DUAL GOLDIE DIMENSION(*)

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SOMMARIO. - *Si fa uso del processo di dualizzazione nella categoria degli R-moduli per provare alcuni teoremi riguardanti i moduli di dimensione duale di Goldie finita, che hanno il loro corrispondente nel caso dei moduli artiniani. In particolare si dà un teorema di struttura per i moduli complementati di dimensione duale di Goldie finita. Come applicazione ai gruppi abeliani, si prova poi che gli $\mathbb{Z}$-moduli di dimensione duale di Goldie finita sono esattamente gli $\mathbb{Z}$-moduli artiniani.*

SUMMARY. - *We make use of the dualization process in the category of R-modules to give a few theorems about modules of finite dual Goldie dimension that have their familiar counterpart in artinian modules. In particular, we give a structure theorem for complemented modules of finite dual Goldie dimension. As an application to abelian groups, we prove that $\mathbb{Z}$-modules of finite dual Goldie dimension are exactly the artinian $\mathbb{Z}$-modules.*

1. Preliminary definitions and Lemmas.

R denotes throughout this article a ring with $1 \neq 0$ and R-modules are unitary left R-modules. A submodule N of an R-module M is said to be *superfluous* in M, in symbols $N \overset{\circ}{\hookrightarrow} M$, if $N + A \neq M$ for every proper submodule A of M.

A *minimal complement* of a submodule A of an R-module M is a minimal member of the family of all submodules X of M such that $X + A = M$.

Minimal complements do not always exist; an R-module M is therefore defined to be *complemented* if every submodule of M has a minimal complement in M. It is easy to see that B is a minimal complement of a submodule A of M if and only if $A + B = M$ and

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$A \cap B \overset{\circ}{\hookrightarrow} B$. The following property of minimal complements is easily verified:

Let X be a submodule of an R-module M. If X is a minimal complement in M, then for all submodules U of X, $X/U \overset{\circ}{\hookrightarrow} M/U$ implies $X = U$. Conversely, if M is complemented and $X/U \overset{\circ}{\hookrightarrow} M/U$ implies $X = U$, then X is a minimal complement in M.

An R-module M is said to be *strongly complemented* if for any two submodules A and B of M such that $M = A + B$, A has a minimal complement contained in B.

A finite family $A_1, \ldots A_n$ of submodules of an R-module M is said to be *coindependent* if $A_i \neq M$ and $A_i + S_{in} = M$ for every $i = 1, \ldots, n$, where $S_{in} = \cap_{j \neq i} A_j$. A family $(A_i)_{i \in I}$ of submodules of an R-module M is *coindependent* if every finite subfamily is coindependent.

Every family of submodules of an R-module M contains a maximal coindependent subfamily. If $(N_i)_{i \in I}$ is a maximal coindependent family of maximal submodules of an R-module M, then the radical rad M is equal to $\cap_i N_i$.

An R-module M is said to be *hollow* if $M \neq 0$ and every proper submodule of M is superfluous in M. Equivalently, an R-module M is hollow if singletons of proper submodules of M are the only coindependent families of submodules of M.

The following three lemmas will be used in the next section. The proofs follow by duality and therefore will not be given.

LEMMA 1.1. *Let $A_1, A_2, \ldots$ be a countable family of submodules of an R-module M. The following four statements are equivalent:*

a) *The family of submodules $A_1, A_2, \ldots$ is coindependent;*

b) *The family of submodules $A_1, A_2, \ldots A_n$ is coindependent for every $n \geq 1$;*

c) *$A_n + S_{nn} = M$ for every $n \geq 1$;*

d) *$M = S_{1n} + \ldots + S_{nn}$ for every $n \geq 1$.*

LEMMA 1.2. *Let $A_1, A_2, \ldots$ be a coindependent family of submodules of an R-module M and $B_1, \ldots, B_n$ another family of submodules*

of M such that $A_i \subseteq B_i$ and $B_i/A_i \overset{\circ}{\hookrightarrow} M/A_i$ for every $i = 1, \ldots, n$. Then $\cap_i B_i/\cap_i A_i \overset{\circ}{\hookrightarrow} M/\cap_i A_i$.

