TOPOLOGICAL EQUIVALENCE AND EXPONENTIAL DICHOTOMY OF LINEAR IMPULSIVE EQUATIONS (*)

by D.D. BAINOV and S.I. KOSTADINOV (in Plovdiv)
NGUYEN VAN MINH and P.P. ZABREIKO (in Minsk)(**)

SOMMARIO. - *In questo lavoro si introducono le nozioni di similarità cinematica e di equivalenza topologica per equazioni differenziali astratte impulsive. Si studiano poi alcune delle loro proprietà legate alla nozione di dicotomia esponenziale.*

SUMMARY. - *In the present paper the notion of kinematical similarity and topological equivalence for abstract impulsive differential equations are introduced and some of their properties related to the notion of exponential dichotomy are investigated.*

1. Introduction.

The work of V.D. Mil'man and A.D. Myshkis [1] (1960) set the beginning of the qualitative investigation of impulsive equations. Of the works which appeared later and were devoted to this subject we shall mention [2]–[6].

In the present paper the dependence between topological equivalence and exponential dichotomy of impulsive equations is investigated and some results of [6]–[9] are generalized.

(*) Pervenuto in Redazione il 17 marzo 1992.

(**) Indirizzo degli Autori: D.D. Bainov and S.I. Kostadinov: Plovdiv University "Paissii Hilendarski", Plovdiv (Bulgaria); N. Van Minh and P.P. Zabreiko: Byelorussian State University, Minsk (Bielorussia).

2. Statement of the problem.

Let X be an arbitrary Banach space with identical operator I. By $L(X)$ we shall denote the space of all linear bounded operators acting in X.

Let $\mathcal{J}$ be an arbitrary interval on the real axis $\mathbb{R} = (-\infty, \infty)$. Consider the impulsive equation

$$\frac{dx}{dt} = A(t) \quad (t \neq t_n) \tag{1}$$

$$x(t_n^+) = Q_n x(t_n) \tag{2}$$

where the sequence of points $T = \{t_n\}_{n=-\infty}^{\infty}$ satisfies the condition

$$t_n < t_{n+1} \ (n = \pm 1, \pm 2, \pm 3, \ldots), \quad \lim_{n \to \pm\infty} t_n = \pm\infty$$

Set $N_{\mathcal{J}} = \{k : t_k \in \mathcal{J}\}$.

We shall say that conditions (A) is met if the following conditions hold:

A1: The operator–valued function $A(\cdot) : \mathcal{J} \to X$ is continuously extendable on each interval $[t_n, t_{n+1}]$.

A2: $Q_n \in L(X)(n \in N_{\mathcal{J}})$.

Definition 1. A *solution* of the impulsive equation (1), (2) we shall call a function $\varphi(t)$ which is continuous for $t \neq t_n$, has discontinuities of the first kind at $t = t_n$, is continuous from the left, for $t \neq t_n$ satisfies equation (1) and at $t = t_n$ meets the condition of a "jump" (2).

Definition 2. An operator $\mathcal{U}(t, \tau)$ associating with each element $\xi \in X$ a solution $x(t) = \mathcal{U}(t, \tau)\xi$ of equation (1), (2) for which $x(\tau) = \xi$ is said to be a *fundamental operator* of (1), (2). Instead of $\mathcal{U}(t, 0)$ we shall write $U(t)$.

The validity of the following formulae is immediately established:

$$U(t,t) = I, \ U(t,s) = U(t,\tau)U(\tau,s) \tag{3}$$

$$U_t'(t,\tau) = A(t)U(t,\tau)(t \neq t_n), U_\tau'(t,\tau) = U(t,\tau)A(\tau)(\tau \neq t_n) \tag{4}$$

$$U(t_n^+,\tau) = Q_n U(t_n,\tau) \tag{5}$$

where $s \leq \tau \leq t$.

LEMMA 1. *Let conditions* (A) *hold.*
Then for $\tau \le t < \infty$ $(t, \tau \in \mathcal{J})$ *the following formula is valid*

$$U(t,\tau) = \begin{cases} U_0(t,\tau), \ t_n < \tau \le t \le t_{n+1} \\ U_0(t,t_n)(\prod_{j=n}^{k+1} Q_j U_0(t_j,t_{j-1}))Q_k U_0(t,\tau), \\ \qquad\qquad t_{k-1} < \tau \le t_k < t_n < t \le t_{n+1} \end{cases}$$

where $U_0(t,\tau)$ *is the fundamental operator of the equation*

$$\frac{dx}{dt} = A(t)x \tag{6}$$

We shall say that condition (B) is met if the operators Q_n have bounded inverse ones.

LEMMA 2. *Let conditions* (A) *and* (B) *hold.*
Then, for $\tau, t \in \mathcal{J}$ *the following formula is valid*

$$U(t,\tau) = \begin{cases} U_0(t,\tau), \ t_n < t \le \tau \le t_{n+1} \\ U_0(t,t_n)(\prod_{j=n}^{k+1} Q_j U_0(t_j,t_{j-1}))Q_k U_0(t,\tau), \\ \qquad\qquad t_{k-1} < \tau \le t_k < t_n < t \le t_{n+1} \\ U_0(t,t_n)(\prod_{j=n}^{k-1} Q_j^{-1} U_0(t_j,t_{j+1}))Q_k^{-1} U_0(t_k,\tau), \\ \qquad\qquad t_{n-1} < t \le t_n \le t_k < \tau \le t_{k+1} \end{cases}$$

where $U_0(t,\tau)$ *is the fundamental operator of* (6).

