SOME CONDITIONS FOR CONTINUITY OF ALMOST CONTINUOUS MULTIFUNCTIONS (*)

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SOMMARIO. - *In questo articolo si mostra come i punti su cui una multifunzione α-regolare a valori α-paracompatti è "almost" continua e "quasi" continua formano un insieme chiuso. Da questo segue una parte della conclusione data nel teorema principale (Teorema 1) di* [3]. *D'altra parte, il Teorema 1 di questo articolo generalizza il Teorema 1 di* [2].

SUMMARY. - *This paper shows that the points on which an α-regular α-paracompact valued multifunction is almost continuous and quasi-continuous is a closed set and then a part of the conclusion given by the main theorem (Theorem 1) from* [3] *comes as a known result. On the other hand the main theorem of this paper (Theorem 1) generalizes Theorem 1 of* [2].

1. Introduction.

It is known from [9, Theorem 5.3] that a multifunction $F : X \to Y$ almost continuous and quasicontinuous is weakly continuous. From [8, Theorem 3] and [6, Theorem 2] we can draw the conclusion that if Y is regular, F almost continuous, quasicontinuous and punctually compact, then F is continuous.

In [3] Holá changes the conditions of quasicontinuity on condition that the set of quasicontinuous points should be dense in X.

We are going to prove for the multifunction $F : X \to Y$ with Y regular space, almost continuous and compact valued, that the set of quasi-continuous points is a closed set in X and therefore a part of the conclusion given by the main theorem from [3] is reduced to the conclusions from [8] and [9].

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2. Definitions and notations.

Let X and Y be two topological spaces. For a multifunction $F : X \to Y$ we shall denote by $F^+(G)$ and $F^-(G)$ the upper and lower inverse of the set G as in [1], and then we have

$$F^+(G) = \{x \in X : F(x) \subset G\},\ F^-(G) = \{x \in X : F(x) \cap G \neq \emptyset\}\ .$$

For a subset A of a topological space denote $Cl(A)$ and $\mathrm{Int}(A)$ the closure and the interior of A, respectively.

DEFINITION 1. (See [7], [11]) A multifunction $F : X \to Y$ is said to be

a) upper almost continuous (u.a.c.) if for each $x \in X$ and for each open set V of Y containing $F(x)$, $x \in \mathrm{Int}\ (Cl(F^+(G)))$.

b) lower almost continuous (l.a.c.) if for each $x \in X$ and each open set G of Y such that $F(x) \cap G \neq \emptyset$, $x \in \mathrm{Int}(Cl(F^-(G)))$.

c) almost continuous if it is upper and lower almost continuous.

DEFINITION 2. (See [6], [11])) A multifunction $F : X \to Y$ is said to be

a) upper weakly continuous (u.w.c.) if for each $x \in X$ and each open set V containing $F(x)$, there exists an open neighbourhood U of x such that $F(U) \subset Cl(V)$.

b) lower weakly continuous (l.w.c.) if for each $x \in X$ and each open set $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists an open set U of X such that $F(u) \cap Cl(V) \neq \emptyset$ for each $u \in U$.

c) weakly continuous if it is upper and lower continuous.

DEFINITION 3. (see [5]). A multifunction $F : X \to Y$ is said to be

a) upper quasicontinuous (u.q.c.) at $x \in X$ if for each open set U containing x and each open set V containing $F(x)$, there exists a nonempty open set G of X such that $G \subset U$ and $F(G) \subset V$.

b) lower quasicontinuous (l.q.c.) at $x \in X$ if for each open set U containing x and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set G of X such that $G \subset U$ and $F(g) \cap V \neq \emptyset$ for every $g \in G$.

c) quasicontinuous (q.c.) at $x \in X$ if it is upper and lower quasicontinuous.

Denote by Q_F the set of all points of X at which F is quasicontinuous. If $Q_F = X$, then F is called quasicontinuous.

DEFINITION 4. (See [4]). A subset of a topological space X is said to be α-regular if for each point $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset Cl(G) \subset U$.

DEFINITION 5. (See [12]). A subset of a topological space X is said to be α-paracompact if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X.

