LECTURES ON GEOMETRIC INTEGRATION AND THE DIVERGENCE THEOREM (*)

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In these lectures I shall present some geometric aspects of the generalized Riemann integral, defined by Henstock and Kurzweil about thirty years ago. In particular, I shall discuss in considerable detail the development of ideas that led to a multidimensional version of the integral, which is coordinate free and provides the divergence theorem for nonlipschitzian vector fields. No a priori knowledge of the subject is assumed.

1. Notation and terminology.

The set of all real numbers is denoted by $\mathbf{R}$. Our ambient space is the m-fold Cartesian product of $\mathbf{R}$, denoted by $\mathbf{R}^m$; here m is a fixed positive integer. For $x = (\xi_1, \ldots, \xi_m)$ and $y = (\eta_1, \ldots, \eta_m)$ in $\mathbf{R}^m$, we let

$$x \cdot y = \sum_{j=1}^{m} \xi_j \eta_j \,, \quad \|x\| = \sqrt{x \cdot x}, \quad \text{and} \quad |x| = \max\{|\xi_1|, \ldots, |\xi_m|\} .$$

In $\mathbf{R}^m$ we use exclusively the metric induced by the norm $|x|$. If $E \subset \mathbf{R}^m$ then $\mathrm{cl}E$, $\mathrm{int}E$, $\mathrm{bd}E$ and $d(E)$ denote, respectively, the closure, interior, boundary, and diameter of E. For $x \in \mathbf{R}^m$ and $\varepsilon > 0$, we set

$$U(x, \varepsilon) = \{y \in \mathbf{R}^m : |x - y| < \varepsilon\} .$$

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As is customary in geometric measure theory, a *measure* is an *outer measure*. If $k \leq m$ is a positive integer, we denote by λ_k the k-dimensional Lebesgue measure in $\mathbf{R}^k$. We write λ instead of λ_1, and $|E|$ instead of $\lambda_m(E)$ for each $E \subset \mathbf{R}^m$. If E is a λ_k-measurable subset of $\mathbf{R}^k$, we denote by $\mathcal{L}_1(E)$ the family of all real-valued λ_k-measurable functions defined on E for which the finite *Lebesgue integral* $(L)\int_E f\,d\lambda_k$ exists. If $k = m$, we write $(L)\int_E f$ instead of $(L)\int_E f\,d\lambda_m$.

Unless specified otherwise, the words "measure" and "measurable", as well as the expressions "almost all" and "almost everywhere", refer to the measure λ_m. Sets $A, B \subset \mathbf{R}^m$ are called *nonoverlapping* if they are almost disjoint, i.e. if $|A \cap B| = 0$.

By an *interval* we always mean a compact nondegenerate subinterval of $\mathbf{R}^m$, i.e. the product $K = \prod_{j=1}^m [a_j, b_j]$ where $a_j < b_j$ are real numbers for $j = 1, \ldots, m$. If $b_j = a_j = d$ for $j = 1, \ldots, m$, we say that K is a *cube* with $d(K) = d$; in particular, $\mathrm{cl}U(x, \varepsilon)$ is a cube of diameter 2ε. A *dyadic cube* is the interval

$$\prod_{j=1}^{m} \left[k_j 2^{-n}, (k_j + 1)2^{-n}\right]$$

where $k_1, \ldots, k_m$ and n are integers with $n \geq 0$. It is easy to see that any family $\mathcal{K}$ of dyadic cubes contains a nonoverlapping subfamily $\mathcal{C}$ such that $\bigcup \mathcal{C} = \bigcup \mathcal{K}$. This property makes dyadic cubes very useful.

Without additional attributes, a *function* is always *real-valued*. The algebraic operations, order, and convergence among functions on the same set are defined *pointwise*. When no confusion is possible, we do not distinguish between a function defined on a set A and its restriction to a set $B \subset A$.

2. Partitions.

A *partition* is a collection (possibly empty)

$$P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$$

where $A_1, \ldots, A_p$ are nonoverlapping intervals and $x_i \in A_i$ for $i = 1, \ldots, p$. The set $\bigcup_{i=1}^p A_i$ is called the *body* of P, denoted by $\bigcup P$. If A is an interval, we say that P is a *partition in* A whenever $\bigcup P \subset A$, and a *partition of* A whenever $\bigcup P = A$. A partition P is called *special* if x_i is a *vertex* of A_i for $i = 1, \ldots, p$.

REMARK 2.1. To each partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ corresponds to a unique special partition P^{sp} defined as follows. Every pair (A_i, x_i) determines uniquely nonoverlapping intervals $A_{i,1}, \ldots, A_{i,k_i}$ where $1 \le k_i \le 2^m$, each $A_{i,k}$ has x_i as a vertex, and $\bigcup_{k=1}^{k_i} A_{i,k} = A_i$. We let

$$P^{sp} = \{(A_{i,k}, x_i) : k = 1, \ldots, k_i;\ i = 1, \ldots, p\}.$$

Given a partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ and a function f on a set $E \subset \mathbf{R}^m$ containing $\{x_1, \ldots, x_p\}$, we let

$$\sigma(f, P) = \sum_{i=1}^{p} f(x_i)\,|A_i|$$

and call this number the *Stieltjes sum* of f over P. Note that $\sigma(f, P) = \sigma(f, P^{sp})$.

Intuitively, an *integral* of a function f defined on an interval A is a real number that can be approximated by Stieltjes sums of f over suitable partitions P in A. Naturally, it is the mode of approximation that determines the utility of the integral.

If δ is a positive function on a set $E \subset \mathbf{R}^m$, then a partition

$$\{(A_1, x_1), \ldots, (A_p, x_p)\}$$

with $\{x_1, \ldots, x_p\} \subset E$ is called *δ-fine* whenever $d(A_i) < \delta(x_i)$ for $i = 1, \ldots, p$. The following existence result, which is fundamental for our exposition, is referred to as *Cousin's lemma.*

LEMMA 2.2. *Let* Δ *be a positive function defined on an interval* $K = \prod_{j=1}^{m} [r_j, s_j]$, *where* $r_j < s_j$ *are integers for* $j = 1, \ldots, m$. *Then there is a* Δ*-fine partition* $\{(K_1, x_1), \ldots, (K_p, x_p)\}$ *of* K *such that* $K_1, \ldots, K_p$ *are dyadic cubes.*

Proof. For $n = 0, 1, \ldots,$ denote by $\mathcal{K}_n$ the family of all dyadic cubes $L \subset K$ such that $d(L) = 2^{-n}$ and there is no Δ-fine partition

$\{(K_1, x_1), \dots, (K_p, x_p)\}$ of L where $K_1, \dots, K_p$ are dyadic cubes. Assuming that the lemma is false and proceeding inductively, it is easy to construct a decreasing sequence $\{L_n\}$ so that $L_n \in \mathcal{K}_n$ for $n = 0, 1, \dots$, Now we have $\bigcap_{n=0}^{\infty} L_n = \{x^*\}$ and $d(L_{n^*}) < \Delta(x^*)$ for a sufficiently large integer $n^* \geq 0$. It follows that $\{(L_{n^*}, x^*)\}$ is a Δ-fine partition of L_{n^*}, a contradiction.

PROPOSITION 2.3. *For each positive function δ on an interval A there is a δ-fine special partition P of A.*

Proof. For $j = 1, \dots, m$, find integers $r_j < s_j$ so that A is contained in the interval $K = \prod_{j=1}^{m} [r_j, s_j]$, and extend δ to a positive function Δ on K by letting

$$\Delta(x) = \inf_{y \in A} |x - y|$$

for each $x \in K - A$. If $\{(K_1, x_1), \dots, (K_p, x_p)\}$ is a Δ-fine partition of K whose existence is guaranteed by Lemma 2.2, then

$$Q = \{(A \cap K_i, x_i) : |A \cap K_i| > 0\}$$

is a δ-fine partition of A. Now it suffices to use Remark 2.1 and let $P = Q^{sp}$.

COROLLARY 2.4. *Let δ be a positive function on an interval A. Each δ-fine partition P in A is a subset of a δ-fine partition Q of A.*

3. The HK-integral.

It appears prudent to discuss first the one-dimensional integral, originally defined by Henstock ([5]) and Kurzweil ([7]). Thus throughout this section we assume that $m = 1$.

DEFINITION 3.1. A function f defined on an interval A is called *HK-integrable* in A if there is a real number I having the following property: given $\varepsilon > 0$, we can find a positive function δ on A so that

$$|\sigma(f, P) - I| < \varepsilon$$

for each δ-fine partition P of A.

It follows from Proposition 2.3 that the number I of Definition 3.1 is determined uniquely by the HK-integrable function f. We call it the *HK-integral* of f over A, denoted by $(HK)\int_A f$ or $(HK)\int_a^b f$ if $A = [a,b]$. By $\mathcal{HK}(A)$ we denote the family of all HK-integrable functions in A.

REMARK 3.2. In view of Remark 2.1, requiring that the inequality $|\sigma(f,P) - I| < \varepsilon$ is satisfied only when P is a δ-fine *special* partition of A does not alter Definition 3.1. This observation will simplify several proofs.

The HK-integral clearly generalizes the classical *Riemann integral*, obtained from Definition 3.1 by restricting δ to a *constant function*. That this generalization is proper follows from the example below.

EXAMPLE 3.3. For each $x \in [0,1]$ set

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

We claim that f belongs to $\mathcal{HK}([0,1])$ and that $(HK)\int_0^1 f = 0$. To see this, choose an $\varepsilon > 0$ and order all the rational numbers from [0,1] into a sequence $\{r_n\}$. Define a positive function δ on [0,1] by setting

$$\delta(x) = \begin{cases} \varepsilon 2^{-n-1} & \text{if } x = r_n,\ n = 1,2,\dots, \\ 1 & \text{if } x \text{ is irrational}. \end{cases}$$

Now if $P = \{(A_1,x_1),\dots,(A_p,x_p)\}$ is a δ-fine partition of [0,1], then

$$0 \le \sigma(f,P) \le \sum_{n=1}^{\infty} \sum_{x_i = r_n} |A_i| < \sum_{n=1}^{\infty} 2\cdot\varepsilon 2^{-n-1} = \varepsilon,$$

which establishes our claim.

PROPOSITION 3.4. *If A is an interval, then the family $\mathcal{HK}(A)$ is a linear space over* $\mathbf{R}$ *and the map $f \mapsto (HK)\int_A f$ is a nonnegative linear functional on $\mathcal{HK}(A)$.*

We leave the standard proof of Proposition 3.4 to the reader, and turn to the important *Cauchy test for integrability.*

LEMMA 3.5. *A function f on an interval A belongs to $\mathcal{HK}(A)$ if and only if for each $\varepsilon > 0$ there is a positive function δ on A such that*

$$|\sigma(f,P) - \sigma(f,Q)| < \varepsilon$$

for all δ-fine partitions P and Q of A.

Proof. For $n = 1,2,\ldots$, choose a positive function δ_n on A so that the condition of the lemma is satisfied for $\varepsilon = 1/n$. Replacing δ_n by $\min(\delta_1,\ldots,\delta_n)$, we may assume that $\delta_1 \geq \delta_2 \geq \ldots$ If P_n is a δ_n-fine partition of A, then it is easy to verify that the sequence $\{\sigma(f,P_n)\}$ is Cauchy. Let $I = \lim \sigma(f,P_n)$, and choose an $\varepsilon > 0$. There is an integer $n^* > 2/\varepsilon$ such that $|\sigma(f,P_{n^*}) - I| < \varepsilon/2$, and we set $\delta = \delta_{n^*}$. Now if P is a δ-fine partition of A, then

$$|\sigma(f,P) - I| \leq |\sigma(f,P) - \sigma(f,P_{n^*})| + |\sigma(f,P_{n^*}) - I| < \frac{1}{n^*} + \frac{\varepsilon}{2} < \varepsilon .$$

Consequently, $f \in \mathcal{HK}(A)$ and $I = (HK)\int_A f$. The converse is obvious.

PROPOSITION 3.6. *If $f \in \mathcal{HK}(A)$, then $f \in \mathcal{HK}(B)$ for each subinterval of B of A.*

Proof. Choose an $\varepsilon > 0$ and find a positive function δ on A so that $|\sigma(f,P) - \sigma(f,Q)| < \varepsilon$ for each pair of δ-fine partitions P and Q of A. With no loss of generality we may assume that B is a proper subinterval of A and that A is the union of nonoverlapping intervals B and C. By Proposition 2.3, there is a δ-fine partition P_C of C. Now if P_B and Q_B are δ-fine partitions of B, then $P = P_B \cup P_C$ and $Q = Q_B \cup P_C$ are δ-fine partitions of A, and we have

$$\varepsilon > |\sigma(f,P) - \sigma(f,Q)| = |\sigma(f,P_B) - \sigma(f,Q_B)| .$$

In view of Cauchy's test, this completes the argument.

PROPOSITION 3.7. *Let f be a function on an interval $[a,b]$, and let $c \in (a,b)$. If f is HK-integrable in $[a,c]$ and $[c,b]$, then it is HK-integrable in $[a,b]$ and*

$$(HK)\int_a^b f = (HK)\int_a^c f + (HK)\int_c^b f.$$

Proof. Set $I = (HK)\int_a^c f + (HK)\int_c^b f$, and choose an $\varepsilon > 0$. There are positive functions δ_- and δ_+ on the intervals $[a,c]$ and $[c,b]$, respectively, such that

$$\left|\sigma(f,P_-) - (HK)\int_a^c f\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|\sigma(f,P_+) - (HK)\int_c^b f\right| < \frac{\varepsilon}{2}$$

for each δ_--fine partition P_- of $[a,c]$, and for each δ_+-fine partition P_+ of $[c,b]$. Define a positive function δ on $[a,b]$ by setting

$$\delta(x) = \begin{cases} \min\{\delta_-(x), c-x\} & \text{if } x < c, \\ \min\{\delta_+(x), x-c\} & \text{if } x > c, \\ \min\{\delta_-(c), \delta_+(c)\} & \text{if } x = c. \end{cases}$$

Now choose a δ-fine *special* partition P of $[a,b]$. From the choice of δ, we see that $P = P_- \cup P_+$, where P_- is a δ_--fine partition of $[a,c]$ and P_+ is a δ_+-fine partitions of $[c,b]$. Since $\sigma(f,P) = \sigma(f,P_-) + \sigma(f,P_+)$, we obtain

$$|\sigma(f,P)-I| \le \left|\sigma(f,P_-)-(HK)\int_a^c f\right| + \left|\sigma(f,P_+)-(HK)\int_c^b f\right| < \varepsilon$$

and the proposition follows from Remark 3.2.

THEOREM 3.8. *If A is an interval, then $\mathcal{L}_1(A) \subset \mathcal{HK}(A)$ and*

$$(HK)\int_A f = (L)\int_A f$$

for each $f \in \mathcal{L}_1(A)$.

