TOPICS IN THE THEORY OF PETTIS INTEGRATION (*)

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Contents

1. Preliminaries.

Throughout (Ω, Σ, μ) denotes a complete probability space, $\mathcal{N}(\mu)$ stands for the collection of all μ-null sets, and Σ_μ^+ – for the family of all sets of positive μ–measure. $\mathcal{P}(\Omega)$ stands for the family of all subsets of

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Ω, and $\sigma(\mathcal{A})$ – for the σ–algebra generated by $\mathcal{A} \subset \mathcal{P}(\Omega)$. μ^* is the outer measure generated by μ. The term "measure" will always denote a finite measure. μ is separable if Σ is separable with respect to the Frechet–Nikodym pseudometric. For a given σ–algebra Σ we shall denote by π_Σ the set of all finite collections π, consisting of pairwise disjoint elements of Σ with strictly positive measure. π_Σ is a directed set if $\pi' \geq \pi$ is defined to mean that each element of π is, except for a null set, a union of elements of π'.

X, Y will always be Banach spaces (with a few explicitly stated exceptions), and X^*, Y^* will be their topological conjugate spaces. $B(X)$ will stand for the closed unit ball of X and $S(X)$ – for the unit sphere. The value of a functional $x^* \in X^*$ on an element $x \in X$ is denoted by $x^*(x)$ or by $\langle x^*, x\rangle$, $\mathcal{F}_X$ is the family of all finite subsets of X and, $\tilde{\mathcal{F}}_X$ is the family of all finite dimensional subspaces of X.

A set function $\nu : \Sigma \to X$ is an X*–valued measure* if it is countably additive in the norm topology, $|\nu|$ is its variation. If $\Gamma \subset X^*$ is a set that is total over X then ν is a Γ*–measure* if for each $x^* \in \Gamma$ the real–valued $x^*\nu$ is a measure. If $X = Y^*$ and $\Gamma = Y$, then we call it a *weak*–measure*. It follows from the Orlicz–Pettis theorem, that an X^*–measure is an X–valued measure. If ν is a Banach space valued finitely additive set function, defined on an algebra of sets, then ν is said to be μ*–continuous* if $\lim_{\mu(E)\to 0} \nu(E) = 0$. We shall denote it by $\nu \ll \mu$. One can show (cf [D-U]), that if ν is a measure, then $\nu \ll \mu$ if and only if $\mathcal{N}(\mu) \subseteq \mathcal{N}(\nu)$. A Γ–measure ν is *scalarly* μ*–continuous* if $x^*\nu \ll \mu$ for all $x^* \in \Gamma$.

N and **R** will denote the set of natural numbers and the set of real numbers respectively. $\mathcal{B}_{\mathbf{R}}$ denotes the algebra of Borel subsets of the real line, *card*(Γ) is the cardinality of Γ and χ_E denotes the indicator of a set E. Given $A \subseteq X$ we shall denote by *conv*(A) the convex hull of A and by *aco*(A)-the absolutely convex hull of A. $\bar{A}^\tau$ will be the closure of A in the topology τ. $A^\perp$ is the *annihilator* of A and A^0 is the polar of A. lin$A$ is the linear subspace of X generated by A. If T is a topological space endowed with a topology $\mathcal{T}$, then $\mathcal{B}a(T, \mathcal{T})$ will denote the σ–algebra of Baire sets (= the σ–algebra generated by all continuous real–valued functions defined on T), and, $\mathcal{B}o(T, \mathcal{T})$ will be the σ–algebra of Borel sets (= the σ–algebra generated by all open subsets of T). In the particular case of a normed space

X considered with the weak or norm topology, we shall use the notations $\mathcal{B}a(X, weak)$, $\mathcal{B}a(X, norm)$, $\mathcal{B}o(X, weak)$, $\mathcal{B}o(X, norm)$.

2. Measures in a normed space.

This chapter contains a few basic facts about Baire and Borel measures defined on a normed space equipped with its weak topology. We need them mainly in order to present a characterization of some function properties of Banach spaces (see Theorem 3-3). We start with a nice description of weakly Baire sets discovered by Edgar [E].

Let I be an arbitrary non-empty set and let $\mathbf{R}^I$ be endowed with the product topology. We denote by $\mathcal{H}_0$ the family of all non-empty open subsets of $\mathbf{R}^I$ depending on finitely many coordinates (i.e. they are of the type $\prod_{i\in I} A_i$, with all A_i open and non-empty, and all but finitely of them are equal to $\mathbf{R}$) and by $\mathcal{H}$ – the collection of all countable unions of sets from $\mathcal{H}_0$.

PROPOSITION 2.1. (Bockenstein's Theorem). *If V_1 and V_2 are disjoint open subsets of $\mathbf{R}^I$, then there exist disjoint sets U_1, U_2 in $\mathcal{H}$ with $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$.*

Proof. If η is a probability measure on $\mathcal{B}_{\mathbf{R}}$ that is positive on each non-empty open subset of $\mathbf{R}$, then its power on $\mathbf{R}^I$ gives positive measure to each member of $\mathcal{H}_0$. It follows, that each collection of pairwise disjoint elements of $\mathcal{H}_0$ is at most countable. If $\mathcal{F}$ is a maximal family of pairwise disjoint elements of $\mathcal{H}_0$ contained in $\mathbf{R}^I \setminus V_2$, then $\mathcal{F}$ is at most countable and so its union U is in $\mathcal{H}$. The inclusion $V_1 \subseteq \bar{U}$ is a consequence of the maximality of $\mathcal{F}$. $U_2 = \mathbf{R}^I \setminus \bar{U}$ depends on countably many coordinates, so it is in $\mathcal{H}$, contains V_2 and is disjoint with V_1. Applying the same procedure to V_1 and U_2 we get the required $U_1 \supseteq V_1$. ◊

COROLLARY 2.1. *Let Y be a dense subset of $\mathbf{R}^I$ and V_1, V_2 be two disjoint open subsets of Y. Then, there exist disjoint elements U_1 and U_2 of $\mathcal{H}$ with $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$.*

Proof. Since Y is dense in $\mathbf{R}^I$ any open sets W_1, W_2 in $\mathbf{R}^I$ with

$W_1 \cap Y = V_1$, $W_2 \cap Y = V_2$ are disjoint. It is enough to apply Proposition 2.1.

$\diamondsuit$

PROPOSITION 2.2. *Let Y be a dense subset of $\mathbf{R}^I$. Then, the Baire σ-algebra of Y is generated by the coordinate functions.*

Proof. Let Ξ be the σ algebra generated by the restrictions to Y of the coordinate functions and, let g be a continuous real-valued function on Y. Fix $a \in \mathbf{R}$. Then Corollary 2.2 implies the existence for each $n \in \mathbf{N}$ of disjoint sets U_n, U_n' in $\mathcal{H}$ with

$$U_n \cap Y \supseteq \{y \in Y : g(y) > a + 2^{-n}\}$$

$$U_n' \cap Y \supseteq \{y \in Y : g(y) < a + 2^{-n}\}$$

Hence $\{y : g(y) > a\} = \bigcup_{n=1}^{\infty} U_n \cap Y \in \Xi$, and this proves the measurability of g.

$\diamondsuit$

THEOREM 2.1. (Edgar) *Let X be a locally convex space. Then, $\mathcal{B}a(X, weak)$ is the σ-algebra generated by the continuous linear functionals.*

Proof. If I is a Hamel basis of X^*, then $(X, weak)$ is linearly homeomorphic to a dense subset of $\mathbf{R}^I$ and so the assertion follows from Proposition 2.2.

$\diamondsuit$

Applying the Hahn–Banach theorem and the above result we get.

COROLLARY 2.2. *If Y is a linear subspace of a locally convex space X, then $\mathcal{B}a(Y, weak)$ is the trace of $\mathcal{B}a(X, weak)$ on Y.*

DEFINITION 2.1. Let X be a normed space. A measure η on $\mathcal{B}a(X, weak)$ is scalarly degenerated if there is a proper closed linear subspace $Y \in \mathcal{B}a(X, weak)$ with $\eta(Y) = \eta(X)$.

It is an immediate consequence of the Hahn–Banach theorem that η is scalarly degenerated if and only if there is a closed hyperplane H in X of full measure.

PROPOSITION 2.3. *Let X be a normed space and η be a scalarly non-degenerated probability on $\mathcal{B}a(X, weak)$. Then the topology of convergence in measure η and the weak*–topology coincide on $B(X^*)$.*

Proof. If $(x_n^*) \subset B(X^*)$ is η–a.e. convergent, then there is $x_0^* \in B(X^*)$ such that $x_n^*(x) \to x_0^*(x)$ for each $x \in X$. To see this, observe that the set $\{x \in X : (x_n^*(x))_{n=1}^{\infty}$ is convergent$\}$ is a closed subspace of X of full measure and so, it has to be the whole space. Define $x_0^* : X^* \to R$ by setting $x_0^*(x) = \lim_n x_n^*(x)$. Clearly $x_0^* \in B(X^*)$.

It follows, that each η–Cauchy sequence $(x_n^*) \subset B(X^*)$ is weak* convergent.

Thus, the convergence of sequences in measure yields the weak*–convergence. As the η-convergence is metrizable on $B(X^*)$ it is stronger then the weak*–topology on the unit ball of X^*. Since two compact comparable topologies coincide, to get the conclusion it is enough to show, that $B(X^*)$ is compact in the topology of convergence in η–measure.

To do it take an arbitrary sequence $(x_n^*) \subset B(X^*)$ and set $f(x) = 1 + \sup_n |x_n^*(x)|$, for each $x \in X$. f is measurable with respect to $\mathcal{B}a(X, weak)$ and the sequence (x_n^*/f) is uniformly bounded. According to a theorem of Komlos [K] there is an increasing sequence (n_k) of natural numbers and a measurable function h such, that for each subsequence (n_{k_p})

$$\lim_p \frac{1}{p}\sum_{i=1}^{p} x_{n_{k_i}}^*(x) = h(x)f(x)$$

η–a.e.

According to the first part of the proof, there is $x^*((n_{k_i})) \in B(X^*)$ with

$$\lim_p \frac{1}{p}\sum_{i=1}^{p} x_{n_{k_i}}^*(x) = x^*((n_{k_i}))(x)$$

everywhere.

In virtue of the scalar non–degeneration of η there is $x_0^* \in B(X^*)$ such that $x^*((n_{k_i})) = x_0^*$ for each sequence (n_{k_i}). Hence, for each sequence (n_{k_i})

$$\lim_p \frac{1}{p}\sum_{i=1}^{p} x_{n_{k_i}}^*(x) = x_0^*(x)$$

everywhere.

It follows, that $(x^*_{n_k})$ is pointwise convergent and hence also in measure. $\diamondsuit$

COROLLARY 2.3. *If there is a scalarly non–degenerated measure on $\mathcal{B}a(X, weak)$, then X is separable.*

DEFINITION 2.2. Let X be a normed space, η be a probability on $\mathcal{B}a(X, weak)$ and

$$L_\eta = \bigcap \{L : L \text{ is a closed linear subspace of } X \text{ and } \eta(L) = 1\}.$$

If L_η is of outer measure 1 then η is said to have a linear support.

LEMMA 2.1. *If X is a normed space, η is a probability on $\mathcal{B}a(X, weak)$ possessing a linear support then $\eta | L_\eta$ is scalarly non-degenerated.*

Proof. Suppose L is a proper closed linear subspace of L_η with $\eta | L_\eta(L) = 1$. Then, according to the Hahn–Banach theorem, there exists a non–zero functional $x_0^* \in L_\eta^*$ such that $\eta | L_\eta \{x \in L_\eta : x_0^*(x) = 0\} = 1$. Let x^* be an extension of x_0^* to the whole X. Then
$\{x \in X : x^*(x) = 0\} \cap L_\eta = \{x \in L_\eta : x_0^*(x) = 0\}$ and so
$\eta\{x \in X : x^*(x) = 0\} = 1$. It follows, that $x^* | L_\eta = 0$ and this contradicts the choice of x_0^*. $\diamondsuit$

DEFINITION 2.3. Let $(T, \mathcal{T})$ be a completely regular topological space and let $\mathcal{A}$ be a σ–algebra on T such that $\mathcal{B}a(T, \mathcal{T}) \subseteq \mathcal{A} \subseteq \mathcal{B}o(T, \mathcal{T})$. A measure η on $\mathcal{A}$ is said to be *τ–smooth* if for any family $(U_\alpha) \in \mathcal{A}$ of open sets stable under finite unions and such that $\bigcup_\alpha U_\alpha \in \mathcal{A}$ the equality $\sup_\alpha \eta(U_\alpha) = \eta(\bigcup_\alpha U_\alpha)$ holds. A measure η on $\mathcal{A}$ is *Radon* if for each $A \in \mathcal{A}$ and each positive ε, there exists a compact set $K \subseteq A$ such that $\eta^*(K) \geq \eta(A) - \varepsilon$.

It is known, that a Baire measure η is τ–smooth if and only if for each net (Z_α) of zero sets decreasing to the empty set one has $\lim_\alpha \eta(Z_\alpha) = 0$ (cf.[W]).

THEOREM 2.2. (Tortrat). *Let X be a normed space and η be a measure on $\mathcal{B}a(X, weak)$ possessing a linear support. Then L_η is separable and η has a unique τ–smooth extension to $\mathcal{B}o(X, norm)$. If X is a Banach space, then η has a unique Radon extension to $\mathcal{B}o(X, norm)$.*

Proof. The separability of L_η is a consequence of Corollary 2.3 and Lemma 2.1. Together with the fact that the balls in a separable space are weakly Baire this yields the equality $\mathcal{B}a(L_\eta, weak) = \mathcal{B}o(L_\eta, norm)$. Hence $\nu = \eta|\mathcal{B}o(L_\eta, norm)$ is τ–smooth. Setting $\hat{\eta}(B) = \nu(B \cap L_\eta)$ for norm Borel subsets of X, we get a τ–smooth measure on $\mathcal{B}o(X, norm)$. If X is a Banach space then $\hat{\eta}$ is Radon.