LEMMA 1.3. *Let M be an R-module in which every family of coindependent submodules is finite. Then for every proper submodule A of M there exists a submodule U of M containing A such that M/U is hollow.*

2. Modules of finite corank.

In Goldie [2], a uniquely determined integer, $\text{rank}M \geq 0$ is associated with every R-module M in which every independent family of submodules is finite. This concept of dimension has been dualized by several authors [1,3,4,5,6,7] in different not always equivalent ways. The treatment in [4,5] is based on dualizing the notion of independence associating thereby an integer, $\text{corank } M \geq 0$ with every R-module M in which every coindependent family of submodules is finite. To these two articles we add under mild restrictions a few theorems that correspond to familiar statements concerning artinian modules. The slightly modified definition of coindependence given here permits dualizing word by word the theorems and proofs on dimension as stated in the original article [2].

With the definition of coindependence of families of submodules of an R-module given in the previous section, the dual of the Goldie Dimension Theorem takes the following form:

THEOREM 2.1. *Let M be an R-module in which every coindependent family of submodules is finite. Then there exists an integer $n \geq 0$ with the following properties:*

a) *M has a coindependent family of submodules $U_1, \ldots, U_n$ such that $M/U_1, \ldots, M/U_n$ are hollow and $U_1 \cap \ldots \cap U_n \overset{\circ}{\hookrightarrow} M$.*

b) *Every coindependent family of submodules of M has at most n elements.*

c) *A submodule U of M is superfluous in M if and only if there exists a coindependent family of submodules $U_1, \ldots, U_n$ of*

M such that $M/U_1, \ldots, M/U_n$ are hollow and $U \subseteq U_1 \cap \ldots \cap U_n$.

The uniquely determined integer n is called the *corank* of the R-module M, and we write corank $M = n$. If M has infinite coindependent families of submodules, we write corank $M = \infty$.

The proof of Theorem 2.1 follows by duality from that of Theorem 1.07 in [2] and therefore will not be given. We point out, however, that it requires a repeated application of Lemmas 1.1, 1.2 and 1.3.

The following remarks are easy consequences of Theorem 2.1 and the definition of coindependence.

i) Isomorphic R-modules have equal coranks.

ii) corank $M = 0$ if and only if $M = 0$.

iii) corank $M = 1$ if and only if M is hollow.

iv) If N is a submodule of the R-module M, then corank $M/N \leq$ corank M. Equality holds if $N \overset{\circ}{\hookrightarrow} M$, and if corank M is finite, the converse holds.

v) If M is a direct sum of a family of submodules $(N_i)_{i \in I}$ then corank $M = \sum_i$ corank N_i.

vi) Let M be an R-module of finite corank. If $N_1, \ldots, N_r$ is a maximal coindependent family of maximal submodules of M, then $0 \leq r \leq$ corank M and rad $M = N_1 \cap \ldots N_r$. By coindependence, $M/\text{rad}M$ is isomorphic to the semisimple R-module $(M/N_1) \oplus \ldots \oplus (M/N_r)$, and so the integer r is uniquely determined and depends only on M. r is called the *torsion corank* of M and will be denoted by $r(M)$. The equality $r(M) = \text{corank}M$ holds if and only if rad $M \overset{\circ}{\hookrightarrow} M$, and since $M/\text{rad}M$ is semi-simple artinian, rad $M \overset{\circ}{\hookrightarrow} M$ if and only if M is finitely generated. It follows that the corank of the R-module R is equal to the corank of the right R-module R. For if either corank is finite, then it is equal to the length of the semi-simple ring $R/\text{rad } R$. Hence it makes sense to define the corank of the ring R as the corank of the R-module R. In case R is commutative and

corank $R < \infty$, it follows that corank R is equal to the number of maximal ideals of R.

vii) An R-module M of finite corank is weakly complemented in the sense that if M is a sum $A_1 + A_2$ of two proper submodules A_1 and A_2, then there exists a submodule B of A_2 such that $M = A_1 + B$ and $A_1 \cap B \overset{\circ}{\hookrightarrow} M$.