Proof. Let $f \in \mathcal{L}_1(A)$ and $\varepsilon > 0$. By the Vitali-Carathéodory theorem ([14, Theorem 2.25]), there are *extended* real-valued functions g and

h defined on A which are, respectively, upper and lower semicontinuous, and such that

$$g \le f \le h \quad \text{and} \quad (L)\int_A (h-g) < \varepsilon\,.$$

Find a positive function δ on A so that

$$g(y) < f(x) + \varepsilon \quad \text{and} \quad h(y) > f(x) > f(x) - \varepsilon$$

for each x and y in A for which $|x-y| < \delta(x)$. Now if $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a δ-fine partition of A, then

$$(L)\int_{A_i} g \le (L)\int_{A_i} f \le (L)\int_{A_i} h\,,$$

$$(L)\int_{A_i} g - \varepsilon|A_i| \le f(x_i)|A_i| \le (L)\int_{A_i} h + \varepsilon|A_i|$$

for $i = 1, \ldots, p$. It follows that

$$\begin{aligned} \left|\sigma(f,P) - (L)\int_A f\right| &\le \sum_{i=1}^{p} \left|f(x_i)|A_i| - (L)\int_{A_i} f\right| \\ &\le \sum_{i=1}^{p} \left[\varepsilon|A_i| + (L)\int_{A_i}(h-g)\right] < \varepsilon(|A|+1) \end{aligned}$$

and the theorem is proved.

COROLLARY 3.9. *Let f and g be functions defined on an interval A and let $f(x) = g(x)$ for almost all $x \in A$. Then f belongs to $\mathcal{HK}(A)$ if and only if g does, in which case*

$$(HK)\int_A f = (HK)\int_A g\,.$$

PROPOSITION 3.10. *If F is a differentiable function in an interval $[a,b]$, then $F' \in \mathcal{HK}([a,b])$ and*

$$(HK)\int_a^b F' = F(b) - F(a)\,.$$

Proof. Given $\varepsilon > 0$, there is a positive function δ on $[a, b]$ such that

$$\left|\frac{F(t) - F(x)}{t - x} - F'(x)\right| < \varepsilon$$

for each $t, x \in [a, b]$ with $|t - x| < \delta(x)$. Now let

$$P = \{([t_0, t_1], x_1), \ldots, ([t_{p-1}, t_p], x_p)\}$$

be a δ-fine *special* partition of $[a, b]$. By our choice of δ,

$$|F(t_i) - F(t_{i-1}) - F'(x_i)(t_i - t_{i-1})| =$$

$$\left|\frac{F(t_i) - F(t_{i-1})}{t_i - t_{i-1}} - F'(x_i)\right| \cdot (t_i - t_{i-1}) < \varepsilon(t_i - t_{i-1}) ,$$

and so

$$\left|\sigma(F', P) - [F(b) - F(a)]\right| =$$

$$\left|\sum_{i=1}^{p} F'(x_i)(t_i - t_{i-1}) - \sum_{i=1}^{p}[F(t_i) - F(t_{i-1})]\right| \leq$$

$$\sum_{i=1}^{p}\left|F'(x_i)(t_i - t_{i-1}) - [F(t_i) - F(t_{i-1})]\right| < \varepsilon(b - a) .$$

Theorem 3.8 asserts that the HK-integral extends the Lebesgue integral over intevals. The next example shows that the extension is proper.

EXAMPLE 3.11. Set $F(0) = f(0) = 0$, and for $x \neq 0$ let

$$F(x) = x^2 \sin x^{-2} \quad \text{and} \quad f(x) = 2x \sin x^{-2} - 2x^{-1} \cos x^{-2} .$$

Since $F'(x) = f(x)$ for each $x \in \mathbf{R}$, it follows from Proposition 3.10 that $f \in \mathcal{HK}([0, 1])$. On the other hand, it is easy to verify that $(L) \int_0^1 f$ does not exist.

The following technical result, referred to as *Henstock's lemma*, is very useful.

LEMMA 3.12. *Let A be an interval and let $f \in \mathcal{HK}(A)$. Given $\varepsilon > 0$, there is a positive function δ on A such that*

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - (HK)\int_{A_i} f \right| < \varepsilon$$

for each δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A.

Proof. Let δ be a positive function on A such that

$$|\sigma(f,P) - (HK)\int_A f| < \frac{\varepsilon}{3}$$

for each δ-fine partition P of A. Choose a δ-fine partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ of A and reorder it so that $f(x_i)|A_i| - (HK)\int_{A_i} f$ is nonnegative for $i = 1, \dots, k$ and negative for $i = k+1, \dots, p$ where k is an integer with $0 \leq k \leq p$. Using Propositions 2.3 and 3.6, we can find a δ-fine partition P_i of A_i so that $|\sigma(f, P_i) - \int_{A_i} f| < \varepsilon/3p$ for $i = 1, \dots, p$. Now

$$P = \{(A_1, x_1), \dots, (A_k, x_k)\} \cup \bigcup_{i=k+1}^{p} P_i,$$

$$Q = \{(A_{k+1}, x_{k+1}), \dots, (A_p, x_p)\} \cup \bigcup_{i=1}^{k} P_i$$

are δ-fine partitions of A. Thus

$$\begin{aligned}
\frac{\varepsilon}{3} > \left|\sigma(f,P) - (HK)\int_A f\right| &\geq \sum_{i=1}^{k}\left[f(x_i)|A_i| - (HK)\int_{A_i} f\right] \\
&\quad - \left|\sum_{i=k+1}^{p}\left[\sigma(f,P_i) - (HK)\int_{A_i} f\right]\right| \\
&\geq \sum_{i=1}^{k}\left|f(x_i)|A_i| - (HK)\int_{A_i} f\right| - (p-k)\frac{\varepsilon}{3p},
\end{aligned}$$

and similarly,

$$\frac{\varepsilon}{3} > \sum_{i=k+1}^{p}\left|f(x_i)|A_i| - (HK)\int_{A_i} f\right| - k\frac{\varepsilon}{3p}.$$

In view of Corollary 2.4, the lemma follows by adding the last two inequalities.

PROPOSITION 3.13. *Let f be an HK-integrable function in an interval $[a,b]$, and set $F(a) = 0$ and $F(x) = (HK)\int_a^x f$ for each $x \in (a,b]$. Then F is continuous in $[a,b]$.*

Proof. Choose an $\varepsilon > 0$ and an $x \in [a,b]$. It follows from Henstock's lemma that there is a $\Delta > 0$ such that $|f(x)|\Delta < \varepsilon$ and

$$|f(x)(y-x) - [F(y) - F(x)]| < \varepsilon$$

for each $y \in [a,b]$ with $|y-x| < \Delta$. Thus $|F(y) - F(x)| < 2\varepsilon$ whenever $|y-x| < \Delta$, and the continuity of F at x is established.

THEOREM 3.14. *Let f be an HK-integrable function in an interval $[a,b]$, and set $F(a) = 0$ and $F(x) = (HK)\int_a^x f$ for each $x \in (a,b]$. Then for almost all $x \in [a,b]$, the function F is differentiable at x and $F'(x) = f(x)$.*

Proof. Let E be the set of all $x \in [a,b]$ for which either F is not differentiable at x or $F'(x) \neq f(x)$. Given $x \in E$, we can find a $\gamma(x) > 0$ so that for each $\beta > 0$ there is a $t \in [a,b]$ with $|x-t| < \beta$ and

$$|f(x)(x-t) - [F(x) - F(t)]| \geq \gamma(x)|x-t|\,.$$

Fix an integer $n \geq 1$ and let $E_n = \{x \in E : \gamma(x) \geq 1/n\}$. Given $\varepsilon > 0$, use Henstock's lemma to find a positive function δ on A so that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - (HK)\int_{A_i} f \right| < \frac{\varepsilon}{5n}$$

for each δ-fine partition $\{(A_1,x_1),\dots,(A_p,x_p)\}$ in A. Let $\mathcal{E}$ be the collection of all subintervals $B = [t,t^*]$ of (a,b) with $d(B) < \delta(x_B)$ and

$$|f(x_B)(t^* - t) - [F(t^*) - F(t)]| \geq \frac{1}{n}|B|$$

for an $x_B \in \{t, t^*\} \cap E_n$. Clearly, $E_n \subset \cup \mathcal{E}$. By [19, Theorem 1.3.1], there is a disjoint subfamily $\{B_1, B_2, \ldots\}$ of $\mathcal{E}$ for which

$$|E_n| \leq 5 \sum_i |B_i| .$$

Let $B_i = [t_i, t_i^*]$, and observe that $\{(B_1, x_{B_1}), \ldots, (B_p, x_{B_p})\}$ is a δ-fine partition in $[a, b]$ whenever the set B_p is defined. Hence

$$\begin{aligned}\sum_{i=1}^{p} |B_i| &\leq n \sum_{i=1}^{p} |f(x_{B_i})(t_i^* - t_i) - [F(t_i^*) - F(t_i)]| \\ &= n \sum_{i=1}^{p} \left| f(x_{B_i})|B_i| - (HK) \int_{B_i} f \right| < \frac{\varepsilon}{5}\end{aligned}$$

and consequently $|E_n| \leq \varepsilon$. By the arbitrariness of ε, the set E_n has measure zero; and so does $E = \cup_{n=1}^{\infty} E_n$.

COROLLARY 3.15. *Each HK-integrable function is measurable.*

Proof. Let f be an HK-integrable function in an interval $[a, b]$, and let F be the continuous function on $[a, b]$ defined in Proposition 3.13. By Theorem 3.14, we have

$$f(x) = \lim n \left[F\left(x + \frac{1}{n}\right) - F(x) \right]$$

for almost all $x \in [a, b)$, and the measurability of f follows.

THEOREM 3.16. *Let f be a function defined on an interval A. Then f belongs to $\mathcal{L}_1(A)$ if and only if both f and $|f|$ belong to $\mathcal{HK}(A)$.*

Proof. As the converse follows from Theorem 3.8, let f and $|f|$ be in $\mathcal{HK}(A)$. Then f is meaurable and

$$\begin{aligned}(L) \int_A |f| &= \lim (L) \int_A \min\{|f|, n\} = (HK) \int_A \min\{|f|, n\} \\ &\leq (HK) \int_A |f| < +\infty .\end{aligned}$$

COROLLARY 3.17. *Let $\{f_n\}$ be a sequence of HK-integrable functions in an interval A, and let* $\lim f_n = f$. *Then*

$$f \in \mathcal{HK}(A) \quad \text{and} \quad (HK)\int_A f = \lim (HK)\int_A f_n$$

whenever either of the following conditions is satisfied:

1. $f_n \leq f_{n+1}$ *for* $n = 1, 2, \ldots$ *and* $\lim (HK) \int_A f_n < +\infty$;
2. $g \leq f_n \leq h$ *for some* $g, h \in \mathcal{HK}(A)$ *and all* $n = 1, 2, \ldots$.

Proof. In view of Theorems 3.8 and 3.16, it suffices to apply the monotone and dominated convergence theorems for the Lebesgue integral to the sequences $\{f_n - f_1\}$ and $\{f_n - g\}$, respectively.

REMARK 3.18. For the reader familiar with the classical *Denjoy-Perron integral* ([15, Chapter VIII), we note that this integral coincides with the HK-integral. A readable proof can be found, e.g., in [4] or [17].

4. Gages and calibers.

The direct generalization of the HK-integral to higher dimensions (see [10]) is both *trivial* and *disappointing*. It is trivial because it amounts to a virtually verbatim repetition of the one-dimensional arguments; it is disappointing since there is no analogue of Proposition 3.10. Indeed, there is a differentiable function F in $\mathbf{R}^2$ such that the partial derivative $\partial F/\partial \xi_1$ is *not* HK-integrable over an interval (see [11, Example 5.7]). A useful generalization of the HK-integral is obtained only by employing more refined partitions.

A *hyperplane* is the set $\{(\xi_1, \ldots, \xi_m) \in \mathbf{R}^m : \xi_j = c\}$ where $1 \leq j \leq m$ is an integer and $c \in \mathbf{R}$. If $m \geq 2$, then each hyperplane is parallel to $m - 1$ coordinate axes of $\mathbf{R}^m$. In this case, using permutations of coordinates and orthogonal projections, we can and will assume that the measure λ_{m-1} is defined on every hyperplane. If $m = 1$, hyperplanes are points on which the *counting measure*, denoted by λ_0, is defined.

A *figure* is a finite (possibly empty) union of intervals. The boundary of each figure is contained in a finite union of hyperplanes. Thus if A is a figure, we give the number $\lambda_{m-1}(\operatorname{bd} A)$ the obvious meaning and call it the *perimeter* of A, denoted by $\|A\|$. Clearly, the perimeter of a cube of diameter d equals $2\,md^{m-1}$. It follows that the perimeter of a one-dimensional figure is equal to two times the number of its connected components. If A and B are figures, then so are the sets $A \cup B$,

$$A \odot B = \mathrm{cl}\,[\mathrm{int}(A \cap B)] \quad \text{and} \quad A \ominus B = \mathrm{cl}(A - B)\,.$$

Since the boundaries of $A \cup B$, $A \odot B$, and $A \ominus B$ are contained in $(\mathrm{bd}\,A) \cup (\mathrm{bd}\,B)$, we have the inequality

$$\max\{\|A \cup B\|, \|A \odot B\|, \|A \ominus B\|\} \le \|A\| + \|B\|\,.$$

A set $T \subset \mathbf{R}^m$ is called *thin* if it is the union of a countable family $\{T_1, T_2, \ldots\}$ where each T_j satisfies the following condition: given $\eta > 0$, there are dyadic cubes $C_1, C_2 \ldots$ of diameters less than η such that

$$T_j \subset \mathrm{int}\left(\bigcup_k C_k\right) \quad \text{and} \quad \sum_k \|C_k\| \le 4\,.$$

We note that the bound 4 is actually attained when showing that the set of dyadic rationals is a thin subset of $\mathbf{R}$.

An easy proof of the next proposition is left to the reader.

PROPOSITION 4.1. *The following statements are true.*

1. *A subset of a thin set is thin.*
2. *A countable union of thin sets is thin.*
3. *Each hyperplane is thin; in particular, a boundary of any figure is thin.*
4. *If* $T \subset \mathbf{R}^m$ *is thin then* $|T| = 0$, *but not vice versa.*
5. *In* $\mathbf{R}$ *the thin sets and countable sets coincide.*

REMARK 4.2. The reader familiar with Hausdorff measures should observe that a set $T \subset \mathbf{R}^m$ is thin if and only if its Hausdorff measure in codimension one is *σ-finite*.

DEFINITION 4.3. A nonnegative function δ defined on a set $E \subset \mathbf{R}^m$ is called a *gage* in E if its *null set* $Z_\delta = \{x \in E : \delta(x) = 0\}$ is thin.