Since in virtue of Corollary 2.2 the σ–algebra $\mathcal{B}a(L_\eta, weak)$ is the trace of $\mathcal{B}a(X, weak)$ on L_η, the measure $\hat{\eta}$ is an extension of η to $\mathcal{B}o(X, norm)$.

To see the uniqueness, consider another τ–smooth extension $\hat{\eta}_1$ of η to $\mathcal{B}o(X, norm)$. Since both measures are τ–smooth on $\mathcal{B}a(X, weak)$ and the σ–algebra of weakly Baire sets contains the cylinder basis of the weak topology of X, we have $\hat{\eta}_1(U) = \hat{\eta}(U)$ for each weakly open U. It is clear that a similar equality holds for all weakly closed sets, and so in particular for L_η. It follows, that the two extensions coincide on $\mathcal{B}o(X, norm)$. ◊

COROLLARY 2.4. *If X is a normed space and η is τ–smooth on $\mathcal{B}a(X, weak)$, then η admits a unique τ–smooth extension to $\mathcal{B}o(X, norm)$.*

Proof. Each τ–smooth measure has a linear support. ◊

COROLLARY 2.5. *If X is a Banach space, then each τ–smooth measure on $\mathcal{B}a(X, weak)$ is Radon on $\mathcal{B}a(X, weak)$ and admits a unique Radon extension to $\mathcal{B}o(X, norm)$.*

In particular, we get the following result:

COROLLARY 2.6. *Let X be a normed space. If η is τ–smooth on $\mathcal{B}o(X, weak)$, then there is a unique τ–smooth extension of η to $\mathcal{B}o(X, norm)$. If X is Banach, the extension is a Radon measure.*

DEFINITION 2.4. A completely regular topological space $(T, \mathcal{T})$ is called measure compact if each Baire measure on it is τ–smooth. It is called strongly measure compact if each Baire measure is Radon.

The main part of Corollary 2.5 can now be formulated in the following manner:

COROLLARY 2.5'. (Tortrat) *If X is a Banach space, then $(X, weak)$ is measure compact if and only if it is strongly measure compact.*

To have an idea what spaces are measure compact, we prove the following

THEOREM 2.3. *Let X be a Banach space. If $(X, weak)$ is Lindelöf, then it is measure compact. In particular weakly compactly generated (WCG) Banach spaces are measure compact.*

Proof. Let $(X, weak)$ be Lindelöf. If $Z_\alpha \searrow \emptyset$ and Z_α are zero sets, then there is an increasing sequence (α_n) such that $\bigcap_{n=1}^\infty Z_{\alpha_n} = \emptyset$. Hence, $\eta(Z_\alpha) \to 0$, where η is a weak Baire measure on X.

Assume now, that X is WCG and take a weakly compact set that is linearly dense in X. Then, there exists an increasing sequence (W_n) of weakly compact sets such, that $\bigcup_{n=1}^\infty W_n$ is dense in $B(X)$. For a sequence $\sigma \in \mathbf{N}^{\mathbf{N}}$ and $k \in \mathbf{N}$ let

$$A_\sigma^k = \bigcap_{n \le k} (W_{\sigma(n)} + 2^{-n} B(X^{**}))$$

$(\sigma(n)$ is the n–th coordinate of $\sigma)$. and

$$A_\sigma = \bigcap_{k=1}^\infty A_\sigma^k$$

As X is a closed subspace of X^{**}, we have $A_\sigma \subset X$. Moreover, since W_n and $B(X^{**})$ are weak*–compact, we get the weak*–compactness of A_σ^k and the weak compactness of A_σ. The density of $\bigcup_{n=1}^\infty W_n$ in $B(X)$ yields the inclusion $B(X) \subset U_\sigma A_\sigma$.

Take now a family $\mathcal{U}$ of weak*–open subsets of $B(X^{**})$ covering $B(X)$. Assuming, that $\mathcal{U}$ is stable under finite unions, we can find for each σ an element U_σ of $\mathcal{U}_\sigma$ such that $A_\sigma \subset U_\sigma$. Since $B(X^{**})\backslash U_\sigma$ is weak*–compact, there is $k(\sigma) \in N$ with $A_\sigma^{k(\sigma)} \subset U_\sigma$. Indeed, if $A_\sigma^k \backslash U_\sigma \neq \emptyset$ for each $k \in N$, then the convergence $A_\sigma^k \searrow A_\sigma$ yields the finite intersection property of $\{A_\sigma^k : k \in N\}$, and then we would have $A_\sigma \backslash U_\sigma \neq \emptyset$, contradicting the choice of U_σ. Since there are only countably many sets of type $A_\sigma^{k(\sigma)}$, $B(X)$ is covered by countably many elements of $\mathcal{U}$. This proves, that $(X, weak)$ is Lindelöf, and so it is measure compact too. $\diamondsuit$

3. Measurable functions.

Three forms of measurability – strong, weak and weak* – form the core in this section.

DEFINITION 3.1. A function $f : \Omega \to X$ is called *simple* if there exist $x_1, \ldots, x_n$ in X and $E_1, \ldots, E_n \in \Sigma$ such that

$$f = \sum_{i=1}^{n} x_i \chi_{E_i} .$$

A function $f : \Omega \to X$ is called *strongly μ–measurable* if there exists a sequence of simple functions $f_n : \Omega \to X$ with $\lim_n \|f_n(\omega) - f(\omega)\| = 0$ μ – a.e..

DEFINITION 3.2. Let Γ be a linear total subset of X^*. A function $f : \Omega \to X$ is said to be Γ-μ–measurable, if $x^* f$ is μ–measurable for each $x^* \in \Gamma$. If $\Gamma = X^*$, then f is called *weakly μ–measurable*. If $X = Y^*$ and $\Gamma = Y$ then f is called *weak*–μ–measurable.*

If μ is fixed the reference to it will be supressed. This will concern also all further definitions.

The following theorem explains the relationship between the strong and weak measurability.

THEOREM 3.1. (Pettis' measurability theorem) *A function $f : \Omega \to X$ is strongly μ–measurable if and only if*

(i) *f is weakly μ–measurable, and*
(ii) *f is μ–essentially separably valued, i.e. there exists $E \in \mathcal{N}(\mu)$ such that $f(\Omega \backslash E)$ is a separable subset of X.*

Proof. Assume that f is strongly μ–measurable and $W = \overline{\text{lin}} \bigcup_{n=1}^{\infty} f_n(\Omega \backslash N)$, where $N \in \mathcal{N}(\mu)$ is such that $f_n \to f$ poitwise on $\Omega \backslash N$. Then, $f(\Omega \backslash N) \subset \overline{\bigcup_{n=1}^{\infty} f_n(\Omega \backslash N)} \subset W$ and so we get (ii).

To prove the necessity of (i), note that $f_n(\omega) \to f(\omega)$ for almost all $\omega \in \Omega$ guarantees that $x^* f_n \to x^* f$ for almost all $\omega \in \Omega$ too. Since each $x^* f_n$ is simple, $x^* f$ is measurable.

To prove the converse, observe first that we may assume X to be separable. Let $D = (x_n)$ be a countable dense subset of X and $B_r(x)$ be the ball in X with center at x and of radius r. With the help of the Hahn–Banach theorem choose for each x_n a functional $x_n^* \in X^*$ such that $||x_n^*|| = 1$ and $x_n^*(x_n) = ||x_n||$. It easily follows that $||f(\omega)|| = \sup_n |x_n^* f(\omega)|$ for each $\omega \in \Omega$. Therefore the function $||f(\cdot)||$ is μ-measurable. By the same argument the functions $||f - x_n||$ are measurable. In particular $f^{-1}(B_r(x)) \in \Sigma$ for each $x \in X$ and positive r. Fix $r > 0$. Since $X = \bigcup_{n=1}^{\infty} B_r(x_n)$, there is $n_r \in \mathbf{N}$ with $\mu(\Omega \backslash \bigcup_{i \le n_r} f^{-1}[B_r(x_n)]) < 2^{-1/r}$. The simple function $g_r : \Omega \to X$ defined by

$$g_r(\omega) = \begin{cases} x_i & \text{if } \omega \in f^{-1}[B_r(x_i)] \backslash \bigcup_{j<i\le n_r} f^{-1}[B_r(x_j)] \\ 0 & \text{otherwise} \end{cases}$$

satisfies the inequality $||f(\omega) - g_r(\omega)|| \le r$ for all $\omega \in E_r = \bigcup_{i \le n_r} f^{-1}\,[B_r(x_i)]$. If $E = \liminf_n E_{1/n}$, then $\mu(\Omega \backslash E) = 0$ and $\lim g_{1/n} = f$ on E. ◇

DEFINITION 3.3. We say that two Γ-μ–measurable functions $f, g : \Omega \to X$ are Γ-*μ–equivalent* if $x^* f = x^* g$ μ–a.e. for each $x^* \in \Gamma$. If $\Gamma = X^*$, we say about weakly–*μ–equivalent* functions and, if $X = Y^*$ and $\Gamma = Y$ then f and g are said to be weak*-*μ–equivalent*. Two strongly measurable functions f and g are *μ–equivalent* if $f = g$ μ– a.e.

EXAMPLE 3.1. *A weakly measurable function that is not strongly measurable but is weakly equivalent to a strongly measurable function.* Let $V \subset [0,1]$ be an uncountable set and let $\{e_t : t \in [0,1]\}$ be the canonical basis for the nonseparable Hilbert space $l_2([0,1))$. Define $f_V : [0,1] \to l_2([0,1])$ by $f_V(t) = e_t$ whenever $t \in V$ and $f_V(t) = 0$ otherwise. It is a consequence of the Riesz Representation Theorem that $x^*f = 0$ λ–a.e. for each $x^* \in l_2([0,1])^*$ (i.e. f is weakly λ–equivalent to the zero function). On the other hand, if $E \subset [0,1]$ is such that $V \backslash E$ is uncountable then $f_V(V \backslash E)$ is nonseparable. Therefore, if $\lambda^*(V) > 0$ then f_V is not essentially separably valued, and so – in virtue of Theorem 3.1 – f_V is not strongly λ–measurable. If V is taken to be a non–measurable set (e.g. the Vitali set) then $||f_V(t)|| = \chi_V(t)$ and so the function $||f_V|| : t \to ||f_V(t)||$ is not measurable either. ◇

The fact that a weakly measurable function may have nonmeasurable norm causes a lot of troubles in the theory of weakly measurable functions.

EXAMPLE 3.2. (Ryll–Nardzewski) *A weak*–measurable function that is not weakly measurable and not weak*–equivalent to any weakly measurable function.* Define f: [0,1] $\to C^*[0,1]$ by $f(s) = \delta_s$. f is obviously weak*–λ–measurable, since for $y \in C[0,1]$, we have $\langle y, f(s)\rangle = y(s)$. To see that f is not weakly measurable denote by η_a the atomic part of $\eta \in C^*[0,1]$, and let V be a non–λ–measurable subset of [0,1]. Define $x^{**} \in C^{**}$ [0,1] by $x^{**}(\eta) = \eta_a(V)$. Since $x^{**}(f) = \chi_V$ the function f is not weakly measurable with respect to λ.

Since $C[0,1]$ is separable, f is not weak*–equivalent to any weakly (hence also strongly) measurable function.

It is worth to notice that the norm of f is a measurable function. ◇

EXAMPLE 3.3. (Hagler) *A weakly measurable function that is not weakly equivalent to a strongly measurable one.* Let (A_n) be a sequence of nonempty subintervals of [0,1], such that:

(i) $A_1 = [0,1]$

(ii) $A_n = A_{2n} \cup A_{2n+1}$ for each $n \in \mathbb{N}$,

(iii) $A_i \cap A_j = 0$ if $i \neq j$ and $2^n \leq i,j \leq 2^{n+1}$,

(v) $\lim_n \lambda(A_n) = 0$.

Define $f : [0,1] \to l_\infty$ by $f(t) = (\chi_{A_n}(t))$ for $t \in [0,1]$ and take any $\eta \in l_\infty^*$. Let η_1 be its countably additive part and η_2 be the purely finitely additive part of η. η_1 is given by $\eta_1(E) = \sum_{n\in E} \eta(\{n\})$ and so

$$\eta(f(t)) = \eta_1(f(t)) + \eta_2(f(t)) = \sum_{n=1}^{\infty} \chi_{A_n}(t)\eta_1(\{n\}) + \eta_2(f(t)) .$$

$\eta_1 f$ is obviously measurable. To prove the measurability of $\eta_2 f$ it is enough to show that $\sum_{t\in[0,1]} |\eta_2 f(t)| < \infty$, since this yields $\eta_2 f = 0$ λ – a.e.. To do it, take arbitrary distinct points $t_1, \ldots, t_k$ in [0,1] and let $B_i = \{n \in \mathbf{N} : t_i \in A_n\}$, $i = 1, \ldots, k$. Clearly, $f(t_i) = \chi_{B_i}$. It follows from the properties of the sequence (A_n) that there is an m such that the sets $B_i \cap \{m, m+1, \ldots\}$, $i = 1, \ldots, k$; are pairwise disjoint. Since η_2 vanishes on finite sets we get the following inequalities:

$$\infty > \|\eta_2\| = |\eta_2|(\Omega) \geq \sum_{i=1}^{k} |\eta_2(B_i \cap \{m, m+1, \ldots\})| =$$

$$= \sum_{i=1}^{k} |\eta_2(B_i)| = \sum_{i=1}^{k} |\eta_2(f(t_i))| .$$

This proves the weak measurability of f.

We have to prove yet that f is not weakly equaivalent to a strongly measurable function. Since l_1 is separable it is enough to show that f itself is not strongly measurable. This follows from Pettis' Meaurability Theorem. Indeed, if $\lambda(E) > 0$ and t, s are two distinct points of E then there is n such that $t \in A_n$ but $s \notin A_n$. Hence $\|f(t) = f(s)\| = 1$ ◊

The above example suggests the question about characterization of those Banach spaces which have the property that each weakly measurable function is weakly equivalent to a strongly measurable one. Such a characterization was discovered by Edgar [E]. Let us consider first a single function.