Artinian R-modules are obvious examples of modules of finite corank. The converse is not in general true. Take , for example, R to be a non-artinian local ring. We show at the end of this article that $\mathbb{Z}$-modules of finite corank are necessarily artinian. However, the following result is true for any ring R.

THEOREM 2.2. *Let M be an R-module. The following statements are equivalent:*

a) *M is artinian;*

b) *corank M is finite and every superfluous submodule of M is artinian;*

c) *Every submodule of M has finite corank and every non-zero subquotient module of M has a simple submodule;*

d) *corank M is finite and M/U is finitely cogenerated for every superfluous submodule U of M.*

Proof. Certainly an artinian module satisfies (b), (c) and (d). We show that each of (b), (c) and (d) implies (a).

b) $\Rightarrow$ a) Suppose that corank $M = n$. By Theorem 2.1, there exists a coindependent family $U_1, \ldots, U_n$ of submodules of M such that $M/U_1, \ldots, M/U_n$ are hollow and $U_1 \cap \ldots \cap U_n \overset{\circ}{\hookrightarrow} M$. By coindependence, there exists an exact sequence

$$0 \longrightarrow U_1 \cap \ldots \cap U_n \longrightarrow M \longrightarrow (M/U_1) \oplus \ldots \oplus (M/U_n) \longrightarrow 0$$

of R-module homomorphisms. Hence $(M/U_1) \oplus \ldots \oplus (M/U_n)$ is isomorphic to $M/(U_1 \cap \ldots \cap U_n)$. Since $U_1 \cap \ldots \cap U_n \overset{\circ}{\hookrightarrow} M$, it follows that every superfluous submodule of $(M/U_1) \oplus \ldots \oplus (M/U_n)$

is artinian. The same is true for the submodules $M/U_1, \ldots, M/U_n$ of $(M/U_1) \oplus \ldots \oplus (M/U_n)$, and since every M/U is hollow, it follows that every M/U_i is artinian. Since $U_1 \cap \ldots \cap U_n \overset{\circ}{\hookrightarrow} M$, it follows that $U_1 \cap \ldots \cap U_n$ is an artinian R-module. The exactness of the above sequence implies that M is an artinian R-module.

c) $\Rightarrow$ a) Let M/N be a non-zero quotient module of M and prove that M/N is finitely cogenerated. It is easily seen that every submodule and every quotient module of M satisfies property (c). In particular, soc M/N is a non-zero finitely generated submodule of M/N. If K/N is a non-zero submodule of M/N, then soc $K/N \neq 0$, and so $K/N \cap$ soc $(M/N) \neq 0$. Thus soc M/N is a finitely generated essential submodule of M/N, that is M/N is finitely cogenerated.

d) $\Rightarrow$ a) Let M/N be a non-zero quotient module of M. Since M is weakly complemented, there exists a submodule K of M such that $M = K + N$ and $K \cap N \overset{\circ}{\hookrightarrow} M$. By assumption $M/(K \cap N)$ is finitely cogenerated. Hence so is the submodule $K/(K \cap N)$. Since $M/N \cong K/(K \cap N)$, M/N is also finitely cogenerated.

THEOREM 2.3. *Let N be a submodule of an R-module M.*

a) *If N and M/N have finite coranks, then M has finite corank.*

b) *corank $M \leq$ corank$M/N +$ corank$N$.*

c) *If N is a minimal complement in M, then*

$$\text{corank}M = \text{corank}M/N + \text{corank}N \ . \qquad (*)$$

If corank M is finite then the converse is also true.

d) *If corank M is finite then $(*)$ is true for every submodule N of M if and only if M is semi-simple.*