If δ is a gage in a set $E \subset \mathbf{R}^m$, then a partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ with $\{x_1, \ldots, x_p\} \subset E$ is called *δ-fine* whenever $d(A_i) < \delta(x_i)$

for $i = 1, \ldots, p$. While formally this is the same concept we used before, gages put more restrictions on P than positive functions: the set $\{x_1, \ldots, x_p\}$ must be contained in $E - Z_\delta$.

If $A = [0, 1]$ and $\delta(x) = x$ for each $x \in A$, then δ is a gage in A and it is easy to verify that no δ-fine partition *of* A exists. We show, however, that for each gage δ in a figure A there is a δ-fine partition P *in* A, i.e., $\cup P \subset A$, for which the figure $A \ominus \cup P$ is "small" (cf. Proposition 4.8).

A *caliber* is a sequence $\{\eta_j\}$ of positive real numbers. Given an $\varepsilon > 0$ and a caliber $\eta = \{\eta_j\}$, we say that a figure B is (ε, η)*-small* if B is the union of nonoverlapping, possibly empty, figures $B_1, \ldots, B_k$ such that $\|B_j\| < 1/\varepsilon$ and $|B_j| < \eta_j$ for $j = 1, \ldots, k$. The concept of (ε, η)-smallness will be motivated by the proofs of Theorems 5.4 and 7.2 below.

EXAMPLE 4.4. The interval $[0, 9/8]$ is (ε, η)-small for $\varepsilon = 1/3$ and the caliber

$$\eta = \left\{\frac{1}{3}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}.$$

Indeed, this can be seen in two ways:

$$\left[0, \frac{9}{8}\right] = \emptyset \cup \emptyset \cup \left[0, \frac{7}{8}\right] \cup \left[\frac{7}{8}, \frac{9}{8}\right] = \left[0, \frac{1}{4}\right] \cup \left[\frac{1}{4}, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{9}{8}\right].$$

REMARK 4.5. In general, a subfigure of an (ε, η)-small figure need not be (ε, η)-small. For example, the interval $[0, 1]$ is (ε, η)-small for $\varepsilon = 1/3$ and the caliber

$$\eta = \left\{\frac{3}{2}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\right\},$$

while the figure $[0, 1/3] \cup [2/3, 1]$ contained in $[0, 1]$ is not (ε, η)-small. On the other hand, it is easy to verify that the intersection of an (ε, η)-small figure and an *interval* is again an (ε, η)-small figure.

DEFINITION 4.6. Let A be a figure, let $\varepsilon > 0$, and let η be a caliber. A partition P in A is called a *partition* of A mod (ε, η) whenever the figure $A \ominus \cup P$ is (ε, η)-small.

The next result, due to J. Howard (see [6]), is a version of Cousin's lemma.

LEMMA 4.7. *Let* Δ *be a gage in an interval* $K = \prod_{j=1}^{m}[r_j, s_j]$ *where* $r_j < s_j$ *are integers for* $j = 1, \ldots, m$. *Then for each positive* $\varepsilon < 1/4$ *and each caliber* η *there is a* Δ*-fine partition* $\{(K_1, x_1), \ldots, (K_p, x_p)\}$ *of* K *mod* (ε, η) *such that* $K_1, \ldots, K_p$ *are dyadic cubes.*

Proof. Let $\eta = \{\eta_j\}$. By our assumption, the null set Z_Δ of Δ is the union of sets $T_1, T_2, \ldots$ such that for each $j = 1, 2, \ldots$, we can find countable families $\mathcal{C}_j$ of dyadic cubes of diameters less than $\eta_j/2$ for which

$$T_j \subset \operatorname{int}\left(\bigcup \mathcal{C}_j\right) \quad \text{and} \quad \sum_{C \in \mathcal{C}_j} \|C\| \leq 4 .$$

Thus, if $\mathcal{E} \subset \mathcal{C}_j$ is a finite family and $E = \cup\mathcal{E}$, then

$$\|E\| \leq \sum_{C \in \mathcal{E}} \|C\| \leq 4 < \frac{1}{\varepsilon},$$

$$|E| < \frac{\eta_j}{2} \sum_{C \in \mathcal{E}} [d(C)]^{m-1} \leq \frac{\eta_j}{4m} \sum_{C \in \mathcal{E}} \|C\| \leq \eta_j .$$

Let $\mathcal{C}$ be a nonoverlapping subfamily of $\cup_j \mathcal{C}_j$ such that $\cup\mathcal{C} = \cup_j(\cup\mathcal{C}_j)$; here we are using the aforementioned property of dyadic cubes. Clearly

$$Z_\Delta = \bigcup_j T_j \subset \bigcup_j \left[\operatorname{int}\left(\bigcup \mathcal{C}_j\right)\right] \subset \operatorname{int}\left(\bigcup \mathcal{C}\right) .$$

Define a positive function Δ_+ on K by setting

$$\Delta_+(x) = \begin{cases} \Delta(x) & \text{if } x \in K - Z_\Delta , \\ \min\{d(C) : C \in \mathcal{C}, x \in C\} & \text{if } x \in Z_\Delta . \end{cases}$$

According to Cousin's lemma, there is a Δ_+-fine partition $\{(K_1, x_1), \ldots, (K_q, x_q)\}$ of K where $K_1, \ldots, K_q$ are dyadic cubes. Denote by $\mathcal{D}$ the family of all $C \in \mathcal{C}$ such that $C \subset K$ and $K_i \subset C$ for an $i = 1, \ldots, p$. Since $\mathcal{C}$ is a nonoverlapping family, $\mathcal{D}$ is finite (it contains at most q elements). If K_i overlaps with a $D \in \mathcal{D}$, then either $K_i \subset D$ or D is a proper subset of

K_i; for both K_i and D are dyadic cubes. The latter case is, however, impossible since D contains a K_j that does not overlap with K_i. We conclude that for $i = 1, \dots, q$, either $K_i \subset \bigcup\mathcal{D}$ or K_i overlaps with no $D \in \mathcal{D}$. Thus after a suitable reordering, $\bigcup\mathcal{D} = K \ominus \bigcup_{i=1}^{p} K_i$ for a nonnegative integer $p \leq q$. As $Z_\Delta \subset \operatorname{int}(\bigcup\mathcal{C})$, our definition of Δ_+ implies that $K_i \subset \bigcup\mathcal{D}$, whenever $x_i \in Z_\Delta$. It follows that $\{x_1, \dots, x_p\}$ is contained in $K - Z_\Delta$, and so $P = \{(K_1, x_1), \dots, (K_p, x_p)\}$ is a Δ-fine partition in K. We complete the proof by showing that the set $K \ominus \bigcup P$ is (ε, η)-small. To this end, let

$$\mathcal{D}_j = \mathcal{D} \cap \mathcal{C}_j - \bigcup_{i=1}^{j-1} \mathcal{D}_i$$

for $j = 1, 2, \dots$. The family $\mathcal{D}$, being a finite subfamily of $\mathcal{C}$, is the disjoint union of finitely many, possibly empty, families $\mathcal{D}_1, \dots, \mathcal{D}_k$. If $D_j = \bigcup\mathcal{D}_j$, then $D_1, \dots, D_k$ are nonoverlapping figures and $\bigcup\mathcal{D} = \bigcup_{j=1}^{k} D_j$. Since $\mathcal{D}_j \subset \mathcal{C}_j$, we have $\|D_j\| < 1/\varepsilon$ and $|D_j| < \eta_j$ for $j = 1, \dots, k$, which establishes our assertion.

PROPOSITION 4.8. *Let δ be a gage in a figure A. Then for each positive $\varepsilon < 1/4$ and each caliber η there is a δ-fine partition $\{(K_1, x_1), \dots, (K_p, x_p)\}$ and A mod (ε, η) such that $K_1, \dots, K_p$ are dyadic cubes.*

Proof. Assume first that A is an interval. For $j = 1, \dots, m$, find integers $r_j < s_j$ so that A is contained in the interval $K = \prod_{j=1}^{m} [r_j, s_j]$. For $x \in K$ denote by $\rho(x)$ the distance from x to bd$A$, and define a gage Δ on K by setting

$$\Delta(x) = \begin{cases} \min\{\delta(x), \rho(x)\} & \text{if } x \in A, \\ \rho(x) & \text{if } x \in K - A. \end{cases}$$

if $Q = \{(K_1, x_1), \dots, (K_q, x_q)\}$ is a Δ-fine partition of K mod (ε, η) whose existence is guaranteed by Lemma 4.7, then $P = \{(K_i, x_i) : x_i \in A\}$ is a δ-fine partition in A. Since

$$A \ominus \bigcup P = A \odot \left(K \ominus \bigcup Q\right)$$

it follows from Remark 4.5 that P is the desired partition.

Let A be a figure. Then A is the union of nonoverapping intervals, say $A_1, \dots, A_n$. If $\eta = \{\eta_j\}_{j=1}^{\infty}$, let $\eta^s = \{\eta_{nj+s}\}_{j=1}^{\infty}$ for $s = 1, \dots, n$. Applying the first part of the proof to A_s and η^s, find a δ-fine partition $P_s = \{(K_{s,1}, x_{s,1}), \dots, (K_{s,p_s}, x_{s,p_s})\}$ of A_s mod (ε, η^s) such that $K_{s,1}, \dots, K_{s,p_s}$ are dyadic cubes. Now it is clear that $P = \bigcup_{s=1}^{n} P_s$ is the desired partition.

5. The HK-integral revisited.

In this section we establish a strong version of the *fundamental theorem of calculus* (Theorem 5.4) that will serve as a model for the *divergence theorem* in higher dimensions. To facilitate an easy proof, we begin by reformulating Definition 3.1. The reformulation is of independent importance as it provides the main ideas for a useful multidimensional generalization of the HK-integral.

THEOREM 5.1. *A function f on an interval A belongs to $\mathcal{HK}(A)$ if and only if there is a real number I with the following property: given $\varepsilon > 0$, we can find a gage δ in A and a caliber η such that*

$$|\sigma(f, P) - I| < \varepsilon$$

for each δ-fine partition P of A mod(ε, η). In this case $(HK) \int_A f = I$.

Proof. Choose a positive $\varepsilon < 1/2$. Assuming that the condition of the theorem is satisfied, there is a gage δ in A and a caliber $\eta = \{\eta_j\}$ such that $|\sigma(f, Q) - I| < \varepsilon/2$ for each δ-fine partition Q of A mod (ε, η). It follows from Proposition 4.1, 5 that $Z_\delta = \{t_1, t_2, \dots, \}$. Find $\varepsilon_j > 0$ so that $\varepsilon_j |f(t_j)| < \varepsilon 2^{-j-1}$, and define a positive function δ_+ on A as follows:

$$\delta_+(x) = \begin{cases} \delta(x) & \text{if } x \in A - Z_\delta\,, \\ \min\{\varepsilon_j, \eta_j\} & \text{if } x = t_j, j = 1, 2 \dots . \end{cases}$$

Let $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ be a δ_+-fine partition of A and let

$$Q = \{(A_i, x_i) \in P : x_i \in A - Z_\delta\}\,.$$

Then Q is a δ-fine partition of A mod (ε, η), and we have

$$|\sigma(f, P) - I| \le |\sigma(f, Q) - I| + \sum_{x_i \in Z_\delta} |f(x_i)| \cdot |A_i| < \frac{\varepsilon}{2} + \sum_{j=1}^{\infty} \varepsilon 2^{-j-1} = \varepsilon\,.$$

Thus $f \in \mathcal{HK}(A)$ and $(HK)\int_A f = I$.

Conversely, let $A = [a,b]$ and assume that $f \in \mathcal{HK}(A)$. By Proposition 3.13,

$$F(x) = \begin{cases} 0 & \text{if } x = a\,, \\ (HK)\int_a^x f & \text{if } x \in (a,b]\,, \end{cases}$$

defines a continuous function F on A. Since F is uniformly continuous on A, we can find $\eta_j > 0$, $j = 1,2,\ldots$, so that for each subinterval $B = [x,y]$ of A with $|B| < \eta_j$ we have

$$\left|(HK)\int_B f\right| = |F(y) - F(x)| < \varepsilon^2 2^{-j}\,.$$

By Henstock's lemma, there is a positive function δ on A such that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - (HK)\int_{A_i} f \right| < \frac{\varepsilon}{2}$$

for each δ-fine partition $P = \{(A_1,x_1),\ldots,(A_p,x_p)\}$ in A. If P is a δ-fine partition of A mod (ε,η) where $\eta = \{\eta_j\}$, then $A \ominus \cup P = \cup_{j=1}^{k} B_j$ where $B_1,\ldots,B_k$ are nonoverapping figures such that $\|B_j\| < 1/\varepsilon$ and $|B_j| < \eta_j$ for $j = 1,\ldots,K$. Hence each B_j is the union of disjoint intervals $K_{j,1},\ldots,K_{j,n_j}$ where $2\,n_j < 1/\varepsilon$. Consequently, for such a partition P we obtain

$$\left|\sum_{i=1}^{p} f(x_i)|A_i| - (HK)\int_A f\right| \le$$

$$\sum_{i=1}^{p}\left|f(x_i)|A_i| - (HK)\int_{A_i} f\right| + \sum_{j=1}^{k}\sum_{n=1}^{n_j}\left|(HK)\int_{K_{j,n}} f\right| <$$

$$\frac{\varepsilon}{2} + \sum_{j=1}^{k}\frac{1}{2\varepsilon}\cdot\varepsilon^2 2^{-j} < \varepsilon\,,$$

and the theorem is established.

REMARK 5.2. Note that in the first part of the proof of Theorem 5.1 we established the existence of a δ-fine partition of A mod (ε,η) directly without refering to Proposition 4.8. It follows from Remark 2.1 that Theorem 5.1 remains valid when the partition P is required to be *special* (cf. Remark 3.2).

LEMMA 5.3. *Let $E \subset \mathbf{R}^m$ have measure zero and let $\varepsilon > 0$. There is a function α defined on all subsets of $\mathbf{R}^m$ satisfying the following conditions:*

1. $0 \leq \alpha(B) \leq \varepsilon$ *for each* $B \subset \mathbf{R}^m$;
2. $\alpha(B \cup C) = \alpha(B) + \alpha(C)$ *for each nonoverlapping measurable sets* $B, C \subset \mathbf{R}^m$;
3. *given* $x \in E$ *and an integer* $n \geq 1$, *there is a* $\delta > 0$ *such that* $\alpha(B) \geq n|B|$ *for each* $B \subset U(x, \delta)$.