THEOREM 3.2. *Let $f : \Omega \to X$ be a weakly measurable function. Then, f is weakly equivalent to a strongly measurable function if and only if the image measure $f(\mu) : \mathcal{B}a(X, weak) \to \mathbf{R}$ is Radon.*

Proof. Assume, that f is weakly equivalent to a strongly measurable $h : \Omega \to X$. Let Y be a separable and closed subspace of X ·

with $\mu h^{-1}(Y) = 1$ and let $\eta : \mathcal{B}o(Y, norm) \to [0,1]$ be given by $\eta(A) = \mu h^{-1}(A)$. Being a measure on a Polish space, η is Radon, so that given $\varepsilon > 0$ and Borel $A \subseteq Y$ there is a norm–compact set $K \subseteq A$ such that $\eta(K) \geq \eta(A) - \varepsilon$. Since f and g are weakly equivalent, one can prove with the help of the Stone–Weierstrass theorem, that $\varphi \circ f = \varphi \circ h$ μ–a.e. for each $\varphi : X \to \mathbf{R}$, that is measurable with respect to the σ–algebra of weakly Baire subsets of X. Hence, if $B \in \mathcal{B}a\,(X, \text{weak})$ and $B \supseteq K$, then

$$f(\mu)(B) = \int_\Omega \chi_B(f(\omega))\mu(d\omega) =$$

$$\int_\Omega \chi_B(h(\omega))\mu(d\omega) = \eta(B) \geq \mu(A) - \varepsilon\,.$$

Thus, $f(\mu)^*(K) \geq \mu(A) - \varepsilon$. Since K is weakly compact, $f(\mu)$ is Radon.

Conversely, assume that $f(\mu)$ is Radon on $\mathcal{B}a(X, \text{weak})$. According to Theorem 2.2, there is a Radon extension η of $f(\mu)$ to $\mathcal{B}o(X, \text{norm})$, and so there is a separable closed subspace Y of X with $\eta(Y) = 1$. If Σ/μ is the measure algebra of (Ω, Σ, μ) then define $\varphi : \mathcal{B}o(Y, \text{norm}) \to \Sigma/\mu$ by setting $\varphi(B) = [f^{-1}(B')]_\mu$, where $[C]_\mu$ is the μ–equivalence class of C, and $B' \in \mathcal{B}a(X, \text{weak})$ is such that $B = Y \cap B'$. Since $\eta(Y) = 1$, φ is well defined. Moreover, φ is a Boolean σ–homomorphism of $\mathcal{B}o(Y, \text{norm})$ into Σ/μ.

It follows from Sikorski's point-mapping theorem [Si, 32.5], that there exists a function $g : \Omega \to Y$ such that $g^{-1}\,\mathcal{B}o(Y, norm) \subset \Sigma$ and $[g^{-1}(A)]_\mu = \varphi(A)$ for all norm–Borel subsets of Y.

In particular, if $A \in \mathcal{B}a(X, \text{weak})$ then

$$[g^{-1}(A)]_\mu = [g^{-1}(A \cap Y)]_\mu = \varphi(A \cap Y) = [f^{-1}(A)]_\mu$$

and so f and g are weakly equivalent. ◊

As a direct consequence of the above theorem, we obtain

THEOREM 3.3 *Let X be a Banach space. Given any (Ω, Σ, μ), each weakly measurable $f : \Omega \to X$ is weakly equivalent to a strongly measurable function if and only if $(X, weak)$ is measure compact.*

Proof. If (X, weak) is measure compact, then it is strongly measure compact by Corollary 2.5'. Hence, Theorem 3.2 yields the equivalence of every weakly measurable X–valued function to a strongly measurable function.

Conversely, take now a measure η on $\mathcal{B}a(X, \text{weak})$. Then the identity function $f(x) = x$ is – by the assumption – weakly equivalent to a strongly measurable function, and so $f(\eta) = \eta$ is Radon, by Theorem 3.2. $\diamond$

DEFINITION 3.4. A function $f : \Omega \to X$ is Γ*–scalarly* μ–bounded provided there is $M > 0$ such that for each $x^* \in \Gamma$ the inequality $|x^*f| \leq M\|x^*\|$ holds μ–a.e. If $\Gamma = X^*$ then we say about *scalarly* μ–bounded function, and in the case of $X = Y^*$ and $\Gamma = Y$ – about weak*–*scalarly* μ–bounded function.

An easy calculation proves that if $f : \Omega \to X$ is strongly measurable and scalarly bounded, then it is bounded (i.e. there is $M > 0$ such that $\sup\{\|f(\omega)\| : \omega \in \Omega\} \leq M \mu$–a.e.).

The following fact permits often to reduce the general situation to the case of scalarly bounded functions.

PROPOSITION 3.1. *If $f : \Omega \to X$ is Γ–measurable then there exists a non–negative measurable function φ_f^Γ with the following properties:*

(i) For each $x^* \in \Gamma$ we have $|\langle x^*, f(\omega)\rangle| \leq \varphi_f^\Gamma(\omega)\|x^*\|$ μ–a.e.,

(ii) $\varphi_f^\Gamma(\omega) \leq \|f(\omega)\|_\Gamma$ $(= \sup\{|\langle x^*, f(\omega)\rangle| : x^* \in \Gamma \cap B(X^*)\})$ μ–a.e.,

(iii) If $\phi : \Omega \to [0, \infty)$ is a measurable function satisfying (i) and (ii) (with φ_f^Γ replaced by ϕ), then $\varphi_f^\Gamma \leq \phi$ μ–a.e..

Proof. Consider the set $\Omega \times \mathbf{R}$ endowed with the σ–algebra $\sigma(\Sigma \times \mathcal{L})$, and the product measure $\mu \times \kappa$, where κ is any probability measure on $\mathcal{L}$ such that $\mathcal{N}(\kappa) = \mathcal{N}(\lambda)$. Let $S(x^*) = \{(\omega, s) : |\langle x^*, f(\omega)\rangle| \geq s\|x^*\|\}$ for $x^* \in \Gamma$, and let $a = \sup\{(\mu \times \kappa)[\bigcup_{n=1}^\infty S(x_n^*)] : x_n^* \in \Gamma \cap B(X^*),\, n \in \mathbf{N}\}$. Since $a < \infty$ there are $x_1^*, x_2^*, \ldots \in \Gamma \cap B(X^*)$ such that $a = (\mu \times \kappa)[\bigcup_{n=1}^\infty S(x_n^*)]$. Now, it is enough to put $\varphi_f^\Gamma = \sup_n |x_n^* f|$, where the supremum is taken pointwise. $\diamond$

COROLLARY 3.1. *If $f : \Omega \to X$ is Γ–measurable, then there* *exists a sequence of Γ-measurable and Γ-scalarly bounded functions $f_n : \Omega \to X$, such that $f = \sum_{n=1}^{\infty} f_n$ and the supports of $f_n - s$ are pairwise disjoint.*

It is easy to give an example of a weakly bounded function that is not bounded. It appears however that the situation can be more complicated.

EXAMPLE 3.4. (Edgar) *A scalarly bounded function which is not weakly equivalent to a bounded function.*

Let $T_n = \{0,1\}^n$ and $T = \bigcup_{n=1}^{\infty} T_n$. Let $\Omega = \{0,1\}^{\mathbf{N}}$. We consider Ω with the ordinary product σ-algebra and the Haar measure μ. For $\omega = (\omega_i) \in \Omega$, define $\omega|n \in T_n$ to be the sequence $(\omega_1, \dots, \omega_n)$ and a seminorm on $l_\infty(T)$ by $||x||_\omega = \limsup_n |x_{\omega|n}|$.

Let $(a_\omega)_{\omega\in\Omega}$ be a collection of numbers such that $a_\omega \geq 1$. For each $x \in l_\infty(T)$ define $N(x) = \sup_\omega(||x||, a_\omega||x||_\omega\}$ and let

$$X = \{x \in l_\infty(T) : N(x) < \infty\}$$

It is obvious that $c_0(T) \subset X$ and that $N(\cdot)$ coincides on $c_0(T)$ with the supremum norm.

Consider now a function $f : \Omega \to l_\infty(T)$ defined by

$$f(\omega) = (y_t))_{t\in T}$$

where $y_t = 1$ if $t = \omega|n$ for some $n \in \mathbf{N}$, and $y_t = 0$ otherwise. Since $N(f(\omega)) = a_\omega$ for all $\omega \in \Omega$, f takes its values in X. We have $l_\infty^*(T) = l_1(T) + c_0^\perp(T)$. If $x^* \in l_1(T)$ then clearly $x^* f$ is measurable. Take now $x^* \in c_0^\perp(T)$ with $||x^*|| \leq 1$ and consider n distinct points $\omega_1, \dots, \omega_n \in \Omega$.

Let $k \in \mathbf{N}$ be such that all elements $\omega_1|k, \dots, \omega_n|k$ are distinct. It follows that for each $i \leq n$ and $m \geq k$ there is exactly one $t_i \in T_m$ such that $y_{t_i}^i = 1$, where $f(\omega_i) = (y_t^i)_{t\in T}$. Thus, if $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1,1\}^n$, then for $t \in \bigcup_{m=k}^{\infty} T_m$ only one component in the sum $\sum_{i=1}^{n} \varepsilon_i y_t^i a_{\omega_i}^{-1}$ is non-zero.

Hence,

$$|\sum_{i=1}^{n} a_{\omega_i}^{-1} \varepsilon_i y_t^i| \leq 1 \text{ if } t \in \bigcup_{m=k}^{\infty} T_m .$$

It follows, that for $x^* \in c_0^{\perp}(T)$ we have

$$|x^*(\sum_{i=1}^{n} a_{\omega_i}^{-1}\varepsilon_i f(\omega_i))| \leq 1 .$$

Setting $\varepsilon_i = \operatorname{sgn} x^* f(\omega_i)$ we get the inequality

$$\sum_{i=1}^{n} a_{\omega_i}^{-1}|x^* f(\omega_i)| \leq 1 .$$

Hence,

$$\sum_{\omega\in\Omega} a_{\omega}^{-1}|x^* f(\omega)| \leq 1$$

and so $x^* f(\omega) \neq 0$ for at most countably many points. It follows that $x^* f$ is measurable. Observe, that X has a countable separating set of functionals (the evaluations at the points of T), so two scalarly equivalent functions are equal μ–a.e. If one chooses (a_ω) such that $\mu^*\{\omega : a_\omega \geq n\} = 1$ for every $n \in \mathbf{N}$, then clearly $\mu^*\{\omega \in \Omega : N(f(\omega)) \geq n\} = 1$ and f is not bounded. Since for $x^* \in c_0^{\perp}(T)$ we have $|x^* f| = 0$ μ–a.e. and for $x^* \in l_1(T)$ we have $|x^* f| \leq 1$ μ–a.e., f is scalarly bounded. ◇

In connection with the above example one can pose the following

PROBLEM 1. Which Banach spaces X have the property that for each (Ω, Σ, μ) and each scalarly bounded and weakly μ–measurable $f : \Omega \to X$ there exists a bounded weakly μ–measurable $g : \Omega \to X$ that is weakly equivalent to f?

PROBLEM 2. Suppose each weakly λ–measurable and scalarly λ–bounded $f : [0,1] \to X$ is weakly λ–equivalent to a bounded weakly λ–measurable g. Is it so for each (Ω, Σ, μ)?

It appears that weak *–scalarly bounded functions behave much better.

PROPOSITION 3.2. *If $f : \Omega \to X^*$ is weak*–scalarly bounded and weak *–scalarly measurable, then it is weak*–scalarly equivalent to a bounded function.*

Proof. Let ρ be a lifting on $L_\infty(\mu)$ and let $\rho(f) : \Omega \to X^*$ be defined by $\langle x, \rho(f)(\omega)\rangle = \rho(\langle x, f\rangle)(\omega)$ for all $x \in X$ and $\omega \in \Omega$ (cf [I-T] for details). Assume that for each $x \in X$ the inequality $|\langle x, f\rangle| \leq M\|x\|$ holds μ–a.e. Hence $|\langle x, \rho(f)\rangle| \leq M\|x\|$ everywhere. This shows that $\rho(f)(\omega) \in X^*$ and $\|\rho(f)(\omega)\| \leq M$ for all $\omega \in \Omega$. It follows from the properties of ρ that f and $\rho(f)$ are weak *–equivalent. ◇

PROBLEM 3. What is the largest set $\Gamma \subset X^*$ such that each Γ–scalarly bounded and Γ–measurable function is Γ–equivalent to a Γ–measurable bounded function? What is the set in the particular case of $X = Y^*$? Is it the union of all weak* closures in Y^{**} of the countable subsets of Y?

4. Pettis integral.

DEFINITION 4.1. Let Γ be a linear total subset of X^*. A function $f : \Omega \to X$ is Γ*–scalarly* μ*–integrable* if $x^*f \in L_1(\mu)$ for each $x^* \in \Gamma$. If $\Gamma = X^*$, then f is called *scalarly* μ*-integrable* and in the case of $f : \Omega \to X^*$ and $\Gamma = X \subseteq X^{**}$, the function f is said to be weak*–*scalarly* μ*–integrable.*

DEFINITION 4.2. A Γ–scalarly μ–integrable $f : \Omega \to X$ is Γ-μ–*integrable* if for each $E \in \Sigma$ there exists $\nu_f(E) \in X$ such that

$$x^*\nu_f(E) = \int_E x^* f d\mu$$

for each $x^* \in \Gamma$. The set function $\nu_f : \Sigma \to X$ is called the *indefinite* Γ*–integral* of f with respect to μ, and $\nu_f(E)$ is called the Γ*–integral* of f over $E \in \Sigma$ with respect to μ. An X^*–integrable function is called *Pettis* μ*–integrable* and an X–integrable function (if $f : \Omega \to X^*$ and $\Gamma = X$) is called *weak*–*μ*–integrable* (or Gelfand μ–integrable). If $f : \Omega \to X$ is considered as an X^{**}–valued function then its weak* integral in X^{**} is called the *Dunford integral.* It is clear that each Γ–integral is uniquely determined and it is an additive set function (provided it exists).

Sometimes, we shall use the following notations: $P - \int_E f d\mu$, weak*–$\int_E f\, d\mu$ and $D - \int_E f d\mu$.