Lemmas 1 and 2 are proved by a straightforward verification.

LEMMA 3. *Let the function* $\varphi(t)$ *be piecewise continuous with points of discontinuity* τ_n $(n = 1, 2, 3, \ldots)$ *of the first kind and for* $t \ge t_0$ *let the following inequality hold*

$$\varphi(t) \le c + \sum_{t_0 < \tau_i < t} a_i \varphi(\tau_i) + \int_{t_0}^{t} v(\tau)\varphi(\tau)d\tau$$

where $c \ge 0$ *and* $a_i \ge 0$ *are constants and the function* $v(t) > 0$ *is locally summable on the half-axis* $t_0 \le t < \infty$.
Then the following estimate is valid

$$\varphi(t) \le c \prod_{t_0 < \tau_i < t} (1 + a_i) e^{\int_{t_0}^{t} v(\tau)d\tau}$$

The proof of the lemma is carried out by means of the lemma of Gronwall-Bellmann.

LEMMA 4. *Let the following conditions hold:*

1. *Condition* (B) *is met.*
2. *The following inequality is valid*

$$\left\| \prod_{j=n}^{k} Q_j \right\| \leq N e^{-\nu(t-s)}$$

$(t_{k-1} < s \leq t_k < t_n < t \leq t_{n+1},\ \nu -$ const., $N > 0$, const.*)*.

Then for the fundamental operator $U(t,s)$ *of the impulsive equation* (1), (2) *the following estimates are valid*

$$N e^{-\nu(t-s)} e^{-N\int_s^t \|A(\tau)\| d\tau} \leq \|U^{\pm 1}(t,s)\| \leq N e^{-\nu(t-s)} e^{N\int_s^t \|A(\tau)\| d\tau} \quad (7)$$

Proof. The fundamental operator of the impulsive equation

$$\frac{dx}{dt} = 0 \ (t \neq t_n) \quad (8)$$

$$x(t_n^+) = Q_n x(t_n) \quad (9)$$

has the form

$$U_1(t,s) = \prod_{j=n}^{k+1} Q_j \ (t_{k-1} < s \leq t_k < t_n < t \leq t_{n+1}) \quad (10)$$

Then for $U_1(t,s)$ the following estimate is valid

$$\|U_1(t,s)\| \leq N e^{-\nu(t-s)} \quad (11)$$

i.e. the condition of Lemma 4 [3] is met. for $U(t,s)$ we obtain the estimate

$$\|U(t,s)\| \leq N e^{-\nu(t-s)} e^{N\int_s^t \|A(\tau)\| d\tau}$$

The estimates

$$e^{-\nu(t-s) - N\int_s^t \|A(\tau)\| d\tau} \leq \|U^{\pm 1}(t,s)\| \leq N e^{-\nu(t-s)} e^{N\int_s^t \|A(\tau)\| d\tau}$$

are proved in an analogous way. ◇

Consider the initial value problem

$$\frac{dx}{dt} = f(t,x) \ (t \neq t_n) \tag{12}$$

$$x(t_n^+) = Q_n x(t_n) \tag{13}$$

$$x(t_0) = \xi \tag{14}$$

LEMMA 5. *Let the following conditions hold:*

1. *The function* $f(t,x) : \mathcal{J} \times X \to X$ *is continuous on* $(\mathcal{J} \backslash T) \times X$ *and* $f(t_n^-, x) = f(t_n, x)(x \in X)$.

2. $\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|(t \in \mathcal{J}; x, y \in X)$ *moreover, the function* $L(t)$ *satisfies the condition*

$$\tilde{L} = \sup_{a<b<\infty} \frac{1}{b-a} \int_a^b L(s)ds < \infty$$

3. $f(t,0) = 0(t \in \mathcal{J})$.

4. $Q_n \in L(X)$, $n \in N_{\mathbb{R}}$, $\mathcal{J} = \mathbb{R}$.

Then there exists a unique solution $x(t) = X(t, t_0, \xi)$ *of the initial value problem* (12), (13), (14) *for* $t \in \mathcal{J}$ *for which the following estimate is valid*

$$\|x(t)\| \leq \|\xi\| \prod_{t \in (\min(t_0,t), \max(t_0,t))} (1 + \|I_i\|) e^{\tilde{L}|t-t_0|}$$

$(I_n = Q_n - I)$.