Proof. a) Suppose, to the contrary, that corank M is not finite, and let $C_1, C_2, \ldots$ be a countably infinite coindependent sequence of submodules of M. By Lemma 1.1, the sequence

$$U_1 = C_1, U_2 = C_2 \cap C_3, \ldots, U_n = C_{t+1} \cap \ldots \cap C_{t+n}, \ldots, t = \frac{n(n-1)}{2}$$

of submodules of M is coindependent. Since corank M/N is finite, the sequence $(N + U_1)$, $(N + U_2), \ldots$ of submodules of M is not coindependent and so $N + U_n = M$ for almost all n. Choose n so that $n >$ corank N and $N + U_n = M$. Then M/U_n is an epimorphic image of N, and so corank $M/U_n \leq \text{corank} N$. On the other hand, $C_{t+1}/U_n, \ldots, C_{t+n}/U_n$ is a coindependent family of submodules of M/U_n having n elements, and so corank $M/U_n \geq n \geq \text{corank} N$. This contradiction shows that corank M is finite.

b) If corank $M = \infty$, it follows from (a) that corank $N = \infty$ or corank $M/N = \infty$. So suppose corank M is finite. Then M is weakly complemented, and hence there exists a submodule K of M such that $M = N + K$ and $N \cap K \overset{\circ}{\hookrightarrow} M$. Hence

$$\text{corank} M = \text{corank} M/(N \cap K) = \text{corank} N/(N \cap K) + \text{corank} K/(N \cap K) .$$

Since $K/(N \cap K) \cong M/N$ and corank $N/(N \cap K) \leq \text{corank} N$, it follows that corank $M \leq$ corank $M/N +$ corank M.

c) If N is a minimal complement in M, then K in (b) may be chosen such that $N \cap K \overset{\circ}{\hookrightarrow} N$, and so corank $N = \text{corank} N/(N \cap K)$. Conversely, if corank M is finite and corank $M =$ corank $N +$ corank M/N, then corank N is finite, and therefore $N \cap K \overset{\circ}{\hookrightarrow} N$ so that N is a minimal complement in M.

d) This follows from (c) since an R-module M is semi-simple if and only if every submodule of M is a minimal complement in M.

LEMMA 2.4. *Let A_1 and A_2 be submodules of an R-module M such that $A_1 \subseteq A_2$. If A_1 is a minimal complement in M and A_2/A_1 is a minimal complement in M/A_1, then A_2 is a minimal complement in M.*

Proof. Let B_1 be a submodule of M such that $A_1 + B_1 = M$ and $A_1 \cap B_1 \overset{\circ}{\hookrightarrow} A_1$ and B_2/A_1 a submodule of M/A_1 such that $A_2 + B_2 = M$ and $(A_2 \cap B_2)/A_1 \overset{\circ}{\hookrightarrow} A_2/A_1$. Then $M = A_2 + (B_1 \cap B_2)$ and it remains to prove that $A_2 \cap B_1 \cap B_2 \overset{\circ}{\hookrightarrow} A_2$. The natural isomorphisms

$$(A_2 \cap B_2)/A_1 \cong (A_2 \cap B_1 \cap B_2)/(A_1 \cap B_1)$$

and

$$A_2/A_1 \cong (A_2 \cap B_1)/(A_1 \cap B_1)$$

together with $(A_2 \cap B_2)/A_1 \overset{\circ}{\hookrightarrow} A_2/A_1$ imply that

$$(A_2 \cap B_1 \cap B_2)/(A_1 \cap B_1) \overset{\circ}{\hookrightarrow} (A_2 \cap B_1)/(A_1 \cap B_1) .$$

Hence $(A_2 \cap B_1 \cap B_2)/(A_1 \cap B_1) \overset{\circ}{\hookrightarrow} A_2/(A_1 \cap B_1)$.

Since $(A_1 \cap B_1) \overset{\circ}{\hookrightarrow} A_1 \subseteq A_2$, it follows that $A_2 \cap B_1 \cap B_2 \overset{\circ}{\hookrightarrow} A_2$.