If $f : \Omega \to X$ is scalarly μ–integrable, then $T_f : X^* \to L_1(\mu)$ will be defined by $T_f(x^*) = x^* f$.

It is one of the main problems in the theory of vector integration to find conditions guaranteeing the existence of the Pettis integral. We shall start with two well known results.

PROPOSITION 4.1. (Gelfand) *Each weak *–scalarly μ–integrable $f : \Omega \to X^*$ is weak*–μ–integrable.*

Proof. For a fixed $E \in \Sigma$ define $T : X \to L_1(\mu)$ by $T(x) = \langle x, f\rangle \chi_E$. It is easily seen that T has closed graph and hence – in virtue of Banach's Closed Graph Theorem – T is continuous. Thus,

$$|\int_E \langle x, f\rangle d\mu| \leq \int_E |\langle x, f\rangle| d\mu = ||T(x)|| \leq ||T||\, ||x||$$

and so the mapping $x \to \int_E \langle x, f\rangle d\mu$ is a continuous linear functional on X and defines and element $\nu_f(E) \in X^*$ satisfying the required in the definition equality. ◇

As an immediate corollary we get the following fact

PROPOSITION 4.2. (Dunford) *Each scalarly μ–integrable function $f : \Omega \to X$ is Dunford μ–integrable.*

If X is reflexive then the Dunford and Pettis integrals coincide. When X is not reflexive, this may not be the case.

EXAMPLE 4.1. *A Dunford integrable function that is not Pettis integrable.* Define $f : (0, 1] \to c_0$ by

$$f(t) = (2\chi_{(2^{-1},1]}(t), 2^2\chi_{(2^{-2},2^{-1}]}(t), \ldots, 2^n\chi_{(2^{-n},2^{-n+1}]}(t), \ldots)\,.$$

If $x^* = (\alpha_1, \alpha_2, \ldots) \in l_1 = c^*$, then

$$x^* f = \sum_{n=1}^{\infty} \alpha_n 2^n \chi_{(2^{-n},2^{-n+1}]} \quad \text{and} \quad \int_0^1 |x^* f| d\lambda \leq \sum_{n=1}^{\infty} |\alpha_n| < \infty\,.$$

It follows from Proposition 4.2, that f is Dunford λ–integrable. On the other hand, it is easily seen that for each $E \in \mathcal{L}$

$$D-\int_E f\,d\lambda = \{2\lambda(E \cap (2^{-1},1]),\ldots,2^n\lambda(E \cap (2^{-n},2^{-n+1}]),\ldots\}.$$

In particular

$$D-\int_{(0,1]} f d\lambda = (1,1,1,\ldots,1,\ldots) \notin c_0$$

and so f is not λ–Pettis integrable. ◇

It is clear that two Γ-equivalent functions are either both Γ-integrable or none of them is. In particular the function f_V considered in Example 3.1 is Pettis integrable with $\nu_{f_0}(E) = 0$ for each $E \in \Sigma$.

But there are also non–trivial examples of Pettis integrable functions.

EXAMPLE 4.2. Let f be the function considered in Example 3.3. Since $\|f(t)\| \leq 1$ everywhere, f is Dunford integrable and for each $E \in \mathcal{L}$

$$\int_E \eta(f)\,d\lambda = \int_E \eta_1(f)\,d\lambda + \int_E \eta_2(f)\,d\lambda = \int_E \sum_{n=1}^{\infty} \chi_{A_n}\eta_1(\{n\})\,d\lambda =$$

$$= \sum_{n=1}^{\infty} \eta_1(\{n\})\lambda(E \cap A_n) = \langle \eta_1, (\lambda(E \cap A_n))\rangle .$$

The last equality follows from the fact that $\lim_n \lambda(A_n) = 0$ and so $\nu(E) = (\lambda(E \cap A_n)) \in c_0$. But η_2 considered as a functional on l_∞ belongs to $c_0^\perp$ and so $\eta_2\nu(E) = 0$ for each $E \in \Sigma$. It follows that

$$\int_E \eta(f)\,d\lambda = \langle \eta, (\lambda(E \cap A_n))\rangle$$

and so f is Pettis λ–integrable. ◇

It follows directly from the definition of the Γ–integrability that the Γ–integral is a Γ–measure. If $\Gamma = X^*$ then much more can be proved.

THEOREM 4.1. *If f is Pettis μ–integrable, then ν_f is a μ–continuous measure of σ–finite variation. Moreover $|\nu_f|(E) = \int_E \varphi_f d\mu$ for each $E \in \Sigma$ (we put here φ_f instead of $\varphi_f^{X^*}$ for the simplicity).*

Proof. The fact that ν_f is an X–valued measure is a consequence of the Orlicz–Pettis theorem, since $x^*\nu_f(E) = \int_E \langle x^*, f\rangle d\lambda$ and so $x^*\nu_f$ is countably additive.

If $E \in \Sigma$, then for every $x^* \in X^*$

$$|\langle x^*, \nu_f(E)\rangle| \leq \int_E |\langle x^*, f\rangle d\mu \leq ||x^*|| \int_E \varphi_f d\mu \,.$$

Hence,

$$|\nu_f|(E) \leq \int_E \varphi_f d\mu$$

and so $|\nu_f|$ is a σ–finite measure.

If $\mu(E) = 0$ then obviously $|\nu_f|(E) = 0$, and so $\nu_f \ll \mu$.

By the classical Radon–Nikodym theorem there exists a non–negative measurable function h on Ω such that

$$|\nu_f|(E) = \int_E h \, d\mu$$

for each $E \in \Sigma$.

The inequality $|\nu_f|(E) \leq \int_E \varphi_f d\mu$ yields now the relation $h \leq \varphi_f$ μ–a.e. If $||x^*|| \leq 1$, then

$$|\langle x^*, \nu_f\rangle|(E) = \int_E |\langle x^*, f\rangle d\mu \leq |\nu_f|(E) = \int_E h d\mu \,.$$

Hence $|\langle x^*, f\rangle| \leq h$ μ–a.e. It follows from the properties of φ_f that $\varphi_f \leq h$ μ–a.e. and this completes the proof. ◊

REMARK 4.1. It can be easily seen that if $f : \Omega \to X$ is strongly measurable and Pettis integrable then $\varphi_f = ||f||$.

PROBLEM 4. Is it true that for each Pettis μ–integrable f there is a Pettis μ–integrable g such that f and g are weakly μ–equivalent and $||g||$

is measurable? Which X or μ have such a property? Which $C(K)$ have such a property?

Observe however, that even if f and g are weakly μ–equivalent and $\|g\|$ is measurable, the equality $\varphi_f = \|g\|$ may fail. So one should look rather for g satisfying this equality.

REMARK 4.2. A result similar to that in Theorem 4.1 for an arbitrary total $\Gamma \subseteq X^*$ is false. If Γ is norming, (i.e. for each $x \in X$ the equality $\|x\| = \sup\{|\langle x^*, x\rangle| : \|x^*\| \leq 1, x^* \in \Gamma\}$ holds) and f is Γ–integrable, then the above proof shows that $|\nu_f|$ is a σ–finite measure and $|\nu_f|(E) = \int_E \varphi_f^\Gamma d\mu$ for all $E \in \Sigma$, but it may happen that ν_f is not countably additive in the norm topology of X, and it is not μ–continuous.

EXAMPLE 4.3. (Diestel, Faires) *A weak*–integrable f with noncountably additive ν_f.* Let $\Omega = \mathbf{N}$ and let Σ be the σ–algebra of all subsets of $\mathbf{N}$. Put $\nu(A) = \chi_A \in l_\infty$ and $\Gamma = l_1$. Then ν is a weak*–measure of σ–finite variation that is not countably additive in the norm topology of l_∞. Let μ be a finite measure on Σ given by $\mu(\{n\}) = 2^{-n}$ and $f : \mathbf{N} \to l_\infty$ be given by $f(n) = 2^n e_n$, where (e_n) is the canonical basis of c_0. If $x = (x_n) \in l_1$ and $E \in \Sigma$, then

$$\int_E \langle x, f\rangle d\mu = \sum_{n\in E} 2^n x_n \mu(\{n\}) = \sum_{n\in E} x_n = \langle x, \nu(E)\rangle$$

and so f is weak* integrable and ν is its weak* integral. It is clear that ν–considered as a finitely additive set function is not μ–continuous. It is however scalarly μ–continuous as a weak *–measure. $\diamondsuit$

EXAMPLE 4.4. For the function presented in Example 4.1 $\nu_f : \Sigma \to l_\infty$ is also weak*–countably additive but not countably additive in the norm topology and not μ–continuous either.

The following result explains the special role of l_∞ in the above two examples.

THEOREM 4.2. *The following statements concerning X are equivalent:*

(i) *X^* does not contain any isomorphic copy of l_∞,*

(ii) *Given any (Ω,Σ,μ) and any weak*–scalarly μ integrable $f:\Omega\to X^*$, $\nu_f:\Sigma\to X^*$ is a measure in the norm topology of X^*.*

Proof. As in the proof of Proposition 4.1 we see that $T_f:X\to L_1(\mu)$ defined by $T_f(x)=xf$ is continuous with respect to the norm topologies of X and $L_1(\mu)$ respectively. Moreover, $\nu_f(E)=T_f^*(\chi_E)$ for each $E\in\Sigma$. Hence $\nu_f(\Sigma)$ is a bounded subset of X^*. In particular, $x^{**}\nu_f$ is a bounded additive scalar–valued set function for each $x^{**}\in X^{**}$, and so the series $\sum_{n=1}^\infty x^{**}\nu_f(A_n)$ is absolutely convergent for each sequence (A_n) of pairwise disjoint elements of Σ. Since X^* does not contain any isomorphic copy of l_∞, it follows from a result of Bessaga and Pelczynski [B-P] that X^* does not contain any isomorphic copy of c_0. Hence, the series $\sum_{n=1}^\infty \nu_f(A_n)$ converges in X^*. The totality of X over X^* yields the equality $\nu_f(\bigcup_{n=1}^\infty A_n)=\sum_{n=1}^\infty \nu_f(A_n)$. Thus ν_f is a measure in X^*.

If $l_\infty\subseteq X^*$, then the set function ν_f considered in Example 4.4 is an example of a non–countably additive weak*–integral. ◊

It is very surprising that the weak* integral ν_f in *(ii)* may be replaced by any weak* measure (in particular such a measure may be of non–σ–finite variation [Th]).

From the integral point of view, the functions with the same indefinite Pettis integrals are non–distinguishable – they are weakly equivalent. We shall denote by $\mathbf{P}(\mu,X)$ (or by $\mathbf{P}((\Omega,\Sigma,\nu),X)$ if necessary) the space of classes of weakly μ–equivalent Pettis μ–integrable X–valued functions. It is a linear space with ordinary algebraic operations. One can define a norm on $\mathbf{P}(\mu,X)$ by

$$\| f \| = \sup\{\int_\Omega |\langle x^*,f\rangle|d\mu : x^*\in B(X^*)\}$$

An easy calculation shows that

$$\sup\{\|\int_E f d\mu\| : E\in\Sigma\}$$

defines an equivalent norm on $\mathbf{P}(\mu,X)$:

$$\sup\{\|\nu_f(E)\| : E\in\Sigma\}=\sup\{\sup[\,|x^*\nu_f(E)|: x^*\in B(X^*)] : E\in\Sigma\}$$

$$\leq \sup\{|x^*\nu_f|(\Omega) : x^* \in B(X^*)\} = \| f \| .$$

If $\pi = \{E_i, \dots, E_n\}$ is a partition of Ω into pairwise disjoint members of Σ and $x^* \in B(X^*)$, then

$$\sum_{E_i \in \pi} |x^*\nu_f(E_i)| = \sum_{E_i \in \pi^+} x^*\nu_f(E_i) - \sum_{E_i \in \pi^-} x^*\nu_f(E_i) =$$

$$= x^*\{\sum_{E_i \in \pi^+} \nu_f(E_i)\} - x^*\{\sum_{E_i \in \pi^-} \nu_f(E_i)\} \leq$$

$$\leq 2 \sup(\|\nu(E)\| : E \in \Sigma)$$

where $\pi^+ = \{E_i : x^*\nu_f(E_i) \geq 0\}$ and $\pi^- = \{E_i : x^*\nu_f(E_i) < 0\}$.

Hence

$$\| f \| \leq 2 \sup(\{\|\nu_f(E)\| : E \in \Sigma\} \qquad \diamond$$

It has been shown by Thomas [Th], that if μ is not purely atomic and X is infinite dimensional, then $\mathbf{P}(\mu, X)$ is non–complete.

We shall finish this section with a classical result that is always a starting point, when one wants to find conditions guaranteeing the Pettis integrability of a function.

THEOREM 4.3. *Let $f : \Omega \to X$ be scalarly integrable. Then the following are equivalent:*
(a) f is Pettis μ–integrable.
(b) $T_f : X^* \to L_1(\mu)$ is weak–weakly continuous.*

Proof. $(a \Rightarrow b)$ By the assumption

$$x^*\nu_f(E) = \int_E x^* f d\mu = \langle T_f x^*, \chi_E \rangle$$

for each $E \in \Sigma$ and $x^* \in X^*$. It follows that $\langle T_f(\cdot), \chi_E \rangle$ is weak* continuous on X^* and hence for each simple $h \in L_\infty(\mu)$ the function $\langle T_f(\cdot), h \rangle$ is weak* continuous. Take now an arbitrary $g \in L_\infty(\mu)$, a net $(x^*_\alpha) \subset B(X^*)$ weak* convergent to a point x^* and a positive ε. Then take a simple $h \in L_\infty(\mu)$ such that $\|g - h\|_{L_\infty(\mu)} < \varepsilon$. By the weak* continuity of $\langle T_f(\cdot), h\rangle$ there exists α_0 such that $|\langle T_f(x_\alpha^*), h\rangle - \langle T_f(x^*), h\rangle| \le \varepsilon$ for each $\alpha \ge \alpha_0$. Hence, for $\alpha \ge \alpha_0$ we have

$$|\langle T_f(x_\alpha^*), g\rangle - \langle T_f(x^*), g\rangle \le |\langle T_f(x_\alpha^*), g\rangle - \langle T_f(x_\alpha^*), h\rangle| +$$

$$+|\langle T_f(x_\alpha^*), h\rangle - \langle T_f(x^*), h\rangle| + |\langle T_f(x^*), h\rangle - \langle T_f(x^*), g\rangle| \le$$

$$\le \varepsilon + \int_\Omega |x_\alpha^* f|\,|g-h|d\mu + \int_\Omega |x^* f|\,|g-h|\,|d\mu \le \varepsilon(1+2\,\|f\|)$$

This proves the weak* continuity of $\langle T_f(\cdot), g\rangle$.

