}(x, \gamma_i)$ then $\omega(F) = U_0 = \pi_0(\pi(F))$. By continuity, $\omega = \pi_0 \circ \pi$ on $\mathcal{C}$. Also $\omega(\mathcal{C}) \subseteq \mathcal{L}_0$ since π_0 takes values in $\mathcal{L}_0$. Now since π is a $*$-morphism of the C^*-algebra $\mathcal{C}$ into the C^*-algebra $\mathcal{L}_A$, $\pi(\mathcal{C})$ is closed in $\mathcal{L}_A$ (Refer to [4], p. 13). Also $\pi(\mathcal{C})$ contains all finite sums $\sum_{i=1}^n M_{\gamma_i}U_{\gamma_i}$. Hence from our Approximation Theorem, $\pi(\mathcal{C}) = \mathcal{L}_A$. This completes the proof of the proposition. ◇

Finally let us observe the following properties of π_0 which will be needed in the sequel.

PROPOSITION 4.9. (i) π_0 is idempotent. That is $\pi_0 \circ \pi_0 = \pi_0$. (ii) Let $B = L^2(G)$. Then for $T \in \mathcal{L}_A$, $\pi_0(T)^*\pi_0(T) \le \pi_0(T^*T)$.

Proof. (i) By the remark at the beginning of this section, $\pi_0(T) \in \mathcal{L}_0$ for any $T \in \mathcal{L}_A$. Also from the limit formula $\pi_0(U) = U$ for any $U \in \mathcal{L}_0$.

Hence $\pi_0(\pi_0(T)) = \pi_0(T)$.

(ii) From the limit formula for π_0, we have for $T \in \mathcal{L}_A$ and $U \in \mathcal{L}_0$.

$\pi_0(UT) = U\pi_0(T)$ and
$\pi_0(TU) = \pi_0(T)U$. By positivity, $\pi_0[(T-\pi_0(T))^*(T-\pi_0(T)] \geq 0$ for any $T \in \mathcal{L}_A$.

Expanding,

$$\pi_0(T^*T) - \pi_0(\pi_0(T)^*T) - \pi_0(T^*\pi_0(T)) + \pi_0(T)^*\pi_0(T) \geq 0 .$$

Using the facts $\pi_0(T)$, $\pi_0(T)^* \in \mathcal{L}_0$, π_0 is an idempotent and $*$-map we get the required inequality.

5. Fourier Series.

Following DeLeeuw [7], we define the Fourier series of any $T \in \mathcal{L}_A$ as $\sum_{\gamma\in\hat{G}} \pi_\gamma(T)$, where $\pi_\gamma(T)$ was defined at the beginning of section 4.

As computed in [7], the Fourier series of M_φ, φ being an almost periodic function, is $\sum_{\gamma\in\hat{G}} \hat{\varphi}(\gamma)M_\gamma$. Here $\hat{\varphi}$ denotes Fourier transform of φ given by

$$\hat{\varphi}(\gamma) = \lim_{\alpha} \frac{1}{m(H_\alpha)} \int_{H_\alpha} (-x,\gamma)\varphi(x)dx$$

It is interesting to study the convergence of $\sum_{\gamma\in\hat{G}} \pi_\gamma(T)$ on $L^2(G)$, where $L^2(G)$ is separable, in the norm constructed by Arveson [3]. We first describe briefly the construction of the norm:

By a state on a unital $*$-algebra of operators on a Hilbert space we mean a positive linear functional φ with $\varphi(I) = 1$. We need a faithful state ρ on $\mathcal{L}_A$ which preserves the invariant mean (i.e. $\rho \circ \pi_0 = \rho$) to define the required norm. To get this we can take a faithful state φ on $\mathcal{L}_0$ such as $\varphi(T) = \sum_{n=1}^{\infty} \frac{1}{2^n}(T\xi_n, \xi_n)$, where $\{\xi_n\}$ is an orthonormal basis for $L^2(G)$. (Recall our assumption that $L^2(G)$ is separable). Then take $\rho = \varphi \circ \pi_0$. That ρ is a faithful state on $\mathcal{L}_A$ follow from the fact that φ is a faithful state on $\mathcal{L}_0$ and that $\pi_0 : \mathcal{L}_A \to \mathcal{L}_0$ is a positive, faithful map with $\pi_0(I) = I$. Thus we get an inner product on $\mathcal{L}_A$ defined by $\langle T, S\rangle_\rho = \rho(S^*, T)$ for $T, S \in \mathcal{L}_A$. Denote the corresponding norm by $\| \quad \|_\rho$. Now we have the following convergence result.

PROPOSITION 5.1. With notations as above, the Fourier series of any $T \in \mathcal{L}_A$ converges to T in the norm $\| \quad \|_\rho$.