Proof. From the theorems of existence and uniqueness of solutions of ordinary differential equations (see e.g. [9]) there follows the existence of a unique solution $x(t)$ $(t \in \mathcal{J})$ of the initial value problem (12), (13). For the solution $x(t)$ the following representation is valid

$$x(t) = \begin{cases} \xi + \int_{t_0}^t f(s,x))ds + \sum_{\tau<t_k<t} I_k(x(t_k)), \ t \geq t_0 \\ \\ \xi + \int_{t_0}^t f(s,x))ds - \sum_{t<t_k<\tau} I_k(x(t_k)), \ t < t_0 \end{cases}$$

Let $t \geq t_0$. Then

$$\|x(t)\| \leq \|\xi\| + \int_{t_0}^t \|f(s, x(s)\|ds + \sum_{t_0<t_k<t} \|I_k(x(t_k))\| \leq$$

$$\|\xi\| + \int_{t_0}^{t} L(s)\|x(s)\|ds + \sum_{t_0<t_k<t} \|I_k\|.\|x(t_k)\|$$

By Lemma 3 we obtain the estimate

$$\|x(t)\| \leq \|\xi\| \prod_{t_0<t_i<t} (1+\|I_i\|)e^{\tilde{L}(t-t_0)} \tag{15}$$

where

$$\tilde{L} = \sup_{a<b<\infty} \frac{1}{b-a}\int_a^b L(s)ds \tag{16}$$

for $t < t_0$ we obtain an estimate analogous to estimate (15).

Lemma 5 is proved. ◊

3. Main results.

Let equations (1), (2) and

$$\frac{dx}{dt} = B(t)x(t \neq t_n) \tag{17}$$

$$x(t_n^+) = \tilde{Q}_n x(t_n) \tag{18}$$

belong to class K.

DEFINITION 3. Equations (1), (2) and (17), (18) are said to be *kinematically similar* on $\mathcal{J}$ if there exists an operator–valued function $S(\cdot)$: $\mathcal{J} \to L(X)$ for which the following conditions are met:

1. $S(t)$ is bounded and continuous for $t \neq t_n$.
2. $S(t)$ has discontinuities of the first kind at $t = t_n$ and it is continuous from the left.
3. $S(t)$ has a bounded inverse one.
4. If $\varphi(t)$ is a solution of (1), (2), then $S(t)\varphi(t)$ is a solution of (17), (18).

REMARK 1. A necessary and sufficient condition for a kinematical similarity of (1), (2) and (17), (18) is the existence of an operator–valued function $S(t)$, $(t \in \mathcal{J})$ with the properties 1,2,3 of Definition 4 for which one of the following conditions is fulfilled

$$B = SAS^{-1} + S'S^{-1} \; (t \neq t_n,\; t \in \mathcal{J}) \tag{19}$$

$$S(t_n^+)Q_n = \tilde{Q}_n S(t_n)\ (t_n \in N_{\mathcal{J}}) \tag{20}$$

or

$$S = XY^{-1}$$

where X and Y are some evolutionary operators of equations (1), (2) and (17) , (18) respectively.

We shall call the function $S(t)$ a *transforming* one.

LEMMA 6. *Let the space X be real Hilbert and let it split up into a direct sum of subspaces X_1 and X_2 with the corresponding projectors P_1 and P_2, i.e. $X_i = P_i X$ $(i = 1, 2)$ and $X = X_1 \dot{+} X_2$.*

Then there exists a splitting $X = X_1 \dot{+} Y_2$ for which the corresponding projectors $\tilde{P}_1 : X \to X_1$ and $\tilde{P}_2 : X \to Y_2$ are Hermitian.

Proof. Let Y_1 be the orthocomplement to X_1. Then the projector $\tilde{P}_1 : X \to X_1$ with $\ker \tilde{P}_1 = Y_1$ is Hermitian. The projector $\tilde{P}_2 = I - \tilde{P}_1$ is also Hermitian and X splits up into a direct sum of the subspaces X_1 and $\tilde{P}_2 X$. ◇

LEMMA 7. *Let the space X be Hilbert and let conditions* (A), (B) *hold. Then equation* (1), (2) *is kinematically similar to the equation*

$$\frac{dx}{dt} = B(t)x$$

where the operator B is Hermitian.

Proof. For the operator A the representation $A = A_R + iA_{\mathcal{J}}$ is valid where $A_R = \frac{1}{2}(A + A^*)$, $A_{\mathcal{J}} = \frac{1}{2i}(A - A^*)$. As a transforming function we take the function $V^{-1}(t)$ $(t \in \mathcal{J})$, where $V(t)$ is a solution of the impulsive equation

$$\frac{dV}{dt} = iA_{\mathcal{J}}V \tag{21}$$

$$V(t_n^+) = Q_n V(t_n) \tag{22}$$

Since the operator $iA_{\mathcal{J}}$ is skew-Hermitian, then the function $V(t)$ is unitary and by the Banach-Steinhaus theorem it is bounded. By formula (19) for the operator B we obtain the representation

$$B = V^{-1} A_R V$$

and by formula (20) for the impulsive operators $\tilde{Q}_n$ of the new equation we obtain,

$$\tilde{Q}_n = V^{-1}(t_n^+)Q_nV(t_n) = V^{-1}(t_n)Q_n^{-1}Q_nV(t_n) = 1$$

Lemma 7 is proved. ◇

THEOREM 1. *Let the following conditions hold.*

1. *The space X is Hilbert with scalar product $(\cdot,\cdot)$ and splits up into a direct sum of the subspaces X_1 and X_2 with projectors P_1 and P_2 respectively.*
2. *Conditions* (A) *and* (B) *hold.*
3. *There exists a constant $M > 0$ for which*