($b \Rightarrow a$) According to Proposition 4.2. f is Dunford integrable. Fix $E \in \Sigma$ and let $\nu(E) \in X^{**}$ be the Dunford integral of f over E. To prove that $\nu(E) \in X$ one has to show that $\nu(E)$ is weak* continuous. But if $x_\alpha^* \to 0$ in the weak*–topology of X^*, then $T_f(x_\alpha^*) \to 0$ weakly in $L_1(\mu)$ and so

$$x_\alpha^*\nu(E) = \int_E x_\alpha^* f d\mu = \int_E T_f x_\alpha^* d_\mu \to 0 \qquad \diamond$$

REMARK 4.3. In condition (b) it is enough to consider the restriction of T_f to $B(X^*)$ (this fact is a consequence of the Banach–Dieudonne theorem, cf. [H, p 154]).

COROLLARY 4.1. *If $f : \Omega \to X$ is Pettis μ–integrable, then $T_f : X^* \to L_1(\mu)$ is weakly compact.*

The following corollary is an immediate consequence of Theorem 4.3.

COROLLARY 4.2. *$f \in \mathbf{P}(\mu, X)$ if and only if the set $\{x^* \in X^* : x^* f = 0\ \mu\text{–a.e.}\}$ is weak* closed.*

5. Integrability of strongly measurable functions.

It is the purpose of this section to describe these strongly measurable functions that are Pettis integrable. We shall start with the following simple lemma:

LEMMA 5.1 *If $f : \Omega \to X$ is strongly measurable then there exists a bounded strongly measurable function $g : \Omega \to X$ and a strongly measurable function $h : \Omega \to X$ of the form $\sum_{n=1}^{\infty} x_n \chi_{E_n}$ with pairwise disjoint $E_n \in \Sigma$, $n \in \mathbf{N}$, such that $f = g + h$.*

Proof. In virtue of the Pettis Measurability Theorem 3.1 we may assume that $f(\Omega)$ is a separable subset of X. Let (x_n) be at most countable dense subset of $f(\Omega)$. Let

$$E_n = \{\omega \in \Omega : f(\omega) \in [x_n + B(X)] \setminus \bigcup_{k=1}^{n-1} [x_k + B(X)]\}$$

and $h = \sum_{n=1}^{\infty} x_n \chi_{E_n}$. Then h is strongly measurable, at most countably valued and, $||f(\omega) - h(\omega)|| \leq 1$ for each $\omega \in \Omega$. It is enough to put $g = f - h$. ◊

PROPOSITION 5.1. *A strongly measurable function such that $\int_\Omega ||f(\omega)||\, d\mu < \infty$ is Pettis integrable.*

Proof. Assume that $f(\Omega)$ is separable and suppose that $f : \Omega \to X$ can be uniformly approximated by X–valued simple functions $f_n : \Omega \to X$, $n \in \mathbf{N}$. Thus, given $\varepsilon > 0$ there is $n_0 \in \mathbf{N}$ such that $||f_n(\omega) - f_m(\omega)|| \leq \varepsilon$ for each $\omega \in \Omega$ and all $m, n \geq n_0$.

In particular, if $E \in \Sigma$ then

$$|| \int_E f_n d\mu - \int_E f_m d\mu || \leq \varepsilon \mu(E)$$

for all $m, n \geq n_0$. It follows that the sequence $(\int_E f_n d\mu)$ is norm Cauchy. Hence it is convergent to an element $\nu(E) \in X$. Since at the same time we have $\lim_n \int_E x^* f_n d\mu = \int_E x^* f d\mu$, the equality $\int_E x^* f d\mu = x^* \nu(E)$ holds.

Assume now, that (f_n) is an arbitrary sequence of simple functions converging μ–a.e. to f and let (Ω_k) be an increasing sequence of measurable sets such that $\mu(\Omega \setminus \Omega_k) < 1/k$ and (f_n) is uniformly convergent on each Ω_k to f.

Define $\nu_k : \Sigma \to X$ by setting

$$\nu_k(E) = P - \int_{E\cap\Omega_k} f d\mu$$

The integrability of $||f||$ yields $\lim_k \int_{\Omega_k^c} ||f||\, d\mu = 0$ and so $(\nu_k(E))$ satisfies the Cauchy condition. Let $\nu(E) = \lim \nu_k(E)$. For a fixed $x^* \in X^*$ we have $\int_\Omega |x^* f| d\mu < \infty$ and so by the classical Lebesgue Convergence Theorem

$$x^*\nu(E) = \lim_k x^*\nu_k(E) = \lim_k \int_{E\cap\Omega_k} x^* f d\mu = \int_E x^* f d\mu \,.$$

Thus f is Pettis μ–integrable ◊

DEFINITION 5.1. A strongly measurable $f : \Omega \to X$ such that $\int_\Omega ||f|| d\mu < \infty$ is said to be *Bochner integrable.*

The next theorem gives a description of Bochner and Pettis integrable functions.

REMARK 5.1. In the terms of Remark 4.2 a strongly measurable f is Bochner integrable if and only if ν_f is of finite variation, whereas in the case of a Pettis integrable function ν_f can be of infinite (but σ–finite) variation.

THEOREM 5.1. (a) *A strongly measurable $f : \Omega \to X$ is Pettis μ–integrable if and only if there exists a bounded strongly measurable function g, a sequence (E_n) of pairwise disjoint members of Σ, and a sequence (x_n) of elements of X such that*

$$f = g + \sum_{n=1}^{\infty} x_n \chi_{E_n}$$

and the series $\sum_{n=1}^{\infty} x_n \mu(E_n)$ converges unconditionally. In this case

$$P - \int_E f d\mu = \sum_{n=1}^{\infty} x_n \mu(E \cap E_n) + P - \int_E g \, d\mu \,.$$

(b) A strongly measurable $f : \Omega \to X$ is Bochner integrable if and only if the elements g, x_n and E_n in (a) can be choosen in such a way that the series $\sum_{n=1}^{\infty} x_n\mu(E_n)$ is absolutely convergent. In such a case

$$B - \int_E f\, d\mu = \sum_{n=1}^{\infty} x_n\mu(E \cap E_n) + B - \int_E g\, d\mu\,.$$

Proof. (a) To prove the necessity of the conditions, assume the Pettis μ–integrability of f and let $f = g + h$ where g and h are as in Lemma 5.1. If $E \in \Sigma$ is an arbitrary set, then the σ–additivity of ν_h gives

$$\int_E h d\mu = \sum_{n=1}^{\infty} \int_{E\cap E_n} h d\mu = \sum_{n=1}^{\infty} x_n\mu(E \cap E_n)$$

and clearly the last series is unconditionally convergent.

To prove the sufficiency, assume for the simplicity that $\mu(E_n) > 0$ for all $n \in \mathrm{N}$. If $E \in \Sigma$, then

$$\sum_{n=1}^{\infty} x_n\mu(E \cap E_n) = \sum_{n=1}^{\infty} x_n\mu(E_n)\frac{\mu(E \cap E_n)}{\mu(E_n)}$$

and $\frac{\mu(E\cap E_n)}{\mu(E_n)} \le 1$ for all n. Hence the series $\sum_{n=1}^{\infty} x_n\mu(E \cap E_n)$ is unconditionally convergent. In particular, for each $x^* \in X^*$ the the inequality $\sum_{n=1}^{\infty} |x^*(x_n)|\mu(E \cap E_n) < \infty$ holds, and so x^*h is μ–integrable. Hence

$$x^*(\sum_{n=1}^{\infty} x_n\mu(E \cap E_n)) = \sum_{n=1}^{\infty} x^*(x_n)\mu(E \cap E_n) = \int_E x^*h\, d\mu$$

It follows that h is Pettis μ–integrable and so the same can be said about f.

(b) If f is Bochner μ–integrable then it is also Pettis μ–integrable. Since g is bounded, it is Bochner integrable too. Hence, we have $\int_\Omega \|g\| d\mu < \infty$ and $\int_\Omega \|\sum_{n=1}^{\infty} x_n\chi_{E_n}\| < \infty$. But $E_n - s$ are pairwise disjoint, so this means that $\sum_{n=1}^{\infty} \|x_n\|\mu(E_n) < \infty$.

In a similar way the inverse implication can be proved. ◇

COROLLARY 5.1. *Let $f : \Omega \to X$ be represented in the form $f = \sum_{n=1}^{\infty} x_n\chi_{E_n}$ with pairwise disjoint $E_n \in \Sigma, n \in \mathrm{N}$. Then:*

(i) *f is Pettis μ–integrable if and only if $\sum_{n=1}^{\infty} x_n\mu(E_n)$ is unconditionally convergent.*

(ii) *f is Bochner μ–integrable if and only if $\sum_{n=1}^{\infty} x_n\mu(E_n)$ is absolutely convergent.*
In the both cases $\int_E f d\mu = \sum_{n=1}^{\infty} x_n\mu(E \cap E_n)$.

The above corollary gives a possibility to formulate the following

REMARK 5.2. If X is an infinite dimensional Banach space, then there exists an X–valued strongly measurable function that is Pettis but not Bochner integrable. Indeed, Dvoretzky–Roger's theorem guarantees the existence of an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ such that $\sum_{n=1}^{\infty} \|x_n\| = \infty$. It is enough to take $\Omega = \mathrm{N}$, $\Sigma = \mathcal{P}(\mathrm{N})$ and to define μ by $\mu(\{n\}) = 2^{-n}$ for each n. The function $f : \mathrm{N} \to X$ given by $f(n) = 2^n x_n$ is suitable for our purpose. Observe, that the variation of ν_f is σ–finite but not finite.

The following result gives a necessary and sufficient condition for the Pettis integrability of strongly measurable function.

THEOREM 5.2. *A strongly measurable and scalarly integrable $f : \Omega \to X$ is Pettis μ–integrable if and only if the set $\{x^*f : x^* \in B(X^*)\}$ is relatively weakly compact in $L_1(\mu)$.*

Proof. If f is an arbitrary Pettis μ–integrable function then $T_f : X^* \to L_1(\mu)$ is weakly compact, as it has been shown in Corollary 4.1.

Assume now that T_f is weakly compact. Since f is strongly μ–measurable we may assume that X is separable. We have to show that T_f is weak*–weakly continuous. But as X is separable it is enough to prove the sequential weak*–weak continuity. To do it, take $x_n^* \to x^*$ weak * in $B(X^*)$. Then $x_n^*f \to x^*f$ pointwise and – by the assumption – (x_n^*f) is a weakly relatively compact subset of $L_1(\mu)$. If $(x_{n_k}^* f)$ is weakly convergent to $h \in L_1(\mu)$ then an appropriately choosen convex combinations of the functions $x_{n_k}^* f$, $k \in \mathrm{N}$, converge to h in the norm of $L_1(\mu)$ and μ–a.e. It follows that $h = x^*f$ μ–a.e. and $(x_{n_k}^* f)$ is convergent to x^*f weakly in

$L_1(\mu)$. Thus, each weakly convergent subsequence of $(x_n^* f)$ converges to $x^* f$ and so $x_n^* f \to x^* f$ weakly in $L_1(\mu)$. $\diamond$

To prove still another characterization of strongly meaurable elements of $\mathbf{P}(\mu, X)$ we need the following

PROPOSITION 5.2. (La Valée Pousin) *A set $W \subset L_1(\mu)$ is weakly relatively compact if and only if it is bounded and there exists an increasing convex function $\varphi : [0, \infty) \to [0, \infty)$ such that*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty \text{ and } \quad M = \sup_{f\in\Omega} \int_\Omega \varphi(|f(\omega)|)\mu(d\omega) < \infty .$$

Proof. Sufficiency. We have to show the uniform integrability of W. To do it let us fix $0 < \varepsilon < 1/2\,M$. Since $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$, there is $\tau > 0$ with $t < \varepsilon^2 \varphi(t)$ for all $t > \tau$.

Consider now for $E \in \Sigma$ and $f \in W$ the set

$$A_f = \{\omega \in E : |f(\omega)| \le \tau\} .$$

It follows, that

$$\int_{A_f} |f| d\mu \le \tau\mu(E) \quad \text{and} \quad \int_{E\setminus A_f} |f| d\mu \le \varepsilon^2 \int_{E\setminus A_f} \varphi(|f|)\, d\mu \le \varepsilon^2 M$$

Hence,

$$\int_E |f| d\mu \le \tau\mu(E) + \varepsilon/2$$

and so $\delta = \varepsilon/2\tau$ gives the inequality $\int_E |f| d\mu < \varepsilon$ for each $f \in W$ and $E \in \Sigma$ with $\mu(E) < \delta$.

Necessity. Let $\alpha = \sup(\int_\Omega |f| d\mu : f \in W\}$. Since W is uniformly integrable, there is a sequence (t_n) of positive numbers, such that $t_0 = 0$, $t_{n+1} > 2t_n$ for each $n \in N$ and $\int_E |f| d\mu < 2^{-n}$ for each $f \in W$, whenever $\mu(E) < \alpha/t_n$.

We define a function $\varphi : [0, \infty) \to [0, \infty)$ by setting

$$\varphi(t) = \frac{(n+1)t_{n+1} - nt_n}{t_{n+1} - t_n}(t - t_n) + nt_n$$

whenever $t \in [t_n, t_{n+1}]$ and $n \in \{0, 1, \ldots\}$.

One can easily see, that $n \leq \frac{\varphi(t)}{t} < n+1$ for $t \in [t_n, t_{n+1})$ and so $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$. Moreover, φ is increasing and convex (this is the place where the relation $t_{n+1} > 2t_n$ is important).