Proof. Find a decreasing sequence $\{U_k\}$ of open sets containing E so that $|U_k| < \varepsilon 2^{-k}$ for $k = 1, 2, \ldots$, and set

$$\alpha(B) = \sum_{k=1}^{\infty} |B \cap U_k|$$

for each $B \subset \mathbf{R}^m$. Clearly, the function α satisfies the first two conditions. Given $x \in E$ and a positive integer n, find $\delta > 0$ so that $U(x, \delta) \subset U_n$. Now if $B \subset U(x, \delta)$, then

$$\alpha(B) = n|B| + \sum_{k=1}^{\infty} |B \cap U_k| \geq n|B|$$

and the lemma is established.

A function F defined on an interval $[a, b]$ is called *almost differentiable* at $x \in (a, b)$ if

$$\limsup_{y \to x} \frac{|F(y) - F(x)|}{|y - x|} < +\infty .$$

By Stepanoff's theorem ([3, Theorem 3.1.9]), if $E \subset (a, b)$ and F is almost differentiable at all $x \in E$, then it is differentiable at almost all $x \in E$.

THEOREM 5.4. *Let T be a thin subset of $\mathbf{R}$, and let F be a continuous function defined on an interval $[a, b]$ that is almost differentiable at each*

$x \in (a,b) - T$. *If* f *is a function on* $[a,b]$ *such that* $f(x) = F'(x)$ *for every* $x \in (a,b) - T$ *at which* F *is differentiable, then* $f \in \mathcal{HK}([a,b])$ *and*

$$(HK)\int_a^b f = F(b) - F(a) .$$

Proof. If a figure $B \subset [a,b]$ is the union of disjoint intervals $[a_1, b_1]$, $\ldots, [a_n, b_n]$ we let

$$\phi(B) = \sum_{k=1}^{n} [F(b_k) - F(a_k)] .$$

Clearly, if B and C are nonoverlapping subfigures of $[a,b]$ then

$$\phi(B \cup C) = \phi(B) + \phi(C) .$$

There is a set $E \subset (a,b) - T$ such that $|E| = 0$ and F is differentiable at each $x \in (a,b) - (E \cup T)$. In view of Corollary 3.9, it suffices to show that a function g on $[a,b]$ defined by

$$g(x) = \begin{cases} F'(x) & \text{if } x \in (a,b) - (E \cup T) , \\ 0 & \text{if } x \in E \cup T \cup \{a,b\} , \end{cases}$$

belongs to $\mathcal{HK}([a,b])$ and $(HK)\int_a^b g = F(b) - F(a)$.

Select an $\varepsilon > 0$ and let α be a function associated to E and ε according to Lemma 5.3. If $x \in E$, then there are positive numbers c_x and δ_x such that

$$|F(y) - F(x)| \le c_x|y - x| \quad \text{and} \quad \alpha(B) \ge c_x|B|$$

for every $y \in [a,b]$ with $|y - x| < \delta_x$ and every interval B contained in $[a,b] \cap U(x, \delta_x)$. Thus for each $x \in E$, the inequality

$$\big|g(x)|B| - \phi(B)\big| = |\phi(B)| \le c_x|B| \le \alpha(B)$$

holds whenever B is a subinterval of $[a,b] \cap U(x, \delta_x)$ and x is a boundary point of B.

If $x \in (a,b) - (E \cap T)$, then there is a $\delta_x > 0$ such that

$$\left| \frac{F(y) - F(x)}{y - x} - g(x) \right| < \varepsilon$$

for every $y \in A$ with $0 < |x - y| < \delta_x$. Hence if $x \in (a,b) - (E \cup T)$, then

$$\left| g(x)|B| - \phi(B) \right| < \varepsilon |B|$$

whenever B is a subinterval of $[a,b] \cap U(x, \delta_x)$ and x is a boundary point of B.

Finally, using the uniform continuity of F in $[a,b]$ it is easy to show that there is a caliber $\eta = \{\eta_j\}$ such that for each $j = 1, 2, \ldots$, we have $|\phi(B)| < \varepsilon 2^{-j}$ whenever $B \subset [a,b]$ is a figure with $\|B\| < 1/\varepsilon$ and $|B| < \eta_j$.

Now we define a gage δ in A by setting

$$\delta(x) = \begin{cases} \delta_x & \text{if } x \in (a,b) - T\,, \\ 0 & \text{if } x \in T \cup \{a,b\}\,, \end{cases}$$

and choose a δ-fine *special* partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ of A mod (ε, η). Then $A \ominus \cup P = \cup_{j=1}^{k} B_j$ where $B_1, \ldots, B_k$ are nonoverlapping figures such that $\|B_j\| < 1/\varepsilon$ and $|B_j| < \eta_j$ for $j = 1, \ldots, k$. Since no x_i belongs to $T \cup \{a,b\}$, we obtain

$$\begin{aligned} \left| \sigma(g,P) - [F(b) - F(a)] \right| &\leq \sum_{i=1}^{p} \left| g(x_i)|A_i| - \phi(A_i) \right| + \sum_{j=1}^{k} |\phi(B_j)| \\ &< \sum_{x_i \in E} \alpha(A_i) + \sum_{x_i \notin E} \varepsilon |A_i| + \sum_{j=1}^{k} \varepsilon 2^{-j} \\ &< \alpha(A) + \varepsilon |A| + \varepsilon \leq \varepsilon(2 + |A|)\,, \end{aligned}$$

and the theorem follows from Remark 5.2.

EXAMPLE 5.5. On the interval [-1,1] define a function F by setting $F(x) = 0$ if $x \in [-1, 0]$, and $F(x) = 1$ if $x \in (0, 1]$. Then F is differentiable at each $x \in [-1,1] - \{0\}$ with $F'(x) = 0$. Thus

$$(HK)\int_{-1}^{1} F' \neq F(1) - F(-1) ,$$

and we see that the continuity of F cannot be omitted in Theorem 5.4.

EXAMPLE 5.6. Let T be the Cantor ternary set in the interval $[0,1]$. If an interval $B \subset [0,1]$ is a connected component of $[0,1] - T$, denote by c_B the midpoint of B and let $F(x) = c_B$ for each $x \in B$. It is easy to see that thus defined function on $[0,1] - T$ can be extended to a continuous function on [0,1], still denoted by F. Clearly, F is differentiable at each $x \in [0,1] - T$ with $F'(x) = 0$. Thus

$$(HK)\int_{0}^{1} F' \neq F(1) - F(0) ,$$

and we see that in Theorem 5.4 the thin set T cannot be replaced by a set of measure zero.

REMARK 5.7. In view of Examples 5.5 and 5.6, it may appear that Theorem 5.4 is the best possible *fundamental theorem of calculus* for the HK-integral (cf. Proposition 3.10). This is not true. The class of almost differentiable functions can be extended to a properly larger family of so called ACG_* functions (an abbreviation for *generalized absoluteley continuous functions in the restricted sense*) for which Theorem 5.4 still holds. We shall not investigate the ACG_* functions. Their study is a highly technical subject restricted only to dimension one; it is unclear at this time how to formulate an ACG_* concept in higher dimensions. The interested reader is referred to [15, Chapter VII], [4] and [17].

6. The gage integral.

The *regularity* of an interval $A \subset \mathbf{R}^m$ is the positive number

$$r(A) = \frac{|A|}{[d(A)]^m} .$$

Clearly, $r(A) \leq 1$ and the upper bound is attained if and only if A is a cube. Given $\varepsilon > 0$, a partition $\{A_1, x_1), \ldots, (A_p, x_p)\}$ is called *ε-regular* if $r(A_i) > \varepsilon$ for $i = 1, \ldots, p$. We note that if P is an ε-regular partition and $m \geq 2$, then the associated special partition P^{sp} of Remark 2.1 *need not* be ε-regular.

DEFINITION 6.1. A function f defined on a figure $A \subset \mathbf{R}^m$ is called *gage integrable* (abbreviated as g-integrable) in A if there is a real number I having the following property: given $\varepsilon > 0$, we can find a gage δ in A and a caliber η such that

$$|\sigma(f, P) - I| < \varepsilon$$

for each δ-fine ε-regular partition P of A mod (ε, η).

It follows from Proposition 4.8 that the number I of Definition 6.1 is determined uniquely by the g-integrable function f. We call it the *g-integral* of f over A, denoted by $(g) \int_A f$. By $\mathcal{G}(A)$ we denote the family of all g-integrable function in A.

Since the regularity of each one-dimensional interval equals 1, Theorem 5.1 implies that the HK- and g-integrals coincide over the subintervals of $\mathbf{R}$. In higher dimensions, however, this is far from truth.

The routine proof of the following proposition is left to the reader.

PROPOSITION 6.2. *If A is a figure, then the family $\mathcal{G}(A)$ is a linear space over $\mathbf{R}$ and the map $f \mapsto (g) \int_A f$ is a nonnegative linear functional on $\mathcal{G}(A)$.*

Let $A \subset \mathbf{R}^m$ be a figure. A *division* of A is a finite family of nonoverlapping figures whose union is A. An *additive function* in A is a function F defined on all subfigures of A such that

$$F(A) = \sum_{D \in \mathcal{D}} F(D)$$

for each division $\mathcal{D}$ of A. An additive function F in A is called *continuous* if given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each figure $B \subset A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta$.

EXAMPLE 6.3. Let $A \subset \mathbf{R}^m$ be a figure and let $v : A \to \mathbf{R}^m$ be a continuous vector field. If $B \subset A$ is a figure, we denote by n_B the usual *exterior normal* of B, defined on all $(m-1)$-dimensional faces of B, and let

$$F(B) = (L) \int_{\mathrm{bd}\, B} v \cdot n_B \, d\lambda_{m-1} \, .$$

We show that F, which is clearly an additive function in A, is continuous. To this end, choose an $\varepsilon > 0$ and find a continuously differentiable vector field $w : \mathbf{R}^m \to \mathbf{R}^m$ so that $\|v(x) - w(x)\| < \varepsilon^2/2$ for each $x \in A$. Select a positive $\eta < \varepsilon/(2c)$ where c is a positive upper bound for $|\mathrm{div}\, w|$ on A. Now let $B \subset A$ be a figure with $\|B\| < 1/\varepsilon$ and $|B| < \eta$. The Schwartz inequality and standard divergence theorem applied to w over B yield

$$\begin{aligned} &\left| (L) \int_{\mathrm{bd}\, B} v \cdot n_B \lambda_{m-1} \right| \leq \\ &\qquad (L) \int_{\mathrm{bd}\, B} \|v - w\| d\lambda_{m-1} + (L) \int_B |\mathrm{div}\, w| d\lambda_m \leq \\ &\qquad \frac{\varepsilon^2}{2} \|B\| + c|B| < \varepsilon \, , \end{aligned}$$

and our assertion is proved.

PROPOSITION 6.4. *Let A be a figure and let $f \in \mathcal{G}(A)$. Then $f \in \mathcal{G}(B)$ for each figure $B \subset A$, and the map $F : B \mapsto (g) \int_B f$ is an additive continuous function in A.*

Proof. Choose a positive $\varepsilon < 1/4$, and find a gage δ in A and a caliber $\eta = \{\eta_j\}_{j=1}^{\infty}$ so that

$$|\sigma(f, P) - (g) \int_A f| < \frac{\varepsilon}{2}$$

for each δ-fine ε-regular partition P of A mod (ε, η).

If B is a subfigure of A and $C = A \ominus B$, let $\eta_B = \{\eta_{2j}\}_{j=1}^{\infty}$ and $\eta_C = \{\eta_{2j-1}\}_{j=1}^{\infty}$. By Proposition 4.8, there is a δ-fine ε-regular partition Q of C mod (ε, η_C). Now if Q_1 and Q_2 are δ-fine ε-regular partitions of

$B \bmod (\varepsilon, \eta_B)$, then $P_1 = Q \cup Q_1$ and $P_2 = Q \cup Q_2$ are δ-fine ε-regular partitions of $A \bmod (\varepsilon, \eta)$. Thus

$$\begin{aligned} |\sigma(f,Q_1) - \sigma(f,Q_2)| &= |\sigma(f,P_1) - \sigma(f,P_2)| \\ &\leq \left|\sigma(f,P_1) - (g)\int_A f\right| + \left|\sigma(f,P_2) - (g)\int_A f\right| < \varepsilon, \end{aligned}$$

and the g-integrability of f in B follows from a Cauchy test similar to that presented in Lemma 3.5. In particular, the function $F : B \mapsto (g)\int_B f$ is defined for all figures $B \subset A$.

If $\{A_1, \ldots, A_n\}$ is a division of A, let $\eta^k = \{\eta_{nj+k}\}_{j=1}^{\infty}$ for $k = 1, \ldots, n$. By the first part of the proof and Proposition 4.8, there are δ-fine ε-regular partitions P_k of $A_k \bmod (\varepsilon, \eta^k)$ such that $|\sigma(f, P_k) - F(A_k)| < \varepsilon/(2n)$. Since $P = \bigcup_{k=1}^{n} P_k$ is a δ-fine ε-regular partition of $A \bmod (\varepsilon, \eta)$, we have

$$\left|F(A) - \sum_{k=1}^{n} F(A_k)\right| \leq |F(A) - \sigma(f,P)| + \sum_{k=1}^{n} |\sigma(f,P_k) - F(A_k)| < \varepsilon,$$

and the additivity of F follows from the arbitrariness of ε.

Choose a figure $B \subset A$ with $|B| < 1/\varepsilon$ and $|B| < \eta_1$. Let $C = A \ominus B$ and $\eta^* = \{\eta_j\}_{j=2}^{\infty}$. There is a δ-fine ε-regular partition Q of $C \bmod (\varepsilon, \eta^*)$ such that $|\sigma(f,Q) - F(C)| < \varepsilon/2$. Since Q is also a δ-fine ε-regular partition of $A \bmod (\varepsilon, \eta)$, we obtain

$$|F(B)| = |F(A) - F(C)| \leq |F(A) - \sigma(f,Q)| + |\sigma(f,Q) - F(C)| < \varepsilon,$$

which establishes the continuity of F.

To illustrate further the manipulation of calibres, we prove the Henstock lemma for the g-integral.

LEMMA 6.5 *Let A be a figure and let $f \in \mathcal{G}(A)$. Given $\varepsilon > 0$, there is a gage δ in A such that*

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - (g)\int_{A_i} f \right| < \varepsilon$$

for each δ-fine ε-regular partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A.

Proof. Choose a positive $\varepsilon < 1/4$, and find a gage δ in A and a caliber $\eta = \{\eta_j\}_{j=1}^{\infty}$ so that

$$\left|\sigma(f,P) - (g)\int_A f\right| < \frac{\varepsilon}{3}$$

for each δ-fine partition P of A mod (ε, η). Let $\eta_{odd} = \{\eta_{2j-1}\}_{j=1}^{\infty}$ and $\eta_{even} = \{\eta_{2j}\}_{j=1}^{\infty}$. It follows from Proposition 4.8 that each δ-fine ε-regular partition in A is a subset of a δ-fine ε-regular partition of A mod $(\varepsilon, \eta_{odd})$. Thus it suffices to prove the inequality in question for a δ-fine ε-regular partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ of A mod $(\varepsilon, \eta_{odd})$.