THEOREM 2.5. *Let M be an R-module.*

a) *If corank M is finite, then M satisfies DCC on minimal complements.*

b) *If M is strongly complemented and satisfies DCC on minimal complements then M satisfies ACC on minimal complements.*

c) *If M is strongly complemented and satisfied ACC on minimal complements then corank M is finite.*

Proof. a) Let $T_1 \supseteq T_2 \supseteq \ldots$ be a strictly descending infinite sequence of minimal complements in M. Then T_n is not superfluous in M and T_n/T_{n+1} is not superfluous in M/T_{n+1} for all $n = 1, 2, \ldots$. It follows that there exists a sequence $A_1, A_2, \ldots$ of proper submodules of M such that $A_n \supseteq T_{n+1}$ and $A_n + T_n = M$ for all $n = 1, 2, \ldots$. Then for all $n \geq 1$, we have $(A_1 \cap \ldots \cap A_{n-1}) + A_n \supseteq T_n + A_n = M$. It follows that M has an infinite coindependent family of submodules which is not so.

b) Let $T_1 \subseteq T_2 \subseteq \ldots$ be an ascending sequence of minimal complements in M. Let K_1 be a minimal complement of T_1 in M. Then $M = T_1 + K_1 = T_2 + K_1$, and so there exists a submodule K_2 of K_1 such that K_2 is a minimal complement in M. By induction, we obtain a descending sequence, $K_1 \supseteq K_2 \supseteq \ldots$ of minimal complements in M. By assumption there exists an integer n such that $K_n = K_{n+1} = \ldots$. We prove that $T_n = T_{n+1} = \ldots$. For this it suffices to prove that $T_{n+1}/T_n \overset{\circ}{\hookrightarrow} M/T_n$. So let $A/T_n \subseteq M/T_n$ be such that $T_{n+1}/T_n + A/T_n = M/T_n$. Then $T_{n+1} + A = M = T_n + K_n$. From $T_{n+1} \supseteq T_n$ we conclude that $T_{n+1} = T_n + (K_n \cap T_{n+1})$ and so

$$M = T_{n+1} + A = T_n + (K_n \cap T_{n+1}) + A = (K_n \cap T_{n+1}) + A .$$

But

$$K_n \cap T_{n+1} = K_{n+1} \cap T_{n+1} \overset{\circ}{\hookrightarrow} K_n \subseteq M \ .$$

It follows that $A = M$.

c) Suppose, to the contrary, that corank M is not finite and let $C_1, C_2, \ldots$ be a countably infinite coindependent sequence of submodules of M. Starting with $A_0 = 0$, we prove by induction the existence of a strictly ascending infinite sequence $A_0 \subseteq A_1 \subseteq \ldots$ of minimal complements in M such that M/A_n has infinite corank for every $n = 0, 1 \ldots$. This is clear when $n = 0$. The transition from the k-th step to the $(k+1)$-st step takes place as follows:

Let $B_1/A_k, B_2/A_k, \ldots$ be a countably infinite coindependent sequence of submodules of M/A_k. Since M is strongly complemented, so is the quotient module M/A_k of M by the minimal complement A_k. Hence $M/A_k = B_1/A_k + B_2/A_k$ implies the existence of a minimal complement A_{k+1}/A_k of B_1/A_k in M/A_k such that $A_{k+1}/A_k \subseteq B_2/A_k$. By coindependence of the sequence B_1/A_k, $B_2/A_k, \ldots$, it follows that A_k is a proper submodule of A_{k+1}, and A_{k+1} is a proper submodule of B_2. By Lemma 2.4, A_{k+1} is a minimal complement in M and it remains to verify that corank M/A_{k+1} is not finite. A straight-forward calculation shows that $A_{k+1} + (B_1 \cap B_n) \neq M$ and

$$(A_{k+1}+(B_1 \cap B_2)) \cap \ldots \cap (A_{k+1}+(B_1 \cap B_{n-1})) + (A_{k+1}+(B_1 \cap B_n)) = M$$

for every $n \geq 1$, so that the infinite sequence $B_2 = A_{k+1} + (B_1 \cap B_2)$, $A_{k+1} + (B_1 \cap B_3), \ldots, A_{k+1} + (B_1 \cap B_n), \ldots$ of submodules M is coindependent. Hence corank M/A_{k+1} is not finite.