Take now $f \in W$ and put $A_n = \{\omega \in \Omega : |f(\omega)| \in [t_n, t_{n+1})\}$. We have $\alpha \geq \int_{A_n} |f| d\mu \geq t_n \mu(A_n)$ and so by the uniform integrability

$$\int_\Omega \varphi(|f|)\, d\mu = \sum_{n=0}^{\infty} \int_{A_n} \varphi(|f|)\, d\mu$$

$$\leq \sum_{n=0}^{\infty} (n+1) \int_{A_n} |f| d\mu \leq \sum_{n=0}^{\infty} (n+1) 2^{-n} \qquad \diamond$$

DEFINITION 5.2. A Young's function is an increasing convex function $\Phi : [0, \infty) \to [0, \infty)$ such that $\Phi(0) = 0$. For such Φ one defines the Orlicz space L^Φ to be the space of all equivalence classes of real–valued functions f such that $\int_\Omega \Phi(k|f|)\, d\mu < \infty$ for some $k > 0$ (depending on f).

It is known, that L^Φ can be equipped with the norm

$$\|f\|_\Phi = \sup \left\{ \int_\Omega f(\omega) g(\omega) \mu(d\omega) : \int_\Omega \Psi(|g(\omega)|)\; \mu(d\omega) \leq 1 \right\}$$

where Ψ is the function complementary to Φ in the sense of Young (cf. [Z]).

Putting together Theorem 5.2 and Proposition 5.2 we obtain now

THEOREM 5.3. *If f is strongly measurable then $f \in \mathbf{P}(\mu, X)$ if and only if there exists a Young's function Φ such that $x^*f \in L^\Phi(\mu)$ for each $x^* \in X^*$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$.*

As a very particular case we get the following:

COROLLARY 5.2. *Let $f : \Omega \to X$ be strongly μ–measurable. If there is $p > 1$ such that $x^*f \in L_p(\mu)$ for each $x^* \in X^*$, then $f \in \mathbf{P}(\mu, X)$.*

Consider now the problem of the integrability of all scalarly integrable and strongly measurable X–valued functions. It appears that c_0 is the only distinguished space.

THEOREM 5.4. *If X does not contain any isomorphic copy of c_0, then each strongly measurable and scalarly integrable X–valued function is Pettis integrable.*

Proof. In view of Theorem 5.1 it suffices to consider functions $f : \Omega \to X$ of the form $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$ with pairwise disjoint $E_n - s$. We have then

$$\sum_{n=1}^{\infty} |x^*(x_n)|\mu(E_n) = \int_{\Omega} |x^* f| d\mu < \infty$$

This means that the series $\sum_{n=1}^{\infty} x_n \mu(E_n)$ is weakly unconditionally Cauchy. Since X does not contain any isomorphic copy of c_0 the series is convergent. Thus, Theorem 5.1 yields the Pettis μ–integrability of f. ◇

Example 4.1 shows that c_0 does not possess such an integration property.

6. Criteria for Pettis integrability.

The principal tool of this section is the *core*, the notion related to Rieffel's essential range of a strongly measurable function.

DEFINITION 6.1. Let $f : \Omega \to X$ be a function and let $E \in \Sigma$. The core f over E, denoted by $cor_f(E)$, is the set given by the formulae

$$cor_f(E) = \bigcap \{\overline{conv}\, f(E \setminus N) : N \in \mathcal{N}(\mu)\}$$

Let us look first at some basic properties of the core.

PROPOSITION 6.1. *If $f \in \mathbf{P}(\mu, X)$, then for each $E \in \Sigma$*

$$cor_f(E) = \overline{conv} \left\{ \frac{1}{\mu(F)} \int_F f d\mu : F \in \Sigma_\mu^+, F \subseteq E \right\}$$

Proof. Fix $E \in \Sigma_\mu^+$ and notice that if $x^* \in X^*$ then $x^*f \geq a$ μ-a.e. on E if and only if $\frac{1}{\mu(F)}\int_F x^*f\,d\mu \geq a$ for each $F \subseteq E$, $F \in \Sigma_\mu^+$. Now if

$H(x^*,a) = \{x \in X : x^*(x) \geq a\}$ then

$$cor_f(E) = \bigcap\{H(x^*,a) : x^*f \geq a \quad \mu - a.e.on\, E\} =$$

$$= \bigcap\{H(x^*,a),\ x^*\left[\frac{1}{\mu(F)}\int_F fd\mu\right] \geq a\,,\ \ F \subseteq E, F \in \Sigma_\mu^+\} =$$

$$= \overline{conv}\left\{\frac{1}{\mu(F)}\int_F fd\mu : F \in \Sigma_\mu^+, F \subseteq E\right\}$$

PROPOSITION 6.2. *Let $f,g : \Omega \to X$ be weakly measurable functions. If f is weakly μ–equivalent to g, then $cor_f(E) = cor_j(E)$ for every $E \in \Sigma$. Conversely, if for each $E \in \Sigma_\mu^+$ the relations $cor_f(E) = cor_g(E) \neq \emptyset$ hold, then f and g are weakly μ–equivalent.*

Proof. If f is weakly equivalent to g, then

$$cor_f(E) = \bigcap(\{H(x^*,a) : x^*f \geq a \quad \mu - a.e.on\, E\} =$$

$$\bigcap(H(x^*,a) : x^*g \geq a \quad \mu - a.e.on\, E\} = cor_g(E)$$

for arbitrary $E \in \Sigma$.

Conversely, suppose now that for some x^* the equality $x^*f = x^*g$ μ-a.e. fails. Let us assume that $\mu\{\omega : x^*f(\omega) < x^*g\} > 0$. Then, there is a set $E \in \Sigma_\mu^+$ such that $\sup\{x^*f(\omega) : \omega \in E\} < \inf\{x^*g(\omega) : \omega \in E\}$. It follows, that $\overline{conv}\,f(E) \cap \overline{conv}\,g(E) = \emptyset$ and this contradicts the assumption $cor_f(E) = cor_g(E) \neq \emptyset$. ◇

The following result is simple but quite interesting. In the case of a Pettis integrable function it follows directly from Theorem 4.3.

LEMMA 6.1. *If $f : \Omega \to X$ is scalarly integrable and $K \subset X^*$ is weak* countably compact (and convex) then $T_f(K)$ is a (weakly) closed subset of $L_1(\mu)$.*

Proof. Suppose g is in the closure of $T_f(K)$ and take $(x_n^*) \subset K$ with $\lim_n x_n^* f = g$ μ–a.e.. Let x^* be a weak* cluster point of (x_n^*). Then $x^* \in K$ and clearly $x^* f = g$ μ–a.e. This means, that $g \in T_f K$ and so $T_f K$ is closed.

If K is convex then – according to Mazur's theorem – its weak closure coincides with the norm one. ◇

Denote now for each $F \subseteq X$ by $\hat{F}$ the set $B(X^*) \cap F^0$.

PROPOSITION 6.3. *If f is scalarly integrable, then f is Pettis integrable if and only if T_f is weakly compact and*

$$W = \bigcap\{T_f(\hat{F}) : F \in \mathcal{F}_X\} = \{0\}$$

Proof. Assume that f is Pettis integrable. Then T_f is weakly compact in virtue of Corollary 4.1. Take an arbitrary g in W and for each $F \in \mathcal{F}_X$ let $x_{\mathcal{F}}^* \in \hat{F}$ be such that $g = T_f x_F^*$. Note that $\lim x_F^* = 0$ in the weak* topology ($\mathcal{F}_X$ is ordered by inclusion). By Theorem 4.3 $g = T_f x_F^* \to 0$ weakly in $L_1(\mu)$ and so $g = 0$ μ–a.e.

Assume now that T_f is weakly compact and $W = \{0\}$. According to Remark 4.3 and Corollary 4.2 it is sufficient to prove the weak*–weak continuity at zero of the restriction of T_f to $B(X^*)$. To do it consider $(x_\alpha^*) \subset B(X^*)$ that is weak*–convergent to zero. Then, for each $F \in \mathcal{F}_X$ the net (x_α^*) is eventually in $\hat{F}$. If g is a weak cluster point of $(T_f(x_\alpha^*))$, then $g \in T_f(\hat{F})$, since $T_f(\hat{F})$ is weakly compact by Lemma 6.1. Hence $g = 0$ μ–a.e. and $T_f x_\alpha^* \to 0$ weakly in $L_1(\mu)$. ◇

REMARK 6.1. In the above Proposition the family $\mathcal{F}_X$ may be replaced by $\tilde{\mathcal{F}}_X$.

COROLLARY 6.1. *A scalarly bounded and weakly measurable function f is Pettis integrable if and only if $\bigcap\{T_f(\hat{F}) : F \in \mathcal{F}_X\} = \{0\}$*

Proof. $T_f(B(X^*))$ is uniformly integrable and bounded in $L_1(\mu)$, hence it is relatively weakly compact ◇

LEMMA 6.2 *Suppose f is weakly measurable and $cor_f(E) \neq \emptyset$ for each $E \in \Sigma_\mu^+$. If $x^* \in X^*$, then $x^*f = 0$ μ–a.e. if and only if $x^* = 0$ on $cor_f(\Omega)$.*

Proof. Assume that x^*f is not μ–a.e. zero. Then we can find $\alpha > 0$ and $E \in \Sigma_\mu^+$ such that $x^*f > \alpha$ (or $< -\alpha$) on E. Let the first inequality holds. Then $\emptyset \neq cor_f(E) \subset \{x \in X : \langle x^*, x\rangle \geq \alpha\}$ and since $cor_f(E) \subseteq cor_f(\Omega)$, the functional x^* is not constantly zero on $cor_f(\Omega)$. ◊

Now we are able to prove the main result of this section.

THEOREM 6.1. *Let $f : \Omega \to X$ be scalarly μ–integrable. Then f is Pettis μ–integrable if and only if $cor_f(E) \neq \emptyset$ for each $E \in \Sigma_\mu^+$ and T_f is weakly compact.*

Proof. Assume that $\mathrm{cor}_f(E) \neq \emptyset$ for each $E \in \Sigma_\mu^+$ and take $g \in \bigcap\{T_f(\hat{F}) : F \in \mathcal{F}_X\}$ with $g = x^*f$ for some $x^* \in X^*$.

According to Proposition 6.3 we have to show that $g = 0$ μ–a.e. If g is not μ–a.e. zero then there is – in view of Lemma 6.2 – an element $x \in cor_f(\Omega)$ such that $\langle x^*, x\rangle \neq 0$.

Consider for each $n \in \mathbf{N}$ a functional $x_n^* \in \{n\hat{x}\}$ with $g = x_n^*f$ μ–a.e. If y^* is a weak* cluster point of (x_n^*), then clearly we have $g = y^*f$ μ–a.e. Moreover, since $x_n^* \in \{n\hat{x}\}$ the inequality $|\langle x_n^*, x\rangle| \leq 1/n$ is satisfied for each n. It follows that $\langle y^*, x\rangle = 0$. On the other hand, setting $z^* = x^* - y^*$ we have $z^*f = 0$ μ–a.e., while $\langle z^*, x\rangle = \langle x^*, x\rangle - \langle y^*, x\rangle = \langle x^*, x\rangle \neq 0$ and this contradicts Lemma 6.2. ◊

REMARK 6.2. The function $f : [0,1] \to c_0$ considered in Example 4.1 has the property that $\mathrm{cor}_f(E) \neq \emptyset$ for each $E \in \mathcal{L}^+$ but T_f is not weakly compact.

PROBLEM 5. Let $f : \Omega \to X^{**}$ be weak* scalarly bounded. Assume that $X \cap cor_f(E) \neq \emptyset$ for each $E \in \Sigma_\mu^+$. Does there exist $g \in \mathbf{P}(\mu, X)$ that is weak* scalarly equivalent to f?

Denote by $\mathcal{W}_X$ the family of all WCG subspaces of X.

LEMMA 6.3. *If H is a convex weak*–countably closed and relatively weak *–compact subset of X^*, then the following statements are equivalent:*

(a) $H \cap F^{\perp} \neq \emptyset$ *for all* $F \in \tilde{\mathcal{F}}_X$.

(b) $0 \in \bar{H}^{w^*}$

(c) $H \cap W^{\perp} \neq \emptyset$ *for each* $W \in \mathcal{W}_X$.

Proof. *(a)* $\Rightarrow$ *(b)*: If $0 \notin \bar{H}^{w^*}$, then there is $x_0 \in X$ such that $x_0(x^*) \neq 0$ for each $x^* \in \bar{H}^{w^*}$. It follows, that $H \cap F^{\perp} = \emptyset$, whenever $F = \operatorname{lin}\{x_0\}$.

(b) $\Rightarrow$ *(c)*: Let W be a WCG subspace of X and let $i : W \to X$ be the identity map. Since $0 \in \bar{H}^{w^*}$ and W is angelic, there is a countable set D in $i^*(H)$ such that $0 \in \bar{D}^{w^*}$. Let $C \subseteq H$ be a countable set such that $i^*(C) = D$. As H is relatively weak*–compact and weak*–countably closed, we have $\bar{C}^{w^*} \subset H$ and $\bar{C}^{w^*}$ is weak *–compact. Hence
$0 \in \bar{D}^{w^*} \subset i^*(\bar{C}^{w^*}) \subset i^*(H)$.

Let $x^* \in H$ be such that $i^*(x^*) = 0$. It is clear, that $x^* \in W^{\perp}$ and so the condition (c) is satisfied.

(c) $\Rightarrow$ *(a)* is obvious. ◇

LEMMA 6.4. *For each scalarly integrable X–valued f the equality*

$$\bigcap\{T_f(\hat{F}) : F \in \tilde{\mathcal{F}}_X\} = \bigcap\{T_f(\hat{G}) : G \in \mathcal{W}_X\}$$

holds.

Proof. Take $h \in \bigcap\{T_f(\hat{F}) : F \in \tilde{\mathcal{F}}_X\}$ and put
$H = B(X^*) \cap T_f^{-1}(h)$. H is a convex weak*–countably closed subset of $B(X^*)$ and $H \cap F^{\perp} \neq \emptyset$ for all $F \in \tilde{\mathcal{F}}_X$. In view of Lemma 6.3 the set $H \cap G^{\perp}$ is non–empty for all $G \in \mathcal{W}_X$. But this exactly means, that $h \in \bigcap\{T_f(\hat{G}) : G \in \mathcal{W}_X\}$ ◇

Now we are in a position to prove the next geometric characterization of the Pettis integration.