$$\|U(t)P_iU^{-1}(t)\| \leq M \; (i = 1,2; \; t \in \mathcal{J})$$

Then the impulsive equation (3), (4) *is kinematically similar to the impulsive equation with an operator–valued function $B(t)$ $(t \in \mathcal{J})$ and impulse operators $\tilde{Q}_n$ which commute with the projectors $P_i(i = 1,2)$ and, moreover, the following estimate is valid*

$$\|B(t)\| \leq \|A(t)\| \; (t \in \mathcal{J}) \tag{23}$$

Proof. By Lemma 6 we shall assume that the projectors P_1 and P_2 are Hermitian. Set

$$R^2 = P_1U^*(t)U(t)P_1 + P_2U^*(t)U(t)P_2 \tag{24}$$

It is not hard to check that the operator $R^2(t)$ $(t \in \mathcal{J})$ is Hermitian and uniformly positive. Let $R(t)$ be the positive square root of $R^2(t)$. The operator $R(t)$ is also Hermitian. Since $R^2(t)$ commutes with $P_i(i = 1,2)$ then $R(t)$ also commutes with P_i. Consider the operator–valued function $S(t) = R(t)U^{-1}(t)(t \in \mathcal{J})$ where $U(t) = U(t,0)$ is the fundamental operator of (1), (2). A straightforward verification yields that the functions $S(t)$ and $S^{-1}(t)$ $(t \in \mathcal{J})$ are continuous and differentiable for $t \neq t_n$, have discontinuities of the first kind at $t = t_n$ and are continuous from the left. From the equalities

$$S^*S = (U^{-1})^*R^*RU^{-1} = (U^{-1})^*R^2U^{-1} =$$

$$(U^{-1})^*P_1U^*UP_1U^{-1} + (U^{-1})^*P_2U^*UP_2U^{-1} =$$

$$(UP_1U^{-1})^*(UP_1U^{-1}) + (UP_2U^{-1})^*UP_2U^{-1}$$

there follows the estimate

$$\|S\|^2 \le \|UP_1U^{-1}\|^2 + \|UP_2U^{-1}\|^2 \le M^2 + (1+M)^2 < \infty$$

We shall show that the function $S^{-1}(t)$ is also bounded

$$P_1(S^{-1})^*S^{-1}P_1 + P_2(S^{-1})^*S^{-1}P_2 =$$
$$P_1R^{-1}U^*UR^{-1}P_1 + P_2R^{-1}U^*UR^{-1}P_2 =$$
$$R^{-1}P_1U^*UP_1R^{-1} + R^{-1}P_2U^*UP_2R^{-1} =$$
$$R^{-1}(P_1U^*UP_1 + P_2U^*UP_2)R^{-1} = I$$

Let $z \in X$ be arbitrarily chosen. Then

$$\begin{aligned}
&\|S^{-1}z\|^2 = \\
&\|S^{-1}P_1z + S^{-1}P_2z\|^2 \le (\|S^{-1}P_1z\| + \|S^{-1}P_2z\|^2) \le \\
&2\|S^{-1}P_1z\|^2 + 2\|S^{-1}P_2z\|^2 = \\
&2(S^{-1}P_1z, S^{-1}P_1z) + 2(S^{-1}P_2z, S^{-1}P_2z) = \\
&2(P_1(S^{-1})^*S^{-1}P_1z, z) + 2(P_2(S^{-1})^*S^{-1}P_2z, z) = \\
&2((P_1(S^{-1})^*S^{-1}P_1 + P_2(S^{-1})^*S^{-1}P_2)z, z) = 2(z,z) = 2\|z\|^2
\end{aligned}$$

Choose the function $S(t)$ $(t \in \mathcal{J})$ as a transforming one. Then by formula (19) for B we obtain

$$B = SAS^{-1} + S'S^{-1}$$

i.e.

$$\begin{aligned}
&B = RU^{-1}AUR^{-1} + (R'U^{-1} + R(U^{-1})')UR' = \\
&RU^{-1}AUR^{-1} + R'R^{-1} - RU^{-1}U'U^{-1}UR^{-1} = \\
&RU^{-1}AUR^{-1} + R'R^{-1} - RU^{-1}AUU^{-1}UR^{-1} = R'R^{-1}
\end{aligned}$$

For the impulsive operators $\tilde{Q}_n$ by (20) we obtain

$$\begin{aligned}
&\tilde{Q}_n = S(t_n^+)Q_nS^{-1}(t_n) = R(t_n^+)U^{-1}(t_n)Q_n^{-1}U(t_n)R^{-1}(t_n) = \\
&R(t_n^+)U^{-1}(t_n)Q_n^{-1}Q_nU(t_n)R^{-1}(t_n) = R(t_n^+)R^{-1}(t_n)
\end{aligned}$$

The fundamental operator of the new equation is $\tilde{U} = SU = R$. It is not hard to check that the operators $B(t)$ and $\tilde{Q}_n$ commute with P_1

hence with P_2 too. We shall prove inequality (23). We differentiate equality (24) and obtain

$$RR' + R'R = P_1U^*(A^* + A)UP_1 + P_2U^*(A^* + A)UP_2$$

For an arbitrary $z \in X$ the following equality is valid

$$((RR' + R'R)z, z) = \\ = ((A^* + A)UP_1z, UP_1z) + ((A^* + A)UP_2z, UP_2z) \tag{25}$$