3. Main Results.

LEMMA 1. (see [4]). *If A is an α-regular α-paracompact subset of a space X and U is an open neighbourhood of A, then there exists an open set G of X such that $A \subset G \subset Cl(G) \subset U$.*

LEMMA 2. (See [10]). *If A is an α-regular set of X then for every open set D which intersects A, there exists an open set D_A so that $A \cap D_A \neq \emptyset$ and $Cl(D_A) \subset D$.*

THEOREM 1. *If a multifunction $F : X \to Y$ is almost continuous and $F(x)$ is α-regular α-paracompact for each $x \in X$, then Q_F is closed.*

Proof. Let $x \in Cl(Q_F)$. Let U and V be open sets such that $x \in U$ and $F(x) \subset V$. As $F(x)$ is α-regular α-paracompact by Lemma 1 there exists an open set W such that $F(x) \subset W \subset Cl(W) \subset V$. Upper almost continuity of F at x implies that there exists an open set $U_1 \subset U$ such that $x \in U_1 \subset Cl(F^+(W))$. From $x \in Cl(Q_F)$ follows that $U_1 \cap Q_F \neq \emptyset$. If $s \in U_1 \cap Q_F$ then

$$s \in F^+(Cl(W)) \ . \qquad (1)$$

Suppose that (1) does not hold.Then there exists $s \in U_1 \cap Q_F$ such that $s \in F^-(Y - Cl(W))$. Lower quasicontinuity of F implies by [9, Theorem 2.2] that there exists a nonempty open set $W_1 \subset U_1$ such that $W_1 \subset F^-(X - Cl(W)) \subset F^-(Y - W)$. That is in contradiction with $U_1 \subset Cl(F^+(W))$. By (1) follows that $F(s) \subset Cl(W) \subset V$. Upper quasi-continuity of F at s implies that there exists a nonempty open set $G \subset U$ such that $F(G) \subset V$. Thus F is u.q.c. at x.

Let U and V be open sets such that $x \in U$ and $F(x) \cap V \neq \emptyset$. As $F(x)$ is α-regular by Lemma 2 there exists an open set W such that $F(x) \cap W = \emptyset$ and $Cl(W) \subset V$. Lower almost continuity of F at x implies that there exists $U_1 \subset U$ such that $x \in U_1 \subset Cl(F^-(W))$. From $x \in Cl(Q_F)$ follows that $U_1 \cap Q_F \neq \emptyset$. If $s \in U_1 \cap Q_F$ then

$$s \in F^-(Cl(W)) \ . \tag{2}$$

Suppose (2) does not hold. Then, there exists $s \in U_1 \cap Q_F$ such that $s \in F^+(Y - Cl(W))$. Upper quasicontinuity of F at s implies by [9, Theorem 2.1] that there exists a non-empty open set $W_1 \subset U_1$ such that $W_1 \subset F^+(Y - Cl(W)) \subset F^+(Y - W)$. That is in contradiction with $U_1 \subset Cl(F^-(W))$.

By (2) follows that $F(s) \cap Cl(W) \neq \emptyset$ and thus $F(s) \cap V \neq \emptyset$. Lower quasicontinuity of F at s implies that there exists a nonempty open set $G \subset U$ such that $F(g) \cap V \neq \emptyset$ for each $g \in G$. Thus F is l.q.c. at x and $x \in Q_F$.

COROLLARY 1. (See [2]) *Let Y be a regular space and $f : X \to Y$ be an almost continuous single valued mapping, then Q_f is closed.*

COROLLARY 2. *Let Y be a regular space and $F : X \to Y$ be an almost continuous multifunction. If F is a compact valued multifunction then Q_F is closed in X.*

COROLLARY 3. (See [3]). *Let Y be a regular space and F be an almost continuous multifunction. If $Cl(Q_F) = X$ and F is a compact valued multifunction, then F is upper semi-continuous.*

Proof. By Corollary 2, F is q.c. and thus F is l.q.c., to F being u.a.c. and l.q.c. by [9, Theorem 5.1] F is u.w.c. and by [8, Theorem 3] F is u.s.c.

THEOREM 2. *If a multifunction $F : X \to Y$ is almost continuous and $F(x)$ is α-regular α-paracompact for each $x \in X$ such that $Cl(Q_F) = X$, then F is continuous.*

Proof. By Theorem 1 $Cl(Q_F) = Q_F = X$ and thus F is quasicontinuous. As F is almost continuous and quasicontinuous by [9, Theorem 5.3] F is weakly continuous. As F is u.w.c. and $F(x)$ is α-regular α-paracompact for each $x \in X$ by [10, Theorem 1] F is u.s.c. As F is l.w.c. and $F(x)$ is α-regular $\forall x \in X$ by [10, Theorem 2] F is l.s.c. Thus F is continuous.

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