THEOREM 6.2. *Let $f : \Omega \to X$ be scalarly integrable. Then*

$f \in \mathbf{P}(\mu, X)$ if and only if there is a WCG space $Y \subseteq X$ such that $x^* f = 0$ μ–a.e. for each $x^* \in Y^\perp$ and T_f is weakly compact.

Proof. To prove the necessity of the condition assume that $f \in \mathbf{P}(\mu, X)$. Then T_f is weakly compact. Hence $T_f^* : L_\infty(\mu) \to X^{**}$ is weakly compact too, by Gantmacher's theorem. In particular T_f^* takes $L_\infty(\mu)$ into X and $Y = \overline{T_f^*(L_\infty(\mu))}$ is a WCG space.

Since $T_f^*(g) = P - \int_\Omega gf \, d\mu$ for each $g \in L_\infty(\mu)$, it is clear that $x^* \in Y^\perp$ yields $x^* f = 0$ μ–a.e.

Conversely, assume that $Y \in \mathcal{W}_X$ is such that $x^* f = 0$ μ–a.e., whenever $x^* \in Y^\perp$ and T_f is weakly compact. But then, $T_f(\hat{Y}) = \{0\}$ and the conclusion follows from Proposition 6.3. $\diamond$

REMARK 6.3. If μ is separable then the WCG subspace of X can be replaced by a separable one in Theorem 6.2.

The following corollary of the above theorem is rather surprising:

THEOREM 6.3. *Let (f_n) be a sequence of X–valued Pettis μ–integrable functions and let $f : \Omega \to X$ be a scalarly integrable function with weakly compact associated operator T_f. If for each $x^* \in X^*$, the condition $\sum_{n=1}^\infty |x^* f_n| = 0$ μ–a.e. yields the equality $x^* f = 0$ μ–a.e., then $f \in \mathbf{P}(\mu, X)$.*

Proof. For each $n \in \mathbf{N}$ let Y_n be a WCG subspace of X such that $x^* f_n = 0$ μ–a.e. whenever $x^* \in Y_n^\perp$. If W_n is a weakly compact set generating Y_n, then $W = \bigcup_{n=1}^\infty 2^{-n} W_n$ is also a weakly compact set, and, it generates a WCG space $Y \subseteq X$. If $x^* \in Y^\perp$, then we have $x^* f = 0$, and so $f \in \mathbf{P}(\mu, X)$. $\diamond$

We shall formulate yet one more condition that is sufficient for the Pettis integrability.

LEMMA 6.5. *Let $\nu : \Sigma \to X^{**}$ and $\kappa : \Sigma \to X$ be finitely additive with $\kappa(\Sigma)$ being relatively weakly compact. If $|x^*\nu|(\Omega) \le |x^*\kappa|(\Omega)$ for every $x^* \in X^*$, then $\nu(\Sigma) \subseteq 2\,\overline{aco}\,\kappa(\Sigma) \subseteq X$.*

Proof. Consider κ as an X^{**}–valued set function. Then, the weak relative compactness of the range of κ implies the equality $\overline{aco}\ \kappa(\Sigma) = \overline{aco}^{w^*}\kappa(\Sigma)$, where the last closure is taken in the weak* topology of X^{**}. Suppose, there is $E \in \Sigma$ such that $\nu(E) \notin 2\,\overline{aco}\kappa(\Sigma)$. Then, according to the Hahn–Banach theorem, there exists $x^* \in X^*$ such that

$$2\ \sup\{\langle x^*, z\rangle : z \in \overline{aco}\kappa(\Sigma)\} < x^*\nu(E)$$

Hence,

$$|x^*\kappa|(\Omega) < |x^*\nu|(E) \leq |x^*\nu|(\Omega)$$

and this contradicts the assumption ◇

THEOREM 6.4. *Let $f : \Omega \to X$ be a scalarly integrable function. If there exists a finitely additive $\kappa : \Sigma \to X$ such that $\kappa(\Sigma)$ is weakly relatively compact (the countable additivity of κ is sufficient but not necessary for it) and*

$$\int_\Omega |x^*f|d\mu \leq |x^*\kappa|(\Omega)$$

for each $x^* \in X^*$, then f is Pettis μ–integrable. If $\kappa(\Sigma)$ is norm relatively compact or separable then, the same holds for $\nu_f(\Sigma)$.

Proof. Let ν be the indefinite Dunford integral of f. Then, ν and κ satisfy the assumptions of Lemma 6.5 and so $\nu(\Sigma) \subseteq 2\,\overline{aco}\kappa(\Sigma) \subseteq X$.
This proves the Pettis μ–integrability of f ◇

As a very particular case we get

COROLLARY 6.3. *Let $f : \Omega \to X$ be weakly measurable and $g : \Omega \to X$ be Pettis μ–integrable. If*

$$\int_\Omega |x^*f|d\mu \leq \int_\Omega |x^*g|d\mu$$

for each $x^* \in X^*$, then $f \in \mathbf{P}(\mu, X)$.

Proof. In view of Corollary 4.1 the set $\nu_g(\Sigma)$ is weakly relatively compact in X ◇

PROBLEM 6. Is it possible to assume that $g \in \mathbf{P}(\mu, X^{**})$ only?

PROBLEM 7. Let $f, g : \Omega \to X$ satisfy the assumptions of Corollary 6.3. Does there exist $h \in \mathbf{P}(\mu, X)$ such that f and h are not weakly μ–equivalent, for no $p \in L_1(\mu)$ the equality $x^*f = px^*(h)$ μ–a.e. holds for all $x^* \in X^*$ and $\int_\Omega |x^*f| d\mu \leq \int_\Omega |x^*h| d\mu$ for each $x^* \in X^*$?

REMARK 6.3. It suffices to assume in Theorem 6.4 that $|x^*\kappa|(\Omega) = 0$ yields $\int_\Omega |x^*f| d\mu = 0$. This generalizes Theorem 6.4 and is a consequence of Theorem 6.2.

We shall finish this section with the following question:

PROBLEM 8. Let $f : \Omega \to X$ be scalarly integrable. Assume that for each $E \in \Sigma$ there exists $\nu(E) \in X$ satisfying for each $x^* \in \operatorname{ext} B(X^*)$ (= extreme points of $B(X^*)$) the equality

$$x^*\nu(E) = \int_E x^* f d\mu$$

For which X the above assumption yields the Pettis μ–integrability of f?

PROBLEM 8'. The same as above but for $X = C(K)$.

REMARK 6.4. Let f and ν be as in Problem 8. It follows easily from the closed graph theorem that $\nu(\Sigma)$ is a bounded subset of X. Appealing to Rainwater's theorem concerning the weak extremal convergence (cf. [D], 1984), we see at once, that ν is an X–valued measure.

If $\sup\{\int |x^*f| d\mu : x^* \in \operatorname{ext} B(X^*)\} < \infty$, then the σ–additivity of ν follows directly from the definition of ν.

If X does not contain any isomorphic copy of l_1 or (X^*, weak^*) is angelic (e.g. X is WCG), then $f \in \mathbf{P}(\mu, X)$. If moreover $f : \Omega \to X$ is strongly measurable, then the inequality $\int_\Omega |x^*f| d\mu < \infty$ for each $x^* \in \operatorname{ext} B(X^*)$ is sufficient ([D], 1984) to have $f \in \mathbf{P}(\mu, X)$.

7. Integration of scalarly bounded functions.

As we have seen a scalarly integrable function need not be Pettis integrable even if it is strongly measurable. What can be said in the situation of scalarly bounded functions? That not all such functions are Pettis integrable was already known to Phillips. We shall begin this section with presenting of Phillips's example.

EXAMPLE 7.1. (CH) Under the continuum hypothesis Sierpinski [S] constructed a set $B \subset [0,1] \times [0,1]$ with the following properties:

(1) For each $t \in [0,1]$ the set $\{s \in [0,1] : (s,t) \in B\}$ is at most countable,

(2) For each $s \in [0,1]$ the set $\{t \in [0,1] : (s,t) \notin B\}$ is at most countable.

Consider $([0,1], \mathcal{L}, \lambda)$ and define $f : [0,1] \to l_\infty[0,1]$ by

$$[f(s)](t) = \chi_B(s,t)$$

We shall prove first that f is weakly measurable. To do it we identify $l_\infty^*[0,1]$ with the space $ba[0,1]$ of additive real–valued set functions defined on all subsets of [0,1] and endowed with the variation norm (cf [D-S]).

If $\eta \in ba$ [0,1] then it can be uniquely represented as the sum $\eta = \eta_1 + \eta_2$, where $\eta_1, \eta_2 \in ba$ [0,1], supp η_1 is at most countable and η_2 vanishes on countable sets.

If $s \in [0,1]$, then

$$\langle f(s), \eta \rangle = \langle f(s), \eta_1 \rangle + \langle f(s), \eta_2 \rangle =$$

$$\int_{supp\ \mu_1} [f(s)](t)\eta_1(dt) + \int_{B_s} [f(s)](t)\eta_2(dt)$$

where $B_s = \{t \in [0,1] : (s,t) \in B\}$.

But since the set $\{s \in [0,1] : \exists t \in supp\eta_1, (s,t) \in B\}$ is at most countable, $\langle f(s), \eta_1 \rangle = 0$ λ–a.e. Moreover, the set $[0,1] \backslash B_s$ is at most countable too, so we have $\langle f(s), \eta_2 \rangle = \eta_2[0,1]$ λ–a.e. It follows that f is weakly λ–measurable (even more is true, $\langle f, \eta \rangle$ is measurable with respect to the Borel σ–algebra of subsets of the unit interval) and being bounded, it is also scalarly integrable.

We shall prove now, that $f \notin \mathbf{P}(\lambda, l_\infty[0,1])$. Observe first, that if $\eta = \delta_t$, and if $g \in l_\infty[0,1]$ is the Pettis integral of f over [0,1], then

$$\int_{[0,1]} \langle f(s), \eta\rangle ds = \int_{[0,1]} \chi_B(s,t)\,ds = 0$$

in view of (1). Thus, $g = 0$ (i.e. $g(s) = 0$ for all $s \in [0,1]$).

On the other hand, if η is any normalized additive extension of λ to $\mathcal{P}[0,1]$ (the existence of such an extension was proved by Łoś and Marczewski [Ł-M], then $\langle f(s), \eta\rangle = \eta[0,1]$ λ–a.e. Hence $\langle g, \eta\rangle = \int_{[0,1]}\langle f(s), \eta\rangle ds = 1$ and clearly $g \neq 0$.

This contradiction shows, that f is not Pettis λ–integrable ◇

Since just considered function f is not Pettis λ–integrable, there must be at least one set $E \in \mathcal{L}^+$ such that $cor_f(E) = \emptyset$. The behaviour of f is however much worse: $cor_f[0,1] = \emptyset$. To see it, let $B^t = \{s \in [0,1] : (s,t) \in B\}$ for each $t \in [0,1]$. The properties of B imply that $\lambda(B_t) = 0$ for each t. If $s \in B^t$ then $[f(s)(t)] = 0$. Hence, if $x \in \overline{conv} f(\Omega \setminus B^t)$ then $x(t) = 0$. It follows that if $x \in cor_f[0,1]$, then $x(t) = 0$ for every t. But $0 \notin cor_f[0,1]$, as for each finite sum $\Sigma\, \alpha_i f(s_i)$ with $\alpha_i \geq 0, \Sigma\, \alpha_i = 1$, we have $\|\Sigma\, \alpha_i f(s_i)\| = \sup_t |\Sigma\, \alpha_i \chi_B(s_i,t)| = 1$.

We shall present one more example of a bounded non–integrable function.

EXAMPLE 7.2. (CH) Once again let $([0,1], \mathcal{L}, \lambda)$ be the basic measure space. Let (a_ξ) be any enumeration of $[0,1]$. We identify [0,1] with $[0,\omega_1]$ considered with the order topology and define

$$f : [0,1] \to C([0,\omega_1])$$

by $f(a_\xi) = \chi_{[0,\xi]}$.

It is easily seen, that if $\eta \in C^*[0,1]$ then η is concentrated on a countable set. Assume, supp $\eta \subseteq [0,\delta] \cup \{\omega_1\}$ with $\delta < \omega_1$. Hence, for $\xi > \delta$ we have

$$\langle f(a_\xi), \eta\rangle = \int_{[0,\omega_1]} [f(a_\xi)](\gamma)\,\eta(d\gamma) = \int_{[0,\omega_1]} \chi_{[0,\xi]}(\gamma)\,\eta(d\gamma) =$$

$$= \int_{[0,\delta]} \chi_{[0,\xi]}(\gamma)\eta(d\gamma) + \int_{\{\omega_1\}} \chi_{[0,\xi]}(\gamma)\eta(d\gamma) = \eta[0,\delta] = \eta[0,\omega_1)$$

Thus, $\langle f, \eta\rangle$ is constant off a countable set. It follows that f is weakly measurable.

Suppose f is Pettis λ–integrable and let $g = \int_{[0,1]} f d\lambda$. Then,

$$\langle g, \eta\rangle = \int_{[0,1]} \langle f(t), \eta\rangle dt = \eta[0,\omega_1) = \int_{[0,1]} \chi_{[0,\omega_1)}(\gamma)\eta(d\gamma)$$

If follows, that $g = \chi_{[0,\omega_1)}$. But $\chi_{[0,\omega_1)} \notin C[0,1]$ and so $f \notin \mathbf{P}(\lambda, C[0,\omega_1])$.

DEFINITION 7.1. X has the *μ–Pettis integral property* (μ–PIP) if each scalarly bounded weakly μ–measurable function $f : \Omega \to X$ is Pettis μ–integrable. If X has the μ–PIP for all finite complete (Ω, Σ, μ), then we say that X has the *Pettis integral property* (PIP).

PROBLEM 9. Which Banach spaces have the μ–PIP (or the PIP)?