For any $t \in \mathcal{J}$ there exists a minimal constant $\beta(t)$ of which for $u \in X$ the following inequality is valid

$$([A^*(t) + A(t)]u, u) \leq \beta(t)(u, u)$$

Then from (25) we obtain

$$((RR' + R'R)z, z) \leq \beta\{(UP_1z, UP_1z) + (UP_2z, UP_2z)\} \ (z \in X)$$

i.e.

$$((RR' + R'R)z, z) \leq \beta(R^2z, z) \tag{26}$$

Set $v = Rz$. Then from (26) it follows that

$$((RR' + R'R)R^{-1}v, R^{-1}v) \leq \beta(Rv, R^{-1}v)$$

i.e.

$$((R'R^{-1} + R^{-1}R')v, v) \leq \beta(v, v)$$

hence

$$((R'R^{-1} + R^{-1}R')v, v) \leq \beta\|v\|^2 \tag{27}$$

Hence the operator

$$\frac{1}{2}(R'R^{-1} + R^{-1}R') = \frac{1}{2}(B + B^*) = B_R$$

is bounded and from (27) it follows that $\|B_R\| \leq \|A\|$. Inequality (23) follows from Lemma 7.

Theorem 1 is proved. ◇

DEFINITION 4. The impulsive equations (1), (2) and (17), (18) are said to be topologically equivalent if there exists a function h: $[0, \infty) \times X \to X$ with the following properties:

1. $h(t,x) \to \infty$ as $x \to \infty$ uniformly on t.
2. The mapping $h_t : X \to X$ defined by the equality

$$h_t(x) = h(t,x)$$

is a homeomorphism for $t \geq 0$.

3. The mapping $g : [0,\infty) \times X \to X$ defined by the equality $g(t,x) = (h_t)^{-1}(x)$ enjoys the property 1.

4. If $x(t)$ is a solution of (1), (2) then $h(t, x(t))$ is a solution of (17), (18).

REMARK 2. Condition 4 implies the equality

$$h(t, X(t)X^{-1}(s)x) = Y(t)Y^{-1}(s)h(s,x)$$

REMARK 3. A straightforward verification shows that the topological equivalence is an equivalence relation in the class of impulsive equations.

REMARK 4. If equations (1), (2) and (17), (18) are kinematically similar, then they are topologically equivalent. Indeed, in the definition of topological equivalence it suffices to set $h(t,x) = S(t)x$, where $S(t)$ is the function realizing the kinematical similarity.

Let $Y \subset X$ be an arbitrary subspace with projector $P_1 (Y = P_1 X)$ and let $P_2 = I - P_1$.

DEFINITION 5. Y is said to *induce an exponential dichotomy* for equation (1), (2) if the following inequalities are valid:

$$\|U(t)P_1U^{-1}(\tau)\| \leq Me^{-\delta(t-\tau)} \ (\tau < t < \infty)$$
$$\|U(t)P_2U^{-1}(\tau)\| \leq Me^{-\delta(\tau-t)} \ (t < \tau < \infty)$$

where M, δ are positive constants. In this case equation (1), (2) is said to be *exponentially dichotomous* (with projectors P_1 and P_2).

Let Y be an arbitrary Banach space. By $B(Y)$ we shall denote the set of functions $f(\cdot) : \mathcal{J} \to Y$ which are continuous for $t \neq t_n$ have points of discontinuity of the first kind at $t = t_n$ and are continuous from the left and which are integrally bounded, i.e.

$$\sup_{t \in \mathcal{J}} \int_t^{t+1} \|f(s)\|_Y ds < \infty$$

The set $B(Y)$ is a Banach space with norm

$$\|f\|_{B(Y)} = \sup_{t\in\mathcal{J}} \int_t^{t+1} \|f(s)\|_Y ds$$

By $M(Y)$ we shall denote the set of sequences $R = \{R_n\}$ whose elements lie in Y and which satisfy the following condition

$$\sup_n \|R_n\| < \infty \tag{28}$$

The set $M(Y)$ is a Banach space with norm

$$\|R\|_{M(Y)} = \sup_n \|R_n\| \tag{29}$$

The space

$$\tilde{B} = B(L(X)) \times M(L(X)) = \\ \{(A,Q) : A \in B(L(X)), Q \in M(L(X))\}$$

is Banach with norm

$$\|(A,Q)\|_{\tilde{B}} = \max\{\|A\|_{B(L(X))}, |Q\|_{M(L(X))}\}$$

DEFINITION 6. The impulsive equation (1), (2) is said to *belong to* $\tilde{B}$ which is written as $(A,Q) \in \tilde{B}$, if $A \in B(L(X))$ and $Q \in M(L(X))$. If the impulsive equations (A,Q) and $(B,\tilde{Q})$ are topologically equivalent, we write $(A,Q) \cong (B,\tilde{Q})$.