THEOREM 2.6 *Let M be a complemented R-module. Then*

a) *corank $M = n < \infty$ if and only if M is an irredundant sum, $M = H_1 + \ldots + H_n$ and n hollow submodules $H_1, \ldots H_n$. If $M = H_1' + \ldots + H_n'$ is another irredundant sum of n hollow submodules $H_1' \ldots H_n'$ then for every integer m, $0 \leq m \leq n$, there exist $n - m$ distinct indices $i_1, \ldots, i_{n-m}$ such that*

$$M = H_1 + \ldots + H_m + H_{i_1}' + \ldots + H_{i_{n-m}}' \ .$$

b) *If* $M = H_1 + \ldots + H_n$ *is an irredundant sum of* n *hollow submodules, then*

i) *rad* $M = \mathrm{rad}H_1 + \ldots + \mathrm{rad}H_n$,

ii) $M/\mathrm{rad}M$ *is isomorphic to the semi-simple* R*-module* $H_1/\mathrm{rad}H_1 \oplus \ldots \oplus H_n/\mathrm{rad}H_n$. *The length* $r(M)$ *of* $M/\mathrm{rad}M$ *is equal to the number of cyclic submodules among* $H_1, \ldots, H_n$. *If these are* $H_1, \ldots, H_r$ *and* $N_i = H_1 + \ldots + H_{i-1} + T_i + H_{i+1} + \ldots + H_n$, *where* T_i *is the maximal submodule of* H_i, $1 \leq i \leq r$, *then every* N_i *is a maximal submodule of* M, *the intersection* $N_1 \cap \ldots \cap N_r$ *is irredundant and is equal to rad* M.

Proof. a) Assume $M = H_1 + \ldots + H_n$ is an irredundant sum of n hollow submodules. The mapping

$$f : H_1 \oplus \ldots \oplus H_n \to M$$

given by

$$f(x_1, \ldots, x_n) = x_1 + \ldots + x_n$$

is an epimorphism with kernel $K = K_1 \oplus \ldots \oplus K_n$ where $K_i = H_i \cap (H_1 + \ldots + H_{i-1} + H_{i+1} + \ldots + H_n)$, $i = 1, \ldots n$. Since the sum $H_1 + \ldots + H_n$ is irredundant, $K_i \neq H_i$ and so $K_i \overset{\circ}{\hookrightarrow} H_i$ for every $i = 1, \ldots, n$. It follows that $K \overset{\circ}{\hookrightarrow} H_1 \oplus \ldots \oplus H_n$ and so corank $M = \mathrm{corank}(H_1 \oplus \ldots \oplus H_n) = n$.

Conversely, assume that corank $M = n$. if $n = 1$, then M is hollow. So assume $n > 1$, By lemma 1.3 there exists a submodule U_1 of M, such that M/U_1 is hollow. Let H_1 be a minimal complement of U_1 in M. By induction, pick at the k-th step in case $M/(H_1 + \ldots + H_{k-1})$ is not hollow, a submodule U_k of M containing $H_1 + \ldots + H_{k-1}$ such that M/U_k is hollow and let H_k be a minimal complement of U_k in M. By Lemma 1.3, the family of submodules $U_1, \ldots U_k$ is coindependent, since

$$U_k + (U_1 \cap \ldots \cap U_{k-1}) \supseteq (H_1 + \ldots + H_{k-1}) + (U_1 \cap \ldots \cap U_{k-1}) ,$$

and by induction, this last sum is equal to M. Hence, by Theorem 2.1, this inductive process must stop for some $k \leq n$. Since corank

$M = n$, it follows from the result of the previous paragraph that the process stops when $k = n$. Thus $M/(H_1 + \ldots + H_{n-1})$ is hollow. If H_n is a minimal complement of $H_1 + \ldots + H_{n-1}$ in M, then H_n is hollow and the sum $M = H_1 + \ldots + H_n$ is irredundant.