The problem in the full generality is open, but several partial answers are known. In particular, it is consistent with ZFC to assume that each X has the λ–PIP ([F-T]), necessarily denying CH, since as we have seen CH yields the non–λ–PIP of $l_\infty[0,1]$. Clearly, all separable Banach spaces have the PIP and, more generally, all measure compact Banach spaces have the PIP. In particular, all WCG spaces have PIP. l_∞ does not have PIP ([F-T]). It is a consequence of Theorem 6.1, that each X with the (C)–property has the PIP (X has the *property* (C) if each family $\mathcal{H}$ of closed convex subsets of X, which is stable under countable intersections and does not contain the empty set has non–empty intersection). The reverse implication fails for $X = l_1(\aleph_1)$ [E].

A non–trivial family of Banach spaces possessing the PIP is also given by the following

PROPOSITION 7.1. *If X does not contain any isomorphic copy of l_1 and X is separable, then X^* has the PIP.*

Proof. We shall prove in fact a stronger property of X^*: if $f : \Omega \to X^*$ is weak*–measurable and weak*–scalarly bounded, then $f \in \mathbf{P}(\mu, X^*)$.

Let $f : \Omega \to X^*$ be a function possessing the just mentioned properties and let $\nu_f : \Sigma \to X^*$ be its weak* integral i.e.

$$\langle x, \nu_f(E) \rangle = \int_E \langle x, f \rangle d\mu$$

for all $x \in X$ and $E \in \Sigma$. If $x^{**} \in X^{**}$, then according to a theorem of Odell and Rosenthal [OR] there exists a sequence $(x_n) \subset X$ such that $||x_n|| \leq ||x^{**}||$ and $\lim_n x_n = x^{**}$ in the weak*–topology, The Lebesgue Dominated Convergence Theorem yields the equality

$$\langle x^{**}, \nu_f(E) \rangle = \int_E \langle x^{**}, f \rangle d\mu \qquad \diamond$$

It should be noted that the separability assumption is essential (at least if one assumes the existence of real–valued measurable cardinals).

PROPOSITION 7.2. *$l_1(\Gamma)$ has the PIP if and only if card Γ is not a real–valued measurable cardinal.*

Proof. Assume, that card Γ is not a real–valued measurable cardinal, and take a weakly measurable and scalarly bounded function $f : \Omega \to l_1(\Gamma)$. Then fix $E \in \Sigma$, define $\eta \in l_1^{**}(\Gamma)$ by

$$\eta(h) = \int_E \langle h, f(\omega) \rangle \mu(d\omega)$$

for all $h \in l_\infty(\Gamma)$ and put

$$\kappa(A) = \eta(\chi_A)$$

for all $A \subseteq \Gamma$.

If $\Gamma \supseteq A_n \downarrow \emptyset$, then $\chi_{A_n} \to 0$ weak* in $l_\infty(\Gamma)$, and so $\langle \chi_{A_n}, f(\omega) \rangle \to 0$ for all $\omega \in \Omega$. Since f is scalarly bounded, we have $\lim_n \kappa(A_n) = 0$. Hence κ is a real–valued measure.

Since, card Γ is less then the first real–valued measurable cardinal, there exists $x = (x_\gamma) \in l_1(\Gamma)$ such that

$$\kappa(A) = \langle x, \chi_A \rangle = \Sigma_{\gamma \in A} x_\gamma$$

It follows, that x is the Pettis integral of f over E.

If card Γ is at least so large as the first real–valued measurable cardinal, then there is a universal real–valued measure μ on $\mathcal{P}(\Gamma)$. One can easily see that the function $f : \Gamma \to l_1(\Gamma)$ defined by $f(\gamma) = \chi_{\{\gamma\}}$ is not Pettis μ–integrable. $\diamondsuit$

PROBLEM 10. Which $C(K)$ spaces have the PIP?

In connection with Theorem 6.4 the following question can be formulated:

PROBLEM 11. Which Banach spaces have the following property: If $\nu : \Sigma \to X^{**}$ is a measure, then for each $x^{***} \in X^{***}$ the inequality

$$|Px^{***}(\nu)|(\Omega) \leq |x^{***}\nu|(\Omega)$$

holds for each $x^{***} \in X^{***}$ (P denotes here the canonical projection of X^{***} onto X^*)?

PROBLEM 12. Which Banach spaces have the μ–PIP for every perfect μ?

The following two problems seem for me to be quite interesting (if the answers are affirmative):

PROBLEM 13. Let $f \in \mathbf{P}(\mu, X^*)$ be weakly measurable and scalarly bounded and let ρ be a lifting on $L_\infty(\mu)$. It is known (cf [I-T] that a function $\rho_1(f) : \Omega \to X^*$ can be defined by $x\rho_1(f) = \rho(xf)$, for each $x \in X$. Is $\rho_1(f)$ Pettis μ–integrable? Is $\rho_1(f)$ at least weakly μ–measurable?

PROBLEM 14. Let $f \in \mathbf{P}(\mu, X)$ be weakly measurable and scalarly bounded and let ρ be a lifting on $L_\infty(\mu)$. In the same way as in the previous

problem we can define $\rho_2(f) : \Omega \to X^{**}$ by setting
$x^*\rho_2(f) = \rho(x^*f)$, whenever $x^* \in X^*$. Is $\rho_2(f)$ Pettis μ–integrable or weakly μ–measurable?

8. Limit theorems.

It is the purpose of this section to prove the convergence theorems of Vitali and Lebesgue type for the Pettis integral.

To prove the main results we need a deep theorem of James that characterizes weakly compact subsets of a Banach space.

PROPOSITION 8.1. (Helly's condition, cf [Da]) *Let Z be a normed space, let $z_1^*, \dots, z_n^*$ be linear continuous functionals of Z and let $c_1, \dots, c_n$ be real numbers. Assume that there is a number M satisfying for each point $(a_1, \dots, a_n) \in R^n$ the inequality*

$$|\sum_{i=1}^{n} a_i c_i| \leq M \| \sum_{i=1}^{n} a_i z_i^* \| .$$

Then, for each $\varepsilon > 0$ there exists $z \in Z$ such that $\|z\| < M + \varepsilon$ and $z_k^*(z) = c_k$ for each $k \in \{1, \dots, n\}$.

PROPOSITION 8.2. (James) [J] *Let C be a weakly closed bounded subset of a Banach space X. If C is not weakly compact, then there is $\theta > 0$, a sequence $(x_n) \subseteq C$ and a bounded sequence $(x_n^*) \subset X^*$ satisfying the following conditions*:

(a) $x_k^*(x_n) > \theta$ if $k \leq n$,

(b) $x_k^*(x_n) = 0$ if $k > n$.

Proof. Let $\tilde{C}$ be the weak* closure of C in X^{**} and let $w \in \tilde{C} \backslash C$. Moreover, let $\Delta = \inf\{\|w - x\| : x \in X\})$ Since X is norm closed in X^{**}, we have $\Delta > 0$.

Take an arbitrary θ between 0 and Δ and then choose any
$x_1^* \in B(X^*)$ such that $\langle w, x_1^* \rangle > \theta$ (the existence of such x_1^* follows from the fact that $\|w\| > 0$). Since $w \in \tilde{C}$ there is $x_1 \in C$ with $\langle x_1^*, x_1 \rangle > \theta$.

The further construction goes by the induction. Suppose, we have already x_n, x_n^* with $n < m$ such that the following are satisfied:

(j) $x_n \in C$ and $||x_n^*|| \leq 1$,
(jj) $x_k^*(x_n) = 0$ if $k > n$,
(jjj) $x_k^*(x_n) > \theta$ if $k \leq n$,
(jv) $\langle w, x_k^* \rangle > \theta$ if $k < m$.

Observe, that if $(a_1, \ldots, a_{m-1}) \in R^{m-1}$ then

$$\Delta \leq || \sum_{i=1}^{m-1} a_i x_i + w||$$

and hence for η such that $\theta < \eta < \Delta$ we get the inequality

$$\eta \leq \frac{\eta}{\Delta} || \sum_{i=1}^{m-1} a_i x_i + w||$$

Setting in Helly's condition $Z = X^*$, $c_1 = c_2 = \ldots = c_{m-1} = 0$, $c_m = \eta$, $f_i = x_i$ if $i < m$, $f_m = w$, $M = \frac{\eta}{\Delta}$ and $\varepsilon = 1 - M$ we get the existence of $z = x_m^* \in B(X^*)$ such that $x_m^*(x_n) = 0$ if $n < m$ and $\langle w, x_m^* \rangle = \eta$. Hence, (jj) is satisfied for $k = m$. The equality $\eta = \langle w, x_m^* \rangle$ yields $\langle w, x^* \rangle > \theta$ and so (jv) holds for $k = m$ too. We have to find yet a point $x_m \in C$ satisfying (jjj). But this follows easily from the fact that $w \in \tilde{C}$. Indeed, if $(y_\alpha) \subseteq C$ is weak* convergent to w, then there is α_0 such that $\langle x_k^*, y_\alpha \rangle > \theta$ for each $k \leq m$ and each $\alpha \geq \alpha_0$. We may take as x_m any point y_α with $\alpha \geq \alpha_0$ $\diamondsuit$

The theorem we are going to present now is an analogue of Vitali's convergence theorem. Conditions (a) and (b) of this theorem guarantee that for each $x^* \in X^*$ and $E \in \Sigma$ the sequence $\left\{ \int_E x^* f_n d\mu : n \in \mathrm{N} \right\}$ is convergent to $\int_E x^* f d\mu$, and that the set $\{x^* f : x^* \in B(X^*)\}$ is weakly relatively compact in $L_1(\mu)$. They may be replaced by any others guaranteeing the above weak compactness and the convergence of the appropriate sequence of scalar integrals.

THEOREM 8.1. (Vitali Convergence Theorem for the Pettis integral) *Let $f : \Omega \to X$ be a function. If there exists a sequence of Pettis integrable functions $f_n : \Omega \to X$ such that:*

(a) The set $\{x^* f_n : ||x^*|| \leq 1, n \in \mathrm{N})$ is uniformly μ–integrable.

(b) $\lim_n x^* f_n = x^* f$ *in μ–measure, for each $x^* \in X^*$,*

then f is Pettis μ–integrable and

$$\lim_n \int_E f_n d\mu = \int_E f d\mu$$

weakly in X, for each $E \in \Sigma$.

Proof. Fix $E \in \Sigma$, and let C be the weak closure of the set $\{\int_E f_n d\mu : n \in \mathbf{N}\}$. Since the classical Vitali's convergence theorem yields the convergence $\lim_n \int_E x^* f_n d\mu = \int_E x^* f d\mu$, for each $x^* \in X^*$, we see that the sequence $(\int_E f_n d\mu)$ is weakly Cauchy.

Hence, it is bounded and $C \setminus \{\int_E f_n d\mu : n \in \mathbf{N}\}$ contains at most one point.

In order to prove our assertion, it is sufficient to show the weak compactness of C, since this will yield the existence of the weak limit of the sequence $(\int_E f_n d\mu)$ in X, and the limit point can clearly be equal only to $\int_E f d\mu$.

Suppose therefore, that C is not weakly compact. Then, according to James's theorem there exist $\theta > 0, (x_n) \subseteq C$, and bounded $(x_k^*) \subset X^*$ such that

$$\langle x_k^*, x_n \rangle = 0 \text{ if } k > n \text{ and } \langle x_k^*, x_n \rangle > \theta \text{ whenever } k \leq n.$$

Consequently, we can find a subsequence (g_m) of (f_n) and a subsequence (y_m^*) of (x_n^*) such that

(i) $\int_E y_k^* g_m d\mu = 0 \quad$ for $k > m$,

(ii) $\int_E y_k^* g_m d\mu > \theta \quad$ for $k \leq m$,

(iii) $\lim_m \int_E x^* g_m d\mu = \int_E x^* f d\mu \quad$ for each $x^* \in X^*$.

Consider now the set $\{y_m^* f : m \in \mathbf{N}\}$. It easily follows from (a) and (b) that this set is uniformly integrable and bounded in $L_1(\mu)$. Hence it is relatively weakly compact. This yields the existence of a function $h \in$

$L_1(\mu)$ and a subsequence (z_j^*) of (y_m^*) such that $\lim_j z_j^* f = h$ weakly in $L_1(\mu)$. Applying (ii) and (iii) for all z_j^* we get the inequality $\int_E z_j^* f d\mu \geq \theta$ for all $j \in \mathbf{N}$. Hence,

$$\int_E h d\mu \geq \theta$$

I want to show that $h = z_0^* f$ μ–a.e. for some z_0^* (this fact follows at once from Lemma 6.1, but since we need some further information, we shall provide more exact calculation).

To do it, we shall appeal to the theorem of Mazur, according to which, there are non–negative numbers $a_1^m \dots a_{k(m)}^m$ such that

$$\sum_{j=1}^{k(m)} a_j^m = 1 \quad \text{and} \quad \lim_m \sum_{j=1}^{k(m)} a_j^m \langle z_{j+m}^*, f\rangle = 0$$

in the norm topology of $L_1(\mu)$.

Without loss of generality, we may assume, that the above convergence holds μ–a.e.

Clearly, if z_0^* is a weak* cluster point of the sequence $(\sum_{j=1}^{k(m)} a_j^m z_{j+m}^* : m \in \mathbf{N})$, then $h = z_0^* f$ μ–a.e. In particular, we have

$$\text{(iv)} \qquad \int_E z_0^* f d\mu \geq \theta$$

On the other hand, if $(w_{n\alpha}^*)_\alpha$ is a subnet of $(\sum_{j=1}^{k(m)} a_j^m z_{j+m}^* : m > n)$ that converges weak* to z_0^* then, applying (i) we get

$$0 = \lim_\alpha \int_E w_{n\alpha}^* g_n d\mu = \lim_\alpha w_{n\alpha}^* \int_E g_n d\mu = z_0^* \int_E g_n d\mu = \int_E z_0^* g_n d\mu$$

The last three equalities are consequences of the Pettis integrability of the functions g_n and since the above collection of inequalities is true for all $n \in \mathbf{N}$, it follows from (iii) that

$$\int_E z_0^* f d\mu = 0 \ .$$

But this contradicts the inequality (iv), and so the