DEFINITION 7. Equation $(A,Q) \in \tilde{B}$ is said to be *structurally stable* if there exists $\rho > 0$ such that if $\|A-B,\ Q-\tilde{Q}\|_{\tilde{B}} \leq \rho$, then the impulsive equations (A,Q) and $(B,\tilde{Q})$ are topologically equivalent.

LEMMA 8. *The set of structurally stable equations forms an open non-empty subset of the Banach space* $\tilde{B}$. *If, moreover* $(A,Q) \in \tilde{B}$ *is structurally stable and* (A,Q) *is kinematically similar to* $(B,\tilde{Q}) \in \tilde{B}$, *then the equation* $(B,\tilde{Q})$ *is also structurally stable.*

Proof. Let the equation $(A,Q) \in \tilde{B}$ be structurally stable. Then there exists a number $\rho > 0$ for which the inequality $\|(A,Q) - (B,\tilde{Q})\|_{\tilde{B}} < \rho$ implies $(A,Q) \cong (B,\tilde{Q})$. The set

$$U(B,\tilde{Q}) = \{(C,\tilde{R}) \in \tilde{B} : \|(C,\tilde{R})-(B,\tilde{Q})\|_{\tilde{B}} < \rho-\|(B,\tilde{Q})-(A,Q)\|_{\tilde{B}}\}$$

is a neighbourhood of $(B,\tilde{Q})$ in $\tilde{B}$. Then

$$\|(C,\tilde{R})-(A,Q)\|\tilde{B} \leq \|(C,\tilde{R})-(B,\tilde{Q})\|_{\tilde{B}}+\|(B,\tilde{Q})-(A,Q)\|_{\tilde{B}} \leq \rho ,$$

i.e. $(C,\tilde{R}) \cong (A,Q)$. Since $(B,\tilde{Q}) \cong (A,Q)$ then $(B,\tilde{Q}) \cong (C,\tilde{R})$.

Hence the equation $(B,\tilde{Q})$ is structurally stable.

Let the function $S(t)$ realize the kinematical similarity between (A,Q) and $(B,\tilde{Q})$. Since the equation (A,Q) is structurally stable, then there exists a number $\delta > 0$ for which $\|(A,Q)-(F,Z)\|_{\tilde{B}} < \delta$ implies $(A,Q) \cong (F,Z)$. The set

$$U(B,\tilde{Q}) = \{(C,L) \in \tilde{B} : \|(C,L)-(B,\tilde{Q})\|_{\tilde{B}} < \frac{\delta}{\sup_{t\in\mathcal{J}}\|S(t)\| \sup_{t\in\mathcal{J}}\|S^{-1}(t)\|}\}$$

is a neighbourhood of $(B,\tilde{Q})$ in the space $\tilde{B}$.

Consider the operator-valued function $D(t) = S^{-1}(t)C(t)S(t) - S^{-1}(t)S'(t)$ and let us estimate th expression $\|A-D\|_{B(L(X))}$:

$$\|A-D\|_{B(L(X))} = \|S^{-1}CS - S^{-1}S' - (S^{-1}BS - S^{-1}S')\|_{B(L(X))} =$$
$$\|S^{-1}CS - S^{-1}BS\|_{B(L(X))} \leq$$
$$\|S^{-1}\|_{B(L(X))}\|CB\|_{B(L(X))}\|S\|_{B(L(X))} < \delta .$$

The equation (C,L) is kinematically similar to the equation (D,R) where $R=\{R_n\}$, $R_n = S^{-1}(t_n^+)L_nS(t_n)$, hence $(C,L) \cong (D,R)$. Let us estimate $\|R-Q\|_{M(L(X))}$:

$$\|R-Q\|_{M(L(X))} = \sup_n \|R_n - Q_n\|_{L(X)} =$$
$$\sup_n \|S^{-1}(t_n^+)L_nS(t_n) - S^{-1}(t_n^+)\tilde{Q}_nS(t_n)\|_{L(X)} \leq$$
$$\sup_{t\in\mathcal{J}} \|S^{-1}(t)\| \sup_{t\in\mathcal{J}} \|S(t)\| \sup_n \|L_n - \tilde{Q}_n\|_{L(X)} =$$
$$\sup_{t\in\mathcal{J}} \|S^{-1}(t)\| \sup_{t\in\mathcal{J}} \|S(t)\|.\|L-\tilde{Q}\|_{M(L(X))} < \delta$$

From the estimate obtained it follows that $\|(D,R)-(A,Q)\|_{\tilde{B}} < \delta$ and $(D,R) \cong (A,Q)$. From the kinematical similarity of (A,Q) and $(B,\tilde{Q})$ it follow that $(A,Q) \cong (B,\tilde{Q})$.

Lemma 8 is proved. ◇

LEMMA 9. *Consider the equation*

$$\frac{dx}{dt} = A(t)x \quad if\ t \neq t_n \tag{30}$$

$$x(t_n^+) = Q_n x(t_n) \tag{31}$$

Assume that (A, Q) satisfies the following conditions:

1. *Equation (A, Q) satisfies the conditions* (A), (B) *and has an exponential dichotomy with projector P_1, furthermore $\|A(t)\|$, $\|Q_n\|$ are bounded.*

2. *X is a Hilbert space.*

3. *$\lim_{T\to\infty} \frac{i(t,t+T)}{T} = p < \infty$ uniformly in $t \in \mathbb{R}$.*

Then equation (30), (31) *is topologically equivalent to the standard equation $\frac{dx_1}{dt} = -x_1$, $\frac{dx_2}{dt} = x_2$, $x_1 \in X_1$, $x_2 \in X_2$.*

Proof. First we assume $P_1 = 0$.