For this purpose, let $\eta^i = \{\eta_{2(pj+i)}\}_{j=1}^{\infty}$ for $i = 1, \ldots, p$, and find δ-fine ε-regular partitions P_i of A_i mod (ε, η^i) such that

$$\left|\sigma(f,P_i) - (g)\int_{A_i} f\right| < \frac{\varepsilon}{3p}.$$

The existence of the P_i's is guaranteed by Propositions 4.8 and 6.4. Reorder P so that $f(x_i)|A_i| - (g)\int_{A_i} f$ is nonnegative for $i = 1, \ldots, k$, and negative for $i = k+1, \ldots, p$, where k is an integer with $0 \le k \le p$. Now observe that

$$Q_+ = \{(A_1, x_1), \ldots, (A_k, x_k)\} \cup \bigcup_{i=k+1}^{p} P_i.$$

$$Q_- = \{(A_{k+1}, x_{k+1}), \ldots, (A_p, x_p)\} \cup \bigcup_{i=1}^{k} P_i$$

are δ-fine ε-regular partitions of A mod (ε, η). Thus

$$\frac{\varepsilon}{3} > \left|\sigma(f,Q_+) - (g)\int_A\right| \ge \sum_{i=1}^{k}\left[f(x_i)|A_i| - (g)\int_{A_i} f\right]$$
$$- \left|\sum_{i=k+1}^{p}\left[\sigma(f,P_i) - (g)\int_{A_i} f\right]\right| \ge \sum_{i=1}^{k}\left|f(x_i)|A_i| - (g)\int_{A_i} f\right|$$
$$- (p-k)\frac{\varepsilon}{3p},$$

and similarly

$$\frac{\varepsilon}{3} > \sum_{i=k+1}^{p} \left| f(x_i)|A_i| - (g)\int_{A_i} f \right| - k\frac{\varepsilon}{3p} .$$

The lemma follows by adding the last two inequalities.

We say that an additive function F in a figure A is *derivable* at $x \in A$ if there exists a finite limit

$$\lim \frac{F(B_n)}{|B_n|}$$

for each sequence $\{B_n\}$ of subintervals of A containing x for which

$$\lim d(B_n) = 0 \quad \text{and} \quad \inf r(B_n) > 0 .$$

When all these limits exist they have the same value, denoted by $F'(x)$. If F is derivable almost everywhere in A, then it follows from [15, Chapter IV, Theorem (4.2)] that the function $x \mapsto F'(x)$ is measurable.

THEOREM 6.6. *Let f be a g-integrable function in a figure A, and let $F(B) = (g)\int_B f$ for each figure $B \subset A$. Then for almost all $x \in A$ the function F is derivable at x and $F'(x) = f(x)$. In particular, the function f is measurable.*

Proof. Let E be the set of all $x \in A$ for which either F is not derivable at x or $F'(x) \neq f(x)$. Given $x \in E$, we can find a $\gamma(x) > 0$ so that for each $\beta > 0$ there is an interval $B \subset A$ with $x \in B$, $d(B) < \beta$, $r(B) > \gamma(x)$, and

$$\left| f(x)|B| - F(B) \right| \geq \gamma(x)|B| .$$

Fix an integer $n \geq 1$ and let $E_n = \{x \in E : \gamma(x) \geq 1/n\}$. Choose a positive $\varepsilon < 1/n$, and use Henstock's lemma to find a gage δ in A so that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \frac{\varepsilon^2}{5^m n}$$

for each δ-fine ε-regular partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. Let $\mathcal{E}$ be the collection of all intervals $B \subset A$ of regularity larger than ε such that $d(B) < \delta(x_B)$ and

$$\left|f(x_B)|B| - F(B)\right| \geq \frac{1}{n}|B|$$

for an $x_B \in B \cap (E_n - Z_\delta)$; here, as always, Z_δ denotes the null set of δ. From the choice of ε we see that $E_n - Z_\delta \subset \cup\mathcal{E}$. Each $B \in \mathcal{E}$ is contained in a cube K_B with $d(K_B) = d(B)$, which implies that $|K_B| < |B|/\varepsilon$. It follows from [19, Theorem 1.3.1] that there is a disjoint subfamily $\{K_{B_1}, K_{B_2}, \ldots\}$ of the family $\{K_B : B \in \mathcal{E}\}$ such that

$$|E_n| = |E_n - Z_\delta| \leq 5^m \sum_i |K_{B_i}| < \frac{5^m}{\varepsilon} \sum_i |B_i| .$$

Observe that $\{(B_1, x_{B_1}), \ldots, (B_p, x_{B_p})\}$ is a δ-fine ε-regular partition in A whenever the set B_p is defined. Hence

$$\sum_{i=1}^{p} |B_i| \leq n \sum_{i=1}^{p} \left|f(x_{B_i})|B_i| - F(B_i)\right| < \frac{\varepsilon^2}{5^m}$$

and consequently, $|E_n| \leq \varepsilon$. By the arbitrariness of ε, the set E_n has measure zero, and so does $E = \cup_{n=1}^{\infty} E_n$.

The proofs of the next theorem and its corollaries are left to the reader; they are completely analogous to the corresponding proofs presented in Section 3.

THEOREM 6.7. *Let f be a function defined on a figure A. Then f belongs to $\mathcal{L}_1(A)$ if and only if both f and $|f|$ belong to $\mathcal{G}(A)$, in which case*

$$(g)\int_A f = (L)\int_A f .$$

COROLLARY 6.8. *Let f and g be functions defined on a figure A such that $f(x) = g(x)$ for almost all $x \in A$. Then f belongs to $\mathcal{G}(A)$ if and only if g does, in which case*

$$(g)\int_A f = (g)\int_A g .$$

COROLLARY 6.9. *Let $\{f_n\}$ be a sequence of g-integrable functions in a figure A, and let* $\lim f_n = f$. *Then*

$$f \in \mathcal{G}(A) \quad and \quad (g)\int_A f = \lim (g)\int_A f_n$$

whenever either of the following conditions are satisfied:

1. $f_n \le f_{n+1}$ *for* $n = 1,2,\dots,$ *and* $\lim (g)\int_A f_n < +\infty$;
2. $g \le f_n \le h$ *for some* $g, h \in \mathcal{G}(A)$ *and all* $n = 1,2,\dots$.

PROPOSITION 6.10. Let f be a function defined on a figure A, and let $\{A_1,\dots,A_n\}$ be a division of A. If f is g-integrable in A_k for $k = 1,\dots,n$, then it is g-integrable in A.

Proof. Set $I = \sum_{k=1}^{n}(g)\int_{A_k} f$ and choose a positive

$$\varepsilon < \frac{1}{\max\{||A_1||,\dots,||A_n||\}}.$$

For $k = 1,\dots,n$, there are gages δ_k in A_k and calibers $\eta^k = \{\eta_j^k\}_j$ such that

$$\left|\sigma(f,P_k) - (g)\int_{A_k} f\right| < \frac{\varepsilon}{n}$$

for each δ_k-fine ε-regular partition P_k of A_k mod $(\varepsilon/2,\eta^k)$. For $x \in A$ denote by $\rho_k(x)$ the distance from x to $\mathrm{bd}A_k$, and define a gage δ in A by setting $\delta(x) = \min\{\delta_k(x),\rho_k(x)\}$ for each $x \in A_k$. Finally, let $\eta = \{\eta_j\}$ be a caliber such that $\eta_j = \min\{\eta_j^1,\dots,\eta_j^n\}$ for all j. Now if P is a δ-fine ε-regular partition of A mod (ε,η), then $P_k = \{(B,x) \in P : x \in A_k\}$ is a δ-fine ε-regular partition in A_k. By definition, $A \ominus \bigcup P = \bigcup_{j=1}^{s} C_j$ where $C_1,\dots,C_s$ are nonoverlapping figures with $||C_j|| < 1/\varepsilon$ and $|C_j| < \eta_j$ for $j = 1,\dots,s$. Since

$$A_k \ominus \bigcup P_k = A_k \odot \left(A \ominus \bigcup P\right) = \bigcup_{j=1}^{s}(A_k \odot C_j)$$

and since for $j = 1,\dots,s$ the following estimates hold

$$||A_k \odot C_j|| \le ||A_k|| + ||C_j|| < \frac{2}{\varepsilon} \quad \text{and} \quad |A_k \odot C_j| < \eta_j \le \eta_j^k,$$

we conclude that P_k is a partition of A_k mod $(\varepsilon/2, \eta^k)$. Thus

$$|\sigma(f,P) - I| \leq \sum_{k=1}^{n} \left| \sigma(f,P_k) - (g)\int_{A_k} f \right| < \varepsilon$$

and the proposition is proved.

7. The divergence theorem.

Let $E \subset \mathbf{R}^m$ and $x \in \operatorname{int} E$. A *vector field* on E is a map $v : E \to \mathbf{R}^m$ whose differentiability at x is defined in the usual way (see [14, Definition 7.22]). So if $v = (f_1, \ldots, f_m)$ is differentiable at x, the partial derivatives $(\partial/\partial\xi_j) f_i(x)$ exist for $i, j = 1, \ldots, m$, and the number

$$\operatorname{div} v(x) = \sum_{j=1}^{m} \frac{\partial}{\partial \xi_j} f_j(x)$$

is called the *divergence* of v at x. The following lemma is the *germ* of the divergence theorem.

LEMMA 7.1. *Let v be a bounded vector field on a set $E \subset \mathbf{R}^m$ that is differentiable at $x \in \operatorname{int} E$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\left| \operatorname{div} v(x)|B| - (L)\int_{\operatorname{bd} B} v \cdot n_B \, d\lambda_{m-1} \right| < \varepsilon d(B)\|B\|$$

for each figure $B \subset E \cap U(x,\delta)$ for which $x \in B$ and v restricted to bdB is λ_{m-1} measureable.

Proof. For each $y \in \mathbf{R}^m$ let $w(y) = v(x) + D(y - x)$ where $D = Dv(x)$ is the differential of v at x. Thus w is a linear vector field defined on $\mathbf{R}^m$, and it is easy to verify that $\operatorname{div} w(y) = \operatorname{div} v(x)$ for each $y \in \mathbf{R}^m$. Moreover, there is a function h defined on E such that

$$\lim_{y \to x} h(y) = 0 \quad \text{and} \quad \|v(y) - w(y)\| \leq h(y)|y - x|$$

for every $y \in E$. Given $\varepsilon > 0$, choose a $\delta > 0$ so that $h(y) < \varepsilon$ whenever $y \in E \cap U(x,\delta)$. Now let $B \subset E \cap U(x,\delta)$ be a figure for which $x \in B$ and v restricted to bd$B$ is λ_{m-1}-measurable. The standard divergence theorem applied to w over B, and Schwartz's inequality yield

$$\begin{aligned}
&\left|\operatorname{div} v(x)|B| - (L)\int_{\operatorname{bd} B} v \cdot n_B \, d\lambda_{m-1}\right| = \\
&\qquad \left|(L)\int_B \operatorname{div} w \, d\lambda_m - (L)\int_{\operatorname{bd} B} v \cdot n_B \, d\lambda_{m-1}\right| = \\
&\qquad \left|(L)\int_{\operatorname{bd} B} (w - v) \cdot n_B \, \lambda_{m-1}\right| \le \\
&\qquad (L)\int_{\operatorname{bd} B} \| w(y) - v(y)\| \, d\lambda_{m-1}(y) \le \\
&\qquad (L)\int_{\operatorname{bd} B} h(y)|y - x| \, d\lambda_{m-1}(y) \le \varepsilon d(B)\|B\| ,
\end{aligned}$$

which proves the lemma.

Let v be a vector field on a set $E \subset \mathbf{R}^m$. We say that v is *almost differentiable* at $x \in \operatorname{int} E$ if

$$\limsup_{y \to x} \frac{|v(y) - v(x)|}{|y - x|} < +\infty .$$

As in the one-dimensional case, Stepanoff's theorem ([3, Theorem 3.1.9]) asserts that if $D \subset \operatorname{int} E$ and v is almost differentiable at all $x \in D$, then it is differentiable at almost all $x \in D$ (cf. Section 5).

THEOREM 7.2. *Let $T \subset \mathbf{R}^m$ be a thin set, and let v be a continuous vector field on a figure A that is almost differentiable at each $x \in \operatorname{int} A - T$. If f is a function on A such that $f(x) = \operatorname{div} v(x)$ for every $x \in \operatorname{int} A - T$ at which v is differentiable, then $f \in \mathcal{G}(A)$ and*

$$(g)\int_A f = (L)\int_{\operatorname{bd} A} v \cdot n_A \, d\lambda_{m-1} .$$

Proof. The proof is similar to that of Theorem 5.4. For each figure $B \subset A$, we let $F(B) = (L)\int_{\operatorname{bd} A} v \cdot n_A d\lambda_{m-1}$. By Example 6.3, this defines an additive continuous function F in A. There is a set $E \subset \operatorname{int} A - T$

such that $|E| = 0$ and v is differentiable at each $x \in \operatorname{int} A - (E \cup T)$. In view of Corollary 6.8, it suffices to show that a function h on A defined by

$$h(x) = \begin{cases} \operatorname{div} v(x) & \text{if } x \in \operatorname{int} A - (E \cup T)\,, \\ 0 & \text{if } \in E \cup T \cup \operatorname{bd} A\,, \end{cases}$$

belongs to $\mathcal{G}(A)$ and $(g) \int_A h = F(A)$.

Select an $\varepsilon > 0$ and let α be a function associated to E and ε according to Lemma 5.3. If $x \in E$, then there are positive numbers c_x and δ_x such that

$$\|v(y) - v(x)\| \leq c_x |y - x| \quad \text{and} \quad \alpha(B) \geq \frac{2mc_x}{\varepsilon}|B|$$

for every $y \in A \cap U(x, \delta_x)$ and every $B \subset A \cap U(x, \delta_x)$. Thus for each $x \in E$, the inequality

$$\begin{aligned} \big|h(x)|B| - F(B)\big| &= \left|(L)\int_{\operatorname{bd} B} v(y) \cdot n_B(y)\, d\lambda_{m-1}(y)\right| \\ &= \left|(L)\int_{\operatorname{bd} B} [v(y) - v(x)] \cdot n_B(y)\, d\lambda_{m-1}(y)\right| \\ &\leq (L)\int_{\operatorname{bd} B} c_x|y - x|\, d\lambda_{m-1}(y) \\ &\leq c_x d(B)\|B\| \leq c_x \cdot 2m[d(B)]^m \\ &< \frac{2mc_x}{\varepsilon}|B| \leq \alpha(B) \end{aligned}$$

holds whenever $B \subset A \cap U(x, \delta_x)$ is an interval with $r(B) > \varepsilon$.