Suppose now that $M = H_1' + \ldots + H_n'$ is another irredundant sum of n hollow submodules. Let $G_m = H_1 + \ldots + H_m$, $0 \leq m \leq n$. Then on the one hand $M/G_m = (G_m + H_{m+1})/G_m + \ldots + (G_m + H_n)/G_m$ is an irredundant sum of $n-m$ hollow submodules of M/G_m implying thereby that corank $M/G_m = n - m$. On the other hand, $M/G_m = (G_m + H_1')/G_m + \ldots + (G_m + H_n')/G_m$ is a sum of n zero or hollow submodules of M/G_m. It follows that m of these submodules may be deleted to yield an irredundant sum of $n - m$ hollow submodules. Thus there exist distinct indices $i_1, \ldots i_{n-m}$ such that $M = H_1 + \ldots + H_m + H_{i_1}' + \ldots + H_{i_{n-m}}'$.

b) Since the kernel K of the epimorphism f defined in (a) is superfluous in $H_1 \oplus \ldots \oplus H_n$, f maps $\mathrm{rad}(H_1 \oplus \ldots \oplus H_n) = \mathrm{rad}H_1 \oplus \ldots \oplus \mathrm{rad}H_n$ onto rad M. Hence rad $M = \mathrm{rad}H_1 + \ldots + \mathrm{rad}\ H_n$. This proves (i). To prove (ii), note that the kernel of the restriction of f to rad $H_1 \oplus \ldots \oplus \mathrm{rad}H_n$ is also equal to K and so f induces an isomorphism from $H_1/\mathrm{rad}H_1 \oplus \ldots \oplus H_n/\mathrm{rad}H_n$ onto $M/\mathrm{rad}M$. $H_i/\mathrm{rad}H_i = 0$ in case H_i is not finitely generated and $H_i/\mathrm{rad}H_i$ is a simple R-module in case H_i is cyclic. This shows that the torsion corank $r(M)$ of M is equal to the number of cyclic submodules among $H_1, \ldots H_n$. To prove the remaining part of (ii) we may assume that $r(M) > 0$. For, if $r(M) = 0$, then rad $M = \mathrm{rad}H_1 + \ldots + \mathrm{rad}H_n = H_1 + \ldots + H_n = M$. For every i, $1 \leq i \leq r$, the quotient module M/N_i is an epimorphic image of the simple R-module H_i/T_i, and since T_i is superfluous in H_i and the sum $M = H_1 + \ldots + H_n$ irredundant, it follows that $N_i \neq M$. Hence M/N_i is also a simple R-module and so N_i is a maximal submodule of M. For the same reason, the intersection $N_1 \cap \ldots \cap N_r$ is irredundant or equivalently the family of maximal submodules $N_1, \ldots, N_r$ is coindependent. The uniqueness of $r(M)$ implies rad $M = N_1 \cap \ldots \cap N_r$.

The irredundant sum representation of a complemented R-module of finite corank is generally not direct. It is direct if the R-module M is projective. In fact, if P is any complemented projective R-module, then rad $P \overset{\circ}{\hookrightarrow} P$, $P/\mathrm{rad}\ P$ is semi-simple and the projection

map $P \to P/\text{rad } P$ lifts the direct sum decomposition of $P/\text{rad}P$ into simple submodules to a direct sum decomposition of P into hollow submodules. Since the endomorphism ring of a hollow projective R-module is local, the direct sum decomposition of a complemented projective R-module into hollow submodules is unique in the sense of the Krull-Schmidt-Remak Theorem. Since a hollow projective R-module is necessarily cyclic, it follows that a complemented projective R-module has finite corank if and only if it is finitely generated.