The proof of this lemma is suggested by Palmer [6].

Consider the function

$$V(t, x) = \int_{-\infty}^{t} \|U(\tau, t)x\|^2 d\tau$$

It is easily shown that $V(t, x)$ enjoys the following properties:

i) There exist positive constants a, b such that

$$a\|x\|^2 \le V(t, x) \le b\|x\|^2$$

ii) If $x(t)$ is any solution of (30), (31) then $V(t, x(t))$ is continuous with respect to $t \in \mathbb{R}$.

iii) V is differentiable with respect to x for every fixed t and differentiable with respect to t for $t \neq t_n$, furthermore $\frac{dV(t,x(t))}{dt} = \|x(t)\|^2$, where $x(t)$ is any solution of (30), (31).

Consider the function

$$f(t, s, x) = V(t, U(t, s)x)$$

It is seen that f is contonuous with respect to t, differentiable with respect to t for $t \neq t_n$, furthermore

$$\frac{\partial f}{\partial t}(t, s, x) = \|U(t, s)x\|^2$$

Under the assumptions of the lemma we can show without difficulty the existence of positive constants c, d such that

$$c\|x\|^2 \le \frac{\partial f}{\partial t}(t, s, x) \le d\|x\|^2 \quad \text{for all } t, s, x$$

Thus the equation

$$1 = f(t, s, x)$$

has a unique solution $t = t(s, x)$ for all $s \in \mathbb{R}$, $x \in X$, furthermore t depends on (s, x) continuously for $x \neq 0$.

Now we are in a position to establish the homeomorphism which transforms (30), (31) into the standard equation. In fact, we defined $h : \mathbb{R} \times X \to \mathbb{R} \times X$ as follows:

$$h(s,x) = h_s(x) = \begin{cases} e^{s-t(s,x)} \cdot \frac{U(t,s,x),s)x}{\|U(t,(s,x),s)x\|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It may be noted that h satisfies the conditions 2 and 4 in Definition 5. It is checked that

$$h_s^{-1}(x) = \begin{cases} U(s,u)(x/\sqrt{V(u,x)}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where

$$u = s - l_n\|x\|$$

Finally, we have to show that

$$\lim_{\|x\|\to\infty} \|h_t(x)\| = \lim_{\|x\|\to\infty} \|h_t^{-1}(x)\| = \infty$$

By definition we have

$$\begin{aligned} V(s,x) - 1 &= V(s,x) - V(t(s,x), X(t,(s,x),s,x)) \\ &= \int_{t(s,x)}^{s} D_{-}V(u, X(u,s,x))du \end{aligned}$$

where $D_{-}V(t, X(t,s,x)) =$
$\lim_{k\to+0} \frac{1}{k}[V(t_n, x(t_n)) - V(t_n - k, x(t_n - k))]$, $x(t) = X(t,s,x)$.
Hence

$$|s - t(s,x)| \leq |V(s,x) - 1| / \inf_{\substack{u\in[t(s,x),s] \\ u\neq t_n}} \frac{\partial f}{\partial u}(u,s,x)$$

If $\|x\| > 1/\sqrt{a}$, we have $V(s,x) \geq 1$. Thus $s \geq t(s,x)$ and we obtain

$$1 \leq f(u,s,x) \leq b\|U(u,s)x\|^2$$

Thus

$$\frac{\partial f}{\partial u}(u,s,x) = \|U(u,s)x\|^2 \geq \frac{1}{b}$$

for $u \in [t(s,x), s]$, $u \neq t_n$. So we get

$$0 \leq s - t(s,x) \leq (V(s,x) - 1)b \leq (a\|x\|^2 - 1)b$$

If $\|x\| \leq 1/\sqrt{b}$, we have $V(s,x) \leq 1$. Then $s \leq t(s,x)$ and we get

$$0 \leq 1 - V(s,x) \leq \bar{c}\|x\|^2 \int_s^{t(s,x)} \exp(K(u-s))du$$

where $\bar{c}, K$ are defined as positive constants satisfying

$$\|U(t,s)\| \leq \bar{c}\exp(K(t-s)) \text{ for all } t \geq s$$

The existence of $\bar{c}, K$ is easily shown by the assumptions of the lemma.