If $x \in \operatorname{int} A - (E \cup T)$, use Lemma 7.1 to find a $\delta_x > 0$ so that

$$\big|h(x)|B| - F(B)\big| < \frac{\varepsilon^2}{2m} d(B)\|B\| \leq \varepsilon^2 [d(B)]^m < \varepsilon|B|$$

for every interval $B \subset A \cap U(x, \delta_x)$ for which $x \in B$ and $r(B) > \varepsilon$.

Finally, since F is a continuous additive function in A, there is a caliber $\eta = \{\eta_j\}$ such that for $j = 1, 2, \ldots$, we have $|F(B)| < \varepsilon 2^{-j}$ for each figure $B \subset A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta_j$.

Now we define a gage δ in A by setting

$$\delta(x) = \begin{cases} \delta_x & \text{if } x \in \operatorname{int} A - T\,, \\ 0 & \text{if } x \in T \cup \operatorname{bd} A\,, \end{cases}$$

and choose a δ-fine ε-regular partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ of A mod (ε, η). Then $A \ominus \cup P = \cup_{j=1}^{k} B_j$ where $B_1, \ldots, B_k$ are nonoverlapping figures such that $\|B_j\| < 1/\varepsilon$ and $|B_j| < \eta_j$ for $j = 1, \ldots, k$. Since no x_i belongs to $T \cup \operatorname{bd} A$, we obtain

$$\begin{aligned} \left|\sigma(h, P) - F(A)\right| &\leq \sum_{i=1}^{p} \left|h(x_i)|A_i| - F(A_i)\right| + \sum_{j=1}^{k} |F(B_j)| \\ &< \sum_{x_i \in E} \alpha(A_i) + \sum_{x_i \notin E} \varepsilon|A_i| + \sum_{j=1}^{\varepsilon} k2^{-j} \\ &< \alpha(A) + \varepsilon|A| + \varepsilon \leq \varepsilon(2 + |A|) \end{aligned}$$

and the theorem is proved.

8. The restricted gage integral.

There is a disturbing asymmetry in the definition of the g-integral: while the partitions involve only intervals, the concept of a figure is needed to define the approximation by Stieltjes sums. Even when the g-integral is restricted to intervals, the use of figures cannot be avoided for the following reasons. We must employ *regular* partitions in order to benefit from Lemma 7.1. Since the regularity of intervals may be ruined by intersections, we cannot prove Proposition 6.10 without introducing *gages*. The use of gages leads, in turn, to partitions *mod* (ε, η) whose definition is based on *figures* in an essential way.

Nonetheless, the natural way of making the g-integral symmetric suggests itself. Indeed it suffices to use "partitions" $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ where $A_1, \ldots, A_p$ are *figures* rather than intervals. This change will produce an integral that is weaker than the g-integral but closer to our final goal of coordinate free integration.

Let A be a figure. We set $r(A) = r^*(A) = 0$ if $|A| = 0$, and let

$$r(A) = \frac{|A|}{[d(A)]^m} \quad \text{and} \quad r^*(A) = \frac{|A|}{d(A)||A||}$$

if $|A| > 0$. The numbers $r(A)$ and $r^*(A)$ are called the *regularity* and **regularity* of A respectively.

LEMMA 8.1. *If A is an interval then* $[2r^*(A)]^m \leq r(A) \leq 2mr^*(A)$.

Proof. Avoiding the trivial case, assume that $m \geq 2$. For $j = 1, \ldots, m$, let A_j be the orthogonal projection of A to the $(m-1)$-dimensional subspace of $\mathbf{R}^m$ perpendicular to the j-th coordinate axis. Clearly

$$2\lambda_{m-1}(A_j) \leq ||A||$$

for $j = 1, \ldots, m$, and also

$$|A|^{m-1} \leq \prod_{j=1}^{m} \lambda_{m-1}(A_j) \ .$$

Combining the above inequalities, we obtain $|A|^{m-1} \leq 2^{-m}||A||^m$. Since $||A|| \leq 2m[d(A)]^{m-1}$ the lemma follows.

REMARK 8.2 In the notation of the previous proof, the inequality

$$|A|^{m-1} \leq \prod_{j=1}^{m} \lambda_{m-1}(A_j)$$

holds also when A is a figure. This is obvious for $m = 2$, and for an arbitrary m it is proved in [8] or [9, Sections 1-6]. As $2\lambda_{m-1}(A_j) \leq ||A||$ is clearly true when A is a figure, we see that the inequality $[2r^*(A)]^m \leq r(A)$ is satisfied for any figure A. On the other hand, if

$$A_n = \left(\bigcup_{k=1}^{n} \left[\frac{1}{2k}, \frac{1}{2k-1} \right] \right) \times [0,1]^{m-1} ,$$

then $r(A_n) \geq 1/2$ and $r^*(A_n) \leq 1/(2n)$ for $n = 1, 2 \ldots$ Thus for figures, *regularity is a *finer* indicator than regularity: it measures both the shape and perimeter of a figure. Moreover, from the proof of Theorem 7.2 we see that *regualrity is precisely taylored to our needs.

An *f-partition* is a collection (possibly empty) $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ where the A_i's are nonoverlapping figures and $x_i \in A_i$ for $i = 1, \ldots, p$. Given $\varepsilon > 0$, we say that P is ε-**regular* whenever $r^*(A_i) > \varepsilon$ for $i = 1, \ldots, p$. All other concepts applied to partitions in the previous sections extend, in the obvious way, to f-partitions.

REMARK 8.3. It follows from Lemma 8.1 that replacing ε-regular partitions by ε-*regular partitions in Definition 6.1 will *not* change the g-integral.

DEFINITION 8.4. A function f defined on a figure $A \subset \mathbf{R}^m$ is called *gage integrable in the resticted sense* (abbreviated as g*-integrable) in A if there is a real number I having the following property: given $\varepsilon > 0$, we can find a gage δ in A and a caliber η such that

$$|\sigma(f, P) - I| < \varepsilon$$

for each δ-fine ε-*regular f-partition P of A mod (ε, η).

If A is a figure, we denote by $\mathcal{G}^*(A)$ the family of all g*-integrable function in A. Remark 8.3 implies that $\mathcal{G}^*(A) \subset \mathcal{G}(A)$ and that for each $f \in \mathcal{G}^*(A)$, the number I from Definition 8.4 equals $(g) \int_A f$. To see that for $|A| > 0$, the inclusion $\mathcal{G}^*(A) \subset \mathcal{G}(A)$ is proper in every dimension requires a complicated example, which is beyond the scope of these lectures (for a one-dimensional example, we refer to [12, Example 8.6]).

It is easy to verify that all statements proved in Sections 6 and 7 remain valid (with identical proofs) when $\mathcal{G}(A)$ is replaced by $\mathcal{G}^*(A)$.

9. Sets of bounded variation.

At this point the way to coordiante free integration is straightforward. We only need to replace figures by more general sets for which the usual divergence theorem holds and which are invariant with respect to diffeomorphisms. While several choices are possible, we shall employ the most general family of sets having these properties.

Let E be a subset of $\mathbf{R}^m$. We say that an $x \in \mathbf{R}^m$ is, respectively, a *density* or *dispersion* point of E whenever

$$\liminf_{\varepsilon \to 0+} \frac{|E \cap U(x,\varepsilon)|}{(2\varepsilon)^m} = 1 \quad \text{or} \quad \limsup_{\varepsilon \to 0+} \frac{|E \cap U(x,\varepsilon)|}{(2\varepsilon)^m} = 0 .$$

The set of all density points of E is called the *essential interior* of E, denoted by $\mathrm{int}^* E$, and the set of all nondispersion points of E is called the *essential closure* of E, denoted by $\mathrm{cl}^* E$. The *essential boundary* of E is the set $\mathrm{bd}^* E = \mathrm{cl}^* E - \mathrm{int}^* E$. If E is measurable, then the sets E, $\mathrm{int}^* E$ and $\mathrm{cl}^* E$ differ only by sets of measure zero ([15, Chapter IV, Theorem (6.1)]); in particular, $|\mathrm{bd}^* E| = 0$. The essential interior, essential closure, and essential boundary are natural measure theoretic analogues of the corresponding topological concepts. Obviously $\mathrm{int} E \subset \mathrm{int}^* E \subset \mathrm{cl}^* E \subset \mathrm{cl}\, E$, and so $\mathrm{bd}^* E \subset \mathrm{bd}\, E$. If $\mathrm{cl}^* E$ equals E or $\mathrm{cl} E$, the set E is called *essentially closed* or *nondispersed*, respectively.

REMARK 9.1. It is clear that any two sets which differ by a set of measure zero have the same density and dispersion points. Hence they have the same essential interior, essential closure, and essential boundary.

The following lemma is useful. It implies, in particular, that

$$\mathrm{cl}^* A \cap \mathrm{int}^* B \subset \mathrm{cl}^*(A \cap B)$$

whenever A and B are measurable subsets of $\mathbf{R}^m$.

LEMMA 9.2 *Let $A \subset \mathbf{R}^m$ be measurable, let $x \in \mathrm{int}^* A$, and let $\{\varepsilon_n\}$ be a sequence of positive numbers with* $\lim \varepsilon_n = 0$. *If $\{A_n\}$ is a sequence of measurable sets such that $A_n \subset U(x, \varepsilon_n)$ and $|A_n| \geq a|U(x,\varepsilon_n)|$ for $n = 1, 2, \ldots$ and a fixed $a > 0$, then*

$$\lim \frac{|A \cap A_n|}{|A_n|} = 1 .$$

Proof. Assume that $\liminf(|A \cap A_n|/|A_n|) < b < 1$, and find a $c < 1$ so that $1 - c < a(1 - b)$. For $n = 1, 2, \dots$, let $U_n = U(x, \varepsilon_n)$. Since $\lim(|A \cap U_n|/|U_n|) = 1$, there is an integer $p \geq 1$ such that

$$|A \cap A_p| \leq b|A_p| \quad \text{and} \quad |A \cap U_p| \geq c|U_p| .$$

A contradiction follows:

$$\begin{aligned} |A \cap (U_p - A_p)| &\geq c|U_p| - b|A_p| \\ &= |U_p| - |A_p| - (1 - c)|U_p| + (1 - b)|A_p| \\ &\geq |A \cap (U_p - A_p)| + [a(1 - b) - (1 - c)]|U_p| \\ &> |A \cap (U_p - A_p)| . \end{aligned}$$

The $(m - 1)$-dimensional *Hausdorff measure* $\mathcal{H}$ in $\mathbf{R}^m$ is defined so that it is the counting measure if $m = 1$, and agrees with the measure λ_{m-1} in $\mathbf{R}^{m-1}$ if $m > 1$ (see [19, Section 1.4]). A bounded set $A \subset \mathbf{R}^m$ is called a BV set (BV for *bounded variation*) whenever the number $||A|| = \mathcal{H}(\text{bd}^* A)$, called the *De Giorgi perimeter* of A is finite. Thus the essential boundary of a BV set is thin (cf. Remark 4.2), although the topological boundary may have a positive measure. It is easy to see that the family BV of all BV sets is a ring and that

$$\max\{||A \cup B||, ||A \cap B||, ||A - B||\} \leq ||A|| + ||B|| .$$

for all $A, B \in BV$.

REMARK 9.3. Denoting by $||A||$ the perimeter of a figure A and also De Giorgi's perimeter of a BV set A is legitimate. Indeed, if A is a figure then $\text{bd}^* A = \text{bd}\, A$. Thus each figure is a BV set whose De Giorgi perimeter is equal to its perimeter defined in Section 4.

REMARK 9.4. By [3, Section 2.10.6 and Theorem 4.5.11], a bounded set $A \subset \mathbf{R}^m$ is a BV set if and only if it is measurable and the *distributional gradient* of its characteristic function is a vector valued measure in $\mathbf{R}^m$ whose variation σ_A is finite (see also [19, Chapter 5]). in this case we have $\sigma_A(E) = \mathcal{H}(E \cap \text{bd}^* A)$ for every set $E \subset \mathbf{R}^m$; in particular, $\sigma_A(\mathbf{R}^m) = ||A||$.

Let A be a BV set. It follows from [3, Chapter 4] that there is a Borel vector field n_A on $\mathbf{R}^m$, called the *Federer exterior normal* of A, such that

$$\mathcal{H}(B \cap \operatorname{bd}^* A) = (L) \int_B \|n_A\| \, d\mathcal{H},$$

$$(L) \int_A \operatorname{div} v \, d\lambda_m = (L) \int_{\operatorname{bd} A} v \cdot n_A \, d\mathcal{H}$$

for every $\mathcal{H}$-measurable set $B \subset \mathbf{R}^m$ and every continuously differentiable vector field v on $\mathbf{R}^m$. An $x \in \mathbf{R}^m$ is called a *perimeter dispersion point* of A whenever

$$\lim_{\varepsilon \to 0+} \frac{\mathcal{H}[\operatorname{bd}^* A \cap U(x,\varepsilon)]}{(2\varepsilon)^{m-1}} = 0 .$$

The set of all $x \in \operatorname{int}^* A$ which are perimeter dispersion points of A is called the *critical interior* of A, denoted by $\operatorname{int}_c A$. By [19, Lemma 5.9.4], we have $\mathcal{H}(\operatorname{int}^* A - \operatorname{int}_c A) = 0$.

The *regularity* and **-regularity* of BV sets is defined by the same formulae we used for figures. It follows from the *isoperimetric inequality* ([19, Theorem 5.4.3]) that there is a constant $c > 0$, depending only on the dimension m, such that $[cr^*(A)]^m \leq r(A)$ for each BV set A. Using [9, Section 8], it is possible to show that $c = 2$ as in Lemma 8.1, but we shall not need this.

LEMMA 9.5. *Let $A \in BV$, $x \in \operatorname{int}_c A$, and let $\{B_n\}$ be a sequence of BV sets such that $x \in \operatorname{cl}^* B_n$ and $r^*(B_n) \geq \varepsilon > 0$ for $n = 1,2,\ldots$ If $\lim d(B_n) = 0$, then $x \in \operatorname{cl}^*(A \cap B_n)$ for $n = 1,2,\ldots,$ and*

$$\liminf \; r^*(A \cap B_n) > \gamma \varepsilon^{m+1} ,$$

where $\gamma > 0$ is a constant depending only on the dimension m.