COROLLARY. *If the ring R is a complemented R-module, then*

a) *corank $R = n < \infty$*

b) *$R = Re_1 \oplus \ldots \oplus Re_n = e_1R \oplus \ldots \oplus e_nR$, where $e_1, \ldots e_n$ are orthogonal idempotents, $1 = e_1 + \ldots + e_n$ and every $Re_i(e_iR)$ is a hollow left (right) ideal of R.*

c) *If R is also commutative, then every Re_i is a local ring with unit element e_i and R is ring isomorphic to the ring product $Re_1 \times \ldots \times Re_n$.*

d) *The R-module R is complemented if and only if the right R-module R is complemented.*

The proof is clear. We only point out that the equality of the coranks of the R-module R and the right R-module R implies that the right ideals $e_1R, \ldots, e_nR$ are hollow. (d) permits defining the notion of a complemented ring without ambiguity. However, the corollary implies that the notion of a complemented ring coincides with the notion of semi-perfectness of the ring. The same corollary gives another proof of the left-right symmetry of semi-perfectness.

A weak point of Theorem 2.6 is the assumption that the R-module M is complemented. Our next result shows that this assumption is redundant in the case of abelian groups.

THEOREM 2.8. *Let G be a non-zero abelian group*

a) *If G is torsion free, then corank $G = \infty$.*

b) *If corank* $G = n < \infty$, *then* G *is a direct sum of* n *hollow subgroups, and the direct summands are uniquely determined.*

c) *If corank* $G = n < \infty$ *then* G *is complemented.*

Proof. a) If G is not reduced, then it contains a direct summand isomorphic to the group $\mathbb{Q}$ of rational numbers which is of infinite corank. So suppose G is reduced, and let P be the set of prime numbers p for which $pG \neq G$. Since for any relatively prime numbers r and s, $rG + sG = G$ and $rsG = rG \cap sG$, it follows that the family of subgroup $(pG)_{p \in P}$ is coindependent. Hence if P is infinite, then corank $G = \infty$. If P is finite, then $pG = G$ for an infinite number of primes. Let these be $p_1, p_2, \ldots$, and let a be a non-zero element of G, Since G is torsion free, division in G by each of $p_1, .p_2, \ldots$ is unique, and hence the elements $\mathbb{Z}a + p_i^{-1}a$, $\mathbb{Z}a + p_i^{-2}a, \ldots$ generate a direct summand of $G/\mathbb{Z}a$ isomorphic to the hollow group $\mathbb{Z}(p_i^\infty)$ for every $i = 1, 2, \ldots$. It follows that corank $G/\mathbb{Z}a = \infty$ and hence corank $G = \infty$.

b) Let T be the torsion part of G. Since G/T is torsion free, it follows from (a) that $T = G$. Being a non-zero torsion group, G has a direct summand isomorphic to the hollow group $\mathbb{Z}(p^k)$, $1 \leq k \leq \infty$. By induction G is a direct sum of n hollow subgroups. Being the primary components of G, these direct summands are uniquely determined.

c) Being a finite sum of hollow, hence complemented subgroups, G is complemented.

Theorem 4.4 may be restated as follows:

A $\mathbb{Z}$*-module has finite corank if and only if it is artinian.*

REFERENCES

[1] FLEURY P., *A note on dualizing Goldie dimension*, Cand. Math. Bull. **17** (1974), 511–517.

[2] GOLDIE A.W., *The structure of Noetherian rings*, Lecture notes in Mathematics, 246, Springer-Verlag, New York, 1972.

[3] GRZESZCZUCK P. and PUCZYLOWSKI E.R., *On Goldie and dual Goldie dimensions*, J. Pure and Applied Algebra **31** (1984), 47–54.

[4] HANNA A. and SHAMSUDDIN A., *Duality in the category of modules. Applications*, Algebra Berichte **49**, (1984).

[5] TAKEUCHI T., *On cofinite dimensional modules*, Hokkaido J. Math. **5** (1976), 1–43.

[6] VARADARAJAN K., *Dual Goldie dimension*, Comm. Alg. **7** (1979), 565–610.

[7] ZÖSCHINGER H., *Minimax-Moduln*, J. Alg. **102** (1986), 1–32.