Proof. Let S be the Hilbert space completion of $\mathcal{L}_A$ with respect of the inner product $\langle \ \ \rangle_\rho$. Let S_γ be the $\| \ \ \|_\rho$-closure of the subspace $\{M_\gamma U : U \in \mathcal{L}_0\}$. If $U, V \in \mathcal{L}_0$ and $\gamma_1 \neq \gamma_2$, then

$$\begin{aligned}\langle M_{\gamma_1}U, M_{\gamma_2}V\rangle_\rho &= \rho(V^* M_{-\gamma_2} M_{\gamma_1} U) = \rho(V^* M_{\gamma_1-\gamma_2} U) \\ &= \rho(V^* U_0 M_{\gamma_1-\gamma_2}), \text{ where } U_0 = M_{\gamma_1-\gamma_2} U M_{\gamma_2-\gamma_1} \varepsilon \mathcal{L}_0, \text{ by 3.2} \\ &= \rho \circ \pi_0(V^* U_0 M_{\gamma_1-\gamma_2}) = \rho(V^* U_0 \pi_0(M_{\gamma_1-\gamma_2})) \\ &= 0 \text{ since } \pi_0(M_\gamma) = 0 \text{ if } \gamma \neq 0\end{aligned}$$

So $S_{\gamma_1}, S_{\gamma_2}$ are orthogonal if $\gamma_1 \neq \gamma_2$. Now the linear subspace generated by $M_\gamma U$ is operator norm dense in $\mathcal{L}_A$. Also if

$$T \in \mathcal{L}_A, \text{ then } \|T\|_\rho^2 = \rho(T^*T) \leq \|\rho\| \, \|T^*T\| = \|\rho\| \, \|T\|^2 .$$

Therefore the subspace generated by $M_\gamma U$'s is $\| \ \ \|_\rho$-dense in $\mathcal{L}_A$. Thus S_γ's span a dense subspace of S. We have decomposed S into orthogonal subspaces S_γ's. Let P_γ be the projection of S onto S_γ. We claim that for $T \in \mathcal{L}_A, P_\gamma(T) = \pi_\gamma(T)$. Actually $\pi_\gamma(T) = M_\gamma \pi_0(M_{-\gamma}T)$ as in the proof of Proposition 4.5.

$$\begin{aligned}(\pi_\gamma \circ \pi_\gamma)(T) &= \pi_\gamma(M_\gamma \pi_0(M_{-\gamma}T)) = M_\gamma \pi_0(M_{-\gamma} M_\gamma \pi_0(M_{-\gamma}T)) \\ &= M_\gamma \pi_0(\pi_0(M_{-\gamma}T)) = M_\gamma \pi_0(M_{-\gamma}T) = \pi_\gamma(T) .\end{aligned}$$

Thus π_γ is an idempotent linear transformation of $\mathcal{L}_A$ onto $\{M_\gamma U : U \in \mathcal{L}_0\}$.

Also for $T \in \mathcal{L}_A$, $\|\pi_\gamma(T)\|_\rho^2 = \rho(\pi_\gamma(T)^* \pi_\gamma(T))$

$$\begin{aligned}&= P\left([\pi_0(M_{-\gamma}T)]^* \, M_{-\gamma} M_\gamma \pi_0(M_{-\gamma}T)\right) \\ &= \rho\left([\pi_0(M_{-\gamma}T)]^* \, \pi_0(M_\gamma T)\right) \\ &\leq \rho\left(\pi_0(T^* M_\gamma M_{-\gamma} T)\right)\end{aligned}$$

using 4.9 (ii) and the fact that ρ is positive.

$$= \rho\left(\pi_0(T^*T)\right) = \rho(T^*T) = \|T\|_\rho^2 .$$

Thus π_γ defined on $\mathcal{L}_A$ is $\| \ \ \|_\rho$ – decreasing and hence has a unique extension $\tilde{\pi}_\gamma$ of S onto S_γ.

Hence $\tilde{\pi}_1 = P_\gamma$.

For every $\xi \in S$, the set of finite sums $\sum_{\gamma \in \hat{G}} P_\gamma \xi$ converges to ξ in $\| \ \|_\rho$-norm. In particular for $T \in \mathcal{L}_A$, $\sum_{\gamma \in \hat{G}} \pi_\gamma(T)$ converges to T in $\| \ \|_\rho$–norm.

In a way $\| \ \|_\rho$ extends the L^2-norm: in fact if G is the circle group and $\varphi(T) = \sum_{n=1}^{\infty} \frac{1}{2^n}(T\xi, \xi_n)$, where (ξ_n) is the standard orthonormal basis $\{e^{int}\}$ for $L^2(G)$ and gf us continuous on G, then $\|M_f\|_\rho = \|f\|_{L^2(G)}$

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