Proof. By Lemma 9.2, we have $x \in \operatorname{cl}^*(A \cap B_n)$ for $n = 1,2,\ldots,$. If $\delta_n = d(B_n)$ and $U_n = U(x, 2\delta_n)$, then $B_n \subset U_n$ and

$$\frac{|B_n|}{|U_n|} = 4^{-m} r(B_n) \geq 4^{-m}[cr^*(B_n)]^m > \left(\frac{c\varepsilon}{4}\right)^m ,$$

where $c > 0$ is a constant depending only on the dimension m. Another application of Lemma 9.2 shows that $\lim[\,|A \cap B_n|/|B_n|] = 1$. In addition we have

$$||A \cap B_n|| \leq \mathcal{H}(U_n \cap \mathrm{bd}^* A) + ||B_n|| \quad \text{and} \quad ||B_n|| \leq \frac{|B_n|}{\varepsilon d(B_n)} \leq \frac{\delta_n^{m-1}}{\varepsilon}.$$

Since x is a perimeter dispersion point of A,

$$\limsup \frac{||A \cap B_n||}{\delta_n^{m-1}} \leq 4^{m-1} \limsup \frac{\mathcal{H}(U_n \cap \mathrm{bd}^* A)}{(4\delta_n)^{m-1}} + \frac{1}{\varepsilon} = \frac{1}{\varepsilon}$$

and consequently,

$$\begin{aligned} &\liminf r^*(A \cap B_n) = \\ &\qquad \liminf \left(\frac{|A \cap B_n|}{|B_n|} \cdot \frac{|B_n|}{d(A \cap B_n)\delta_n^{m-1}} \cdot \frac{\delta_n^{m-1}}{||A \cap B_n||} \right) \geq \\ &\qquad \varepsilon \liminf r(B_n) \geq \varepsilon [c r^*(B_n)]^m > \gamma \varepsilon^{m+1}, \end{aligned}$$

where $\gamma = c^m/2$ is a positive constant depending only on m.

The following proposition is due to I. Tamanini and C. Giacomelli (see [18, Theorem 2.3] where a more complete result is proved).

PROPOSITION 9.6 *For each BV set A there are nondispersed BV sets $A_n \subset A$ such that $||A_n|| \leq ||A||$ and $|A - A_n| \leq ||A||/n$ for $n = 1, 2, \ldots$.*

Proof. Fix an integer $n \geq 1$ and let $\alpha = \inf(\,||B|| - n|B|)$ where the infimum is taken over all BV sets $B \subset A$. If $\{B_k\}$ is a sequence of BV subsets of A and $\lim(\,||B_k|| - n|B_k|) = \alpha$, then $\sup||B_k|| < +\infty$. By [19, Corollary 5.3.4], there is a subsequence of $\{B_k\}$, still denoted by $\{B_k\}$, and a set $E \in BV$ such that $E \subset A$, $\lim|B_k - E|$, and $\lim|E - B_k|$ equal zero. From [19, Theorem 5.2.1] we obtain

$$\alpha \leq ||E|| - n|E| \leq \liminf ||B_k|| - \lim |B_k| = \lim(\,||B_k|| - n|B_k|) = \alpha,$$

hence $||E|| - n|E| \leq ||A|| - n|A|$. As the last inequality yields $||E|| \leq ||A||$ and $n|A - E| \leq ||A||$, it suffices to show that E is not dispersed.

Since the set of all $x \in E$ such that $|E \cap U(x, \varepsilon)| = 0$ for an $\varepsilon > 0$ has measure zero, we may assume that $|E \cap U(x, \varepsilon)| > 0$ for each $\varepsilon > 0$.

Select an $x \in \operatorname{cl} E$, and for $t > 0$ let $U_t = U(x,t)$. The inequalities and equations which follow hold for λ-almost all $t > 0$; some hold for all $t > 0$, but this is irrelevant. The isoperimetric inequality ([19, Theorem 5.4.3]) provides a constant $c > 0$, depending only on the dimension m, such that

$$c \leq \|E \cap U_t\| \cdot |E \cap U_t|^{\frac{1}{m}-1} .$$

The minimality of E gives

$$\|E\| - \eta|E| \leq \|E - U_t\| - \eta|E - U_t| ,$$

hence using the equations

$$\|E \cap U_t\| = \mathcal{H}(U_t \cap \operatorname{bd}^* E) + \mathcal{H}(E \cap \operatorname{bd} U_t) ,$$

$$\|E - U_t\| = \|E\| - \mathcal{H}(U_t \cap \operatorname{bd}^* E) + \mathcal{H}(E \cap \operatorname{bd} U_t)$$

established in [2], we obtain

$$c \leq 2\mathcal{H}(E \cap \operatorname{bd} U_t)|E \cap U_t|^{\frac{1}{m}-1} + \eta|E \cap U_t|^{\frac{1}{m}} .$$

If $x = (\xi_1, \ldots, \xi_m)$, then applying Fubini's theorem to the sets

$$\{(\eta_1, \ldots, \eta_m) \in E \cap U_t : |\eta_j - \xi_j| \leq |\eta_i - \xi_i|,\ j = 1, \ldots, m\}$$

for $i = 1, \ldots, m$, it is easy to verify that

$$|E \cap U_t| = (L) \int_0^t \mathcal{H}(E \cap \operatorname{bd} U_s)\, d\lambda(s)$$

or equivalently

$$\mathcal{H}(E \cap \operatorname{bd} U_t) = \frac{d}{dt}|E \cap U_t| .$$

Thus

$$c \leq 2m \frac{d}{dt}\left(|E \cap U_t|^{\frac{1}{m}}\right) + \eta|E \cap U_t|^{\frac{1}{m}} .$$

Dividing the last inequality by an $\varepsilon > 0$ and integrating over the interval $(0, \varepsilon)$ yields

$$c \leq m \left(\frac{|E \cap U_\varepsilon|}{(2\varepsilon)^m}\right)^{\frac{1}{m}} + \eta|E \cap U_\varepsilon|^{\frac{1}{m}} ,$$

from which we conclude that $x \in \mathrm{cl}^*E$.

REMARK 9.7. We note that the existence of a BV set $E \subset A$ minimizing the value of $||B|| - n|B|$ over all BV sets $B \subset A$ is established by interpreting BV sets according to Remark 9.4. If this can be proved directly from our geometric definition of BV sets is unclear.

10. The integral.

A BV* set is an *essentially closed* BV set. In view of Remark 9.1, each BV set differs from a BV* set by a set of measure zero. It is easy to verify that a BV* set is *nondispersed* if and only if it is *closed.* If A and B are BV* sets then so are the sets $A \cup B$,

$$A \odot B = \mathrm{cl}^*(A \cap B) \quad \text{and} \quad A \ominus B = \mathrm{cl}^*(A - B)\ .$$

Note that for figures the above defined operations $\odot$ and $\ominus$ agree with those introduced in Section 4. The family of all BV* sets is denoted by BV^*.

Given an $\varepsilon > 0$ and a caliber $\eta = \{\eta_j\}$, we say that a BV* set B is (ε, η)*-*small* if B is the union of nonoveraapping, possibly empty, BV^* sets $B_1, \ldots, B_k$ such that $||B_j|| < 1/\varepsilon$ and $|B_j| < \eta_j$ for $j = 1, \ldots, k$. If $A, B \in BV^*$ and B is (ε, η)-*small, then $A \odot B$ and $A \ominus B$ are (ε^*, η)-*small for $\varepsilon^* = \varepsilon/(\varepsilon||A|| + 1)$.

REMARK 10.1. If a figure is (ε, η)-small, then it is (ε, η)-*small. It would be interesting to know whether the converse is also true.

A **partition* is a collection (possibly empty)

$$P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$$

where the A_i's are nonoverlapping BV* sets and $x_i \in A_i$ for $i = 1, \ldots, p$. As before, let $\cup P = \cup_{i=1}^{p} A_i$. Given a BV* set A, an $\varepsilon > 0$, and a caliber η, we say that P is a *partition of A **mod* (ε, η) whenever $\cup P \subset A$ and $A \ominus \cup P$ is (ε, η)-*small. Thus if A is a figure, then by Remark 10.1, each f-partition of A mod (ε, η) is a *partition of A *mod (ε, η). All other concepts applied to partitions and f-partitions in the previous sections extend, in the obvious way, to *partitions.

At this point, a word concerning notation is in order. While using all the asterisks is a clear notational nuisance, at the introductory level, it is the only safe way to avoid confusion. Thus, with an apology, we ask the reader to bear with it.

THEOREM 10.2. *Let A be a BV^* set. There is a constant $\kappa > 0$, depending only on the dimension m, such that given a gage δ in A and a caliber η, we can find a δ-fine κ-*regular *partition of A *mod (κ, η).*

Proof. Assume first that $\|A\| \leq 2$. If $\eta = \{\eta_1, \eta_2, \ldots\}$, then by Proposition 9.6, there is a closed BV* set $B \subset A$ with $\|B\| \leq 2$ and $|A \ominus B| < \eta_1$. For $j = 1, \ldots, m$, find integers $r_j < s_j$ so that the interval $K = \prod_{j=1}^{m}[r_j, s_j]$ contains B, and for each $x \in K$ let $\rho(x)$ be the distance from x to B. Let $\kappa = \min\{1/7, \gamma(2m)^{-m-1}\}$ where γ is the positive constant from Lemma 9.5. If $x \in \operatorname{int}_c B$, then Lemma 9.5 implies the existence of an $\alpha(x) > 0$ such that $r^*(B \odot C) > \kappa$ for each cube C with $x \in C$ and $d(C) < \alpha(x)$. Since $B - \operatorname{int}_c B$ is a thin set, letting

$$\Delta(x) = \begin{cases} \rho(x) & \text{if } x \in K - \operatorname{int}_c B\,, \\ \min\{\alpha(x), \delta(x)\} & \text{if } x \in \operatorname{int}_c B\,, \end{cases}$$

defines a gage Δ in K. Let $\vartheta = \{\eta_2, \eta_3, \ldots\}$ and use Lemma 4.7 to find a Δ-fine partition $Q = \{(K_1, x_1), \ldots, (K_q, x_q)\}$ of K mod $(1/5, \vartheta)$ such that $K_1, \ldots, K_q$ are dyadic cubes. By Lemma 9.5, the collection $P = \{(B \odot K_i, x_i) : x_i \in \operatorname{int}_c B\}$ is a δ-fine κ-*regular *partition in B and

$$B \ominus \bigcup P = B \odot \left(K \ominus \bigcup Q\right).$$

Since $K \ominus \bigcup Q$ is $(1/5, \vartheta)$-small and $\|B\| \leq 2$, the set $B \ominus \bigcup P$ is (κ, ϑ)-*small. From this we conclude that P is a *partition of A *mod (κ, η).

It is easy to show that an arbitrary BV* set A is the union of nonoverlapping BV* sets $A_1, \ldots, A_n$ whose De Giorgi perimeters are less than or equal to 2. For $k = 1, \ldots, n$, let $\eta^k = \{\eta_{nj+k}\}_{j=1}^{\infty}$ and use the first part of the proof to find a δ-fine κ-*regular *partition P_k of A_k *mod (κ, η^k). Now it is clear that $P = \bigcup_{k=1}^{n} P_k$ is the desired *partition, and the theorem is proved.

DEFINITION 10.3. A function f on a BV* set A is called *integrable* in A if there is a real number I having the following property: given $\varepsilon > 0$, we can find a gage δ in A and a calibre η such that

$$|\sigma(f, A) - I| < \varepsilon$$

for each δ-fine ε-*regular *partition P of A *mod (ε, η).

It follows from Theorem 10.2 that the number I of Definition 10.3 is determined uniquely by the integrable function f. We call it the *integral* of f over A, denoted by $\int_A f$. By $\mathcal{I}(A)$ we denote the family of all integrable functions in A.

In dimension one, the figures and BV* sets coincide, and so do the g*-integral and integral. Let $m \geq 2$ and let A be a figure. It follows directly from Definitions 8.4 and 10.3 that $\mathcal{I}(A) \subset \mathcal{G}^*(A)$ and $\int_A f = (g) \int_A f$ for each $f \in \mathcal{I}(A)$. Using Theorem 11.4 below, it is possible to show that the inclusion $\mathcal{I}(A) \subset \mathcal{G}^*(A)$ is proper whenever $|A| > 0$ (cf. [11, Example 6.2]).

Let A be a BV* set. A **division* of A is a finite family of nonoverlapping BV* sets whose union is A. A **additive* function in A is a function F defined on all BV* subsets of A such that

$$F(A) = \sum_{D \in \mathcal{D}} F(D)$$

for each *division $\mathcal{D}$ of A. A *additive function F in A is called **continuous* if given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(B)| < \varepsilon$ for each BV* set $B \subset A$ with $\|B\| < 1/\varepsilon$ and $|B| < \eta$, If A is a figure, then by restriction, a *additive function F in A defines a unique additive function in A, which is continuous whenever F is *continuous.

PROPOSITION 10.4 *Let A be a BV* set and let $f \in \mathcal{I}(A)$. Then $f \in \mathcal{I}(B)$ for each BV* set $B \subset A$, and the map $F : B \mapsto \int_A f$ is a *additive *continuous function in A.*

The proof is left to the reader; modulo the obvious adjustments, it is the same as that of Proposition 6.4.

Replacing figures by BV* sets, Proposition 6.2, Lemma 6.5, Theorems 6.6 and 6.7 as well as Corollaries 6.8 and 6.9 hold for the integral defined in this section. The exact reformulations and proofs are left to the reader.

PROPOSITION 10.5 *Let f be a function defined on a BV^* set A, and let $\{A_1, \ldots, A_n\}$ be a *division of A consisting of closed sets. If f is integrable in A_k for $k = 1, \ldots, n$, then it is integrable in A.*

Proof. The proof is similar to that of Propostition 6.10. Set $I = \sum_{k=1}^{n} \int_{A_k} f$ and choose a positive $\varepsilon < 1/\max\{||A_1||, \ldots, ||A_n||\}$. For $k = 1, \ldots, n$, there are gages δ_k in A_k and calibers $\eta^k = \{\eta_j^k\}_j$ such that

$$\left|\sigma(f, P_k) - \int_{A_k} f\right| < \frac{\varepsilon}{n}$$

for each δ_k-fine ε-*regular *partition P_k of A_k *mod $(\varepsilon/2, \eta^k)$. For $x \in A$ denote by $\rho_k(x)$ the distance from x to the closed set $D_k = \bigcup_{i \neq k} A_i$, and define a nonnegative function δ on A by setting $\delta(x) = \min\{\delta_k(x), \rho_k(x)\}$ for each $x \in A_k$. Since $D_k \cap \operatorname{int}^* A_k = \emptyset$ (see the comment preceding Lemma 9.2) and since $\mathrm{bd}^* A_k$ is a thin set, the function δ is a gage in A. Now let $\eta = \{\eta_j\}$ be a caliber such that $\eta_j = \min\{\eta_j^1, \ldots, \eta_j^n\}$ for all j. If P is a δ-fine ε-*regular *partition of A *mod (ε, η), then $P_k = \{(B, x) \in P : x \in A_k\}$ is a δ-fine ε-*regular *partition in A_k. By definition, $A \ominus \bigcup P$ is (ε, η)-*small. As

$$A_k \ominus \bigcup P_k = A_k \odot \left(A \ominus \bigcup P\right),$$

it is easy to verify that P_k is a *partition of A_k *mod $(\varepsilon/2, \eta^k)$. Thus

$$|\sigma(f, P) - I| \leq \sum_{k=1}^{n} \left|\sigma(f, P_k) - \int_{A_k} f\right| < \varepsilon$$

and the proposition is proved.