We get

$$0 \leq 1 - b\|x\|^2 \leq \frac{\bar{c}}{K}\|x\|^2(\exp(K(t(s,x)-s)) - 1)$$

$$\frac{1}{K}\ell n[\frac{K}{\bar{c}\|x\|}(1 - b\|x\|^2 + \frac{\bar{c}}{K}\|x\|^2) \leq t(s,x) - 1$$

from this and the above arguments it follows that there exists an increasing function $L_1 : [0,\infty) \to [0,\infty)$, with $L_1(0) = 0$, continuous at 0 such that

$$\|h_s(x)\| \leq L_1(\|x\|) \quad \text{for } s,x$$

Similarly, it is shown that there exists $L_2 : [0,\infty) \to [0,\infty)$ with the same properties as L_1 such that

$$\|h_s^{-1}\| \leq L_2(\|x\|) \quad \text{for all } s,x$$

Finally, from this we deduce without difficulty that

$$\lim_{|x\|\to\infty} \|h_s(x)\| = \lim_{|x|\to\infty} \|h_s^{-1}(x)\| = \infty$$

Now we are in a position to prove Lemma 9 without the assumption $P_1 = 0$.

From Theorem 1 it follows that (30), (31) is kinematically similar to a reducible equation. Applying repeatedly the above result we obtain the proof of Lemma 9. ◊

THEOREM 2. *Let the conditions of Lemma 9 be met.*
Then equation (A, Q) is *structurally stable.*

Proof. We complete the definition of the function $A(t)$ $(t \in \mathcal{J})$ on $\mathbb{R}$ in such a way that the new function $\tilde{A}(T)$ is bounded and the equation

$$\frac{dx}{dt} = \tilde{A}(t)x \; (t \neq t_n) \tag{32}$$

$$x(t_n^+) = Q_n x(t_n) \; (t_n \in \mathcal{J}) \tag{33}$$

is exponentially dichotomous on $\mathbb{R}\backslash\mathcal{J}$ with projector P_2 for which $\ker P_2 + J_m P_1 = X$. Then (see [8]) equation (32), (33) will be exponentially dichotomous on $\mathbb{R}$ with projector P, $\ker P = \ker P_2$. $J_m P = J_m P_1$. By Lemma 9 equation (32), (33) is topologically equivalent of $\mathbb{R}$ to the system

$$\frac{dx_1}{dt} = -x_1 \quad (x_1 \in X_1) \tag{34}$$

$$\frac{dx_2}{dt} = x_2 \quad (x_2 \in X_2) \tag{35}$$

If we consider the restrictions of all functions of system (34), (35) on $\mathcal{J}$, we obtain that $(A, Q) \cong$ (34), (35).

By Theorem 3 [11] there exists a number $\delta > 0$ such that for $\|(B, R) - (A, Q)\|_{\tilde{B}} < \delta$ equation (B, R) is exponentially dichotomous with subspaces Y_1 and Y_2. From Lemma 9 it follows that (B, R) is topoloigcally equivalent of the system

$$\frac{dx_1}{dt} = -x_1 \quad (x_1 \in Y_1) \tag{36}$$

$$\frac{dx_2}{dt} = x_2 \quad (x_2 \in X_2) \tag{37}$$

Moreover, (34), (35) $\cong$ (36), (37) therefore $(B, R) \cong$ (34), (35), i.e. $(A, Q) \cong (B, R)$.

Theorem 2 is proved. ◊

REMARK 5. Following the ideas of the exposition in [7] and lemmas 4,5 it is not hard to check that for dim $X < \infty$ if equation (A, Q) is structurally stable, then it is exponentially dichotomous too.

REFERENCES

[1] MIL'MAN V.D. and MYSHKIS A.D., *On the stability of motion in the presence of impulses*, Sib. Math. J. **1** (1960), 233-237 (in Russian).

[2] BAINOV D.D., ZABREIKO P.P. and KOSTADINOV S.I., *Stability of the general exponent of nonlinear impulsive differential equations in a Banach space*, International J. of Theoretical Physics **27** (3) (1988), 374-380.

[3] BAINOV D.D., ZABREIKO P.P. and KOSTADINOV S.I., *Characteristic exponent of impulsive differential equations in a Banach space*, International J. of Theoretical Physics **27** (6) (1988), 731-743.

[4] BAINOV D.D., KOSTADINOV S.I. and ZABREIKO P.P., *L_p-equivalence of impulsive equations*, International J. of Theoretical Physics **27** (11) (1988), 1411-1423.

[5] BAINOV D.D., KOSTADINOV S.I. and MYSHKIS A.D., *Bounded and periodic solutions of differential equations with impulse effect in a Banach space*, Differential and Integral Equations **1** (2) (1988), 223-230.

[6] PALMER J.K., *A characterization of exponential dichotomy in terms of topological equivalence*, J. Math. Anal. Appl. **69** (1) (1989), 8-16.

[7] PALMER K.J., *The structurally stable linear systems on the half-line are those with exponential dichotomies*, J. of Differential Equations **33** (1979), 16-25.

[8] COPPEL W.A., *Dichotomies in Stability Theory*, Lecture Notes in Mathematics, v. **629**, Springer-Verlag, Berlin (1978), p.97.

[9] DALECKII YU. L. and KREIN M.G., *Stability of solutions of differential equations in Banach space*, AMS, Books and Journals in Advanced Mathematics (1974), pp. 386.

[10] COPPEL W.A., *Stabillity and asymptotic behaviour of differential equations*, Heath., Boston (1965).

[11] BAINOV D.D., KOSTADINOV S.I. and ZABREIKI P.O., *Exponential dichotomy of linear impulsive differential equations in a Banach space*, International J. of Theoretical Physics **28** (1989) No.**7**, 797-814.