REMARK 10.6. The assumption that the A_i's are closed sets is essential for the validity of Proposition 10.5 (see [13, Example 5.21]). This is, however, only a minor deficiency that can be easily corrected by various extensions of the integral. A particularly useful extension can be found in [13, Sections 8 and 9].

LEMMA 10.7. *Let v be a bounded vector field on a set $E \subset \mathbf{R}^m$ that is differentiable at $x \in \operatorname{int} E$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\left| \operatorname{div} v(x)|B| - (L)\int_{\operatorname{bd} B} v \cdot n_B \, d\mathcal{H} \right| < \varepsilon d(B)\|B\|$$

for each BV set $B \subset E \cap U(x,\delta)$ for which $x \in B$ and v restricted to* $\mathrm{bd}^* B$ *is $\mathcal{H}$-measurable.*

The previous lemma is the *germ* of the divergence theorem for the BV* sets. Its proof, which is identical to that of Lemma 7.1, is left to the reader. For an alternative proof we refer to [12, lemma 5.5].

LEMMA 10.8. *Let E be a measurable subset of an open set $U \subset \mathbf{R}^m$ and let g and h be functions defined on U that have partial derivatives at each $x \in E$. If $g(x) = h(x)$ for all $x \in E$, then $(\partial/\partial\xi_i)g(x) = (\partial/\partial\xi_i)h(x)$ for $i = 1,\dots,m$ and almost all $x \in E$.*

Proof. Suppose that the set

$$C = \{x \in E : (\partial/\partial\xi_1)g(x) \neq (\partial/\partial\xi_1)h(x)\}$$

has a positive measure. By Fubini's theorem, there is a $z \in \mathbf{R}^{m-1}$ such that the set $S \subset \mathbf{R}$ of those s for which $(s,z) \in C$ has a positive measure λ. In particular, there is a $t \in S$ and a sequence $\{t_n\}$ in $S - \{t\}$ with $\lim t_n = t$. From this we obtain that $(\partial/\partial\xi_1)g(t,z) = (\partial/\partial\xi_1)(t,z)$, a contradiction. The lemma follows by symmetry.

Let v be a vector field on an *arbitrary* set $C \subset \mathbf{R}^m$ and let $E \subset C$ be a measurable set. We say that v is *almost differentiable* on E if there is a set $D \subset \mathbf{R}^m$ and a vector field w on D such that the following conditions are satisfied:

1. $C \subset D$ and $E \subset \operatorname{int} D$;
2. $w(x) = v(x)$ for every $x \in C$;
3. w is almost differentiable at each $x \in E$.

By Stepanoff's theorem ([3, Theorem 3.1.9]), the vector field w is differentiable at almost all $x \in E$. For each $x \in E$ we set

$$\operatorname{div} v(x) = \begin{cases} w(x) & \text{if } w \text{ is differentiable at } x \\ 0 & \text{otherwise}. \end{cases}$$

It follows from Lemma 10.8 that upto a set of measure zero, the function $\operatorname{div} v : E \to \mathbf{R}$ is determined uniquely by the vector field v and does not depend on the choice of w.

THEOREM 10.9. *Let A be a BV* set and let T be a thin set. Suppose that v is a continuous vector field on $\mathrm{cl}A$ which is almost differentiable on* $\operatorname{int}^* A - T$. *Then* div v *is integrable in A and*

$$\int_A \operatorname{div} v = (L)\int_{\operatorname{bd} A} v \cdot n_A \, d\mathcal{H} .$$

Proof. By our assumption there is a vector field w on a set $D \subset \mathbf{R}^m$ such that $\mathrm{cl}A \subset D$ and $w(x) = v(x)$ for every $x \in \mathrm{cl}\, A$, and at the same time $\operatorname{int}^* A - T \subset \operatorname{int} D$ and w is almost differentiable at each $x \in \operatorname{int}^* A - T$. Letting

$$F(B) = (L)\int_{\operatorname{bd} B} v \cdot n_B \, d\mathcal{H} = (L)\int_{\operatorname{bd} B} w \cdot n_B \, d\mathcal{H}$$

for every BV* set $B \subset A$ defines a *additive *continuous function in A (cf. Example 6.3). There is a set $E \subset \operatorname{int}^* A - T$ such that $|E| = 0$ and w is differentiable at each $x \in \operatorname{int}^* A - (E \cup T)$. It suffices to show that a function f on A defined by

$$f(x) = \begin{cases} \operatorname{div} w(x) & \text{if } x \in \operatorname{int}^* A - (E \cup T) , \\ 0 & \text{if } \in E \cup T \cup \operatorname{bd}^* A , \end{cases}$$

belongs to $\mathcal{I}(A)$ and $\int_A f = F(A)$. In view of Lemma 10.7, the proof of this is analogous to that of Theorem 7.2. We leave the details to the reader.

11. Change of variables.

Let $E \subset \mathbf{R}^m$ and let $\Phi : E \to \mathbf{R}^m$. The map Φ is called *Lipschitzian* if there is a real number c, called a *Lipschitz constant* of Φ, such that

$$|\Phi(x) - \Phi(y)| \le c|x - y|$$

for each $x, y \in E$. We say that Φ is a *lipeomorphism* if it is bijective and Lipschitzian and the inverse map $\Phi^{-1} : \Phi(E) \to E$ is also Lipschitzian.

LEMMA 11.1. *Let Φ be a lipeomorphism of a set $E \subset \mathbf{R}^m$ into $\mathbf{R}^m$. Then Φ extends uniquely to a lipeomorphism of* cl$E$ *into $\mathbf{R}^m$, also denoted by Φ, and* $\Phi(\mathrm{cl}^*E) = \mathrm{cl}^*\Phi(E)$.

Proof. There are positive constants a and b such that

$$a|x - x'| \leq |\Phi(x) - \Phi(x')| \leq b|x - x'|$$

for all $x, x' \in E$. By the completeness of $\mathbf{R}^m$, the map Φ extends uniquely to cl$E$. As the extended map, still denoted by Φ, satisfies the above inequalities for all x, $x' \in \mathrm{cl}\, E$, it is a lipeomorphism. Let $x \in \mathrm{cl}\, E$ be a dispersion point of E, and let $y = \Phi(x)$. Since

$$\Phi(E) \cap U(y, \varepsilon) \subset \Phi(E \cap U(x, \varepsilon/a)) ,$$

$$|\Phi(E \cap U(x, \varepsilon/a))| \leq b^m |E \cap U(x, \varepsilon/a)|$$

for each $\varepsilon > 0$, we obtain

$$\limsup_{\varepsilon \to 0+} \frac{|\Phi(E) \cap U(y, \varepsilon)|}{\varepsilon^m} \leq \left(\frac{b}{a}\right)^m \limsup_{\varepsilon \to 0+} \frac{|E \cap U(x, \varepsilon/a)|}{(\varepsilon/a)^m} = 0 .$$

Thus y is a dispersion point of $\Phi(E)$, and the lemma follows by symmetry

REMARK 11.2. Under the assumptions of Lemma 11.1, it is also true that $\Phi(\mathrm{int}^*E) = \mathrm{int}^*\Phi(E)$ whenever E is measurable. It turns out, however, that this fact requires a surprisingly complex proof, recently found by Z. Buczolich (see [1]).

LEMMA 11.3. *Let $A \in BV$ and let $\Phi : A \to \mathbf{R}^m$ be a lipeomorphism with a Lipschitz constant c. Then $B = \Phi(A)$ is a BV set with $|B| \leq c^m|A|$ and $||B|| \leq c^{m-1}||A||$.*

A proof of Lemma 11.3 based on interpreting BV sets as *integral currents* (see [3, Chapter 4]) is given in [13, Lemma 6.6]. Alternately, Buczolich's result mentioned in Remark 11.2 implies that

$$\Phi(\mathrm{bd}^*A) = \mathrm{bd}^*\Phi(A) ,$$

from which the lemma follows.

If Φ is a Lipschitzian map of a measurable set $E \subset \mathbf{R}^m$ we denote by $\det \Phi$ the *determinant* of the differential $D\Phi$ of Φ. By the Kirszbraun and Rademacher theorems ([3, Theorems 2.10.43 and 3.1.6]), we see that the function $\det \Phi$ is defined almost everywhere in E and it is determined uniquely up to a set of measure zero according to Lemma 10.8. If Φ is a lipeomorphism then $\det \Phi(x) \neq 0$ for almost all $x \in E$.

THEOREM 11.4. *Let A be a BV^* set, let $\Phi : A \to \mathbf{R}^m$ be a lipeomorphism, and let $f \in \mathcal{I}(\Phi(A))$. Then $f \circ \Phi \cdot |\det \Phi|$ belongs to $\mathcal{I}(A)$ and*

$$\int_A f \circ \Phi \cdot |\det \Phi| = \int_{\Phi(A)} f \, .$$

Proof. We let $x^\bullet = \Phi(x)$ for each $x \in A$ and $B^\bullet = \Phi(B)$ for each $B \subset A$. There is a constants $c \geq 1$, depending only on Φ, such that $c^{-1}|x - y| \leq |x^\bullet - y^\bullet| \leq c|x - y|$ for each $x, y \in A$. By Lemma 11.3, if $B \subset A$ is a BV* set, then so is $B^\bullet$ and

$$c^{-m}|B| \leq |B^\bullet| \leq c^m|B| \quad \text{and} \quad c^{1-m}\|B\| \leq \|B^\bullet\| \leq c^{m-1}\|B\| \, .$$

Let $\varepsilon > 0$ and let $\tau = \varepsilon c^{-2m}$. There is a gage δ in $A^\bullet$ and acaliber $\eta = \{\eta_j\}$ such that

$$\left| \sigma(f, Q) - \int_{A^\bullet} f \right| < \varepsilon$$

for each δ-fine τ-*regular *partition Q of $A^\bullet$ *mod (τ, η). Given $x \in A$, select an $\varepsilon_x > 0$ so that $\varepsilon_x |f(x)| < \varepsilon$. Since

$$|B^\bullet| = (L) \int_B |\det \Phi| \, d\lambda_m$$

for every measurable set $B \subset A$ ([3, Theorem 3.2.3)(1)]) it follows from [15, Chapter 4, Theorem (6.3)] that there is a set $E \subset A$ with $|E| = 0$ and a positive function φ in A such that

$$\big| |\det \Phi(x)| \cdot |B| - |B^\bullet| \big| < \varepsilon_x |B|$$

for each $x \in A - E$ and each BV* set $B \subset A \cap U(x, \varphi(x))$ with $x \in B$ and $r^*(B) > \varepsilon$. In view of Corollary 6.8 for the integral, we may assume that det $\Phi(x) = 0$ for every $x \in E$. Finally, let α be the function from Lemma 5.3 associated with the set $E^\bullet$ and $\varepsilon > 0$. There is a positive function θ in $A^\bullet$ such that $|f(y)| \cdot |C| \leq \alpha(C)$ for each $y \in E^\bullet$ and each set $C \subset U(y, \theta(y))$.

Observing that a subset of $\mathbf{R}^m$ is thin if and only if its $\mathcal{H}$ measure is σ-finite (cf. Remark 4.2), it is easy to show that a Lipshitzian image of a thin set is again thin. Thus the function

$$\Delta = \min\left\{\frac{\delta \circ \Phi}{c}, \varphi, \frac{\theta \circ \Phi}{c}\right\}$$

is a gage in A. If $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a Δ-fine ε-*regular *partition of A *mod (ε, ϑ) where $\vartheta = \{\eta_j c^{-m}\}$, then it is easy to verify that $Q = \{(A_i^\bullet, x_1^\bullet), \ldots, (A_p^\bullet, x_p^\bullet)\}$ is a δ-fine τ-*regular *partition of $A^\bullet$ *mod (τ, η). Consequently

$$\left|\sigma(f \circ \Phi \cdot |\det \Phi|, P) - \int_{A^\bullet} f\right| \leq$$

$$\sum_{i=1}^{p} |f(x_i^\bullet)| \cdot \big||\det \Phi(x_i)| \cdot |A_i| - |A_i^\bullet|\big| + \left|\sigma(f, Q) - \int_{A^\bullet} f\right| <$$

$$\sum_{x_i \in E} |f(x_i^\bullet)| \cdot |A_i^\bullet| + \sum_{x_i \notin E} \varepsilon_{x_i} |f(x_i^\bullet)| \cdot |A_i| + \varepsilon \leq$$

$$\sum_{x_i \in E} \alpha(A_i^\bullet) + \varepsilon \sum_{x_i \notin E} |A_i| + \varepsilon \leq \alpha(A^\bullet) + \varepsilon|A| + \varepsilon \leq \varepsilon(2 + |A|),$$

and the theorem follows.

At this point, it is routine to lift the integral to Lipschitzian manifolds and prove the *Stokes theorem* for compact Lipschitzian manifolds with boundaries. The details of this task are left to the reader.

12. Some open problems.

I shall close these lectures by describing breifly three major problems, which to my knowledge are still completely open.

PROBLEM 12.1. Find versions of *Fubini's theorem* that are valid for the integral. The usual Fubini theorem cannot be proved as it is incompatible with the divergence theorem ([11, Example 5.7]), but weaker versions may hold.

PROBLEM 12.2. A *multiplier* is a function g defined on a BV* set A such that $fg \in \mathcal{I}(A)$ whenever $f \in \mathcal{I}(A)$. Characterize the family of all multipliers, or at least describe a sufficiently large class of them. It appears that BV functions ([19, Definition 5.1.1]) are possible candidates, but we do not even know whether infinitely differentiable functions are multipliers. It was proved in [16] that a function is a multiplier for the one-dimensional HK-integral if and only if it is equal almost everywhere to a function of finite variation.

PROBLEM 12.3. The nonabsolutely convergent nature of the integral suggests that the integration process is, in some way, based on summation of *conditionally convergent series*. Specific examples corroborate this suspicion. Explain the role of conditionally convergent series in the absence of a natural linear order in $\mathbf{R}^m$ for $m \geq 2$. For the one-dimensional HK-integral, a full explanation is given by the *Denjoy constructive definition* ([15, Chapter VIII, Sections 4 and 5]) of the Denjoy-Perron integral, which is equivalent to the HK-integral (cf. Remark 3.18).

It is likely that Problems 12.1 and 12.2 are closely related.

ADDED IN PROOF. Recently, J.W. Mortensen and the author proved that the Lipschitz functions are multipliers for the g^*-integral as well as for the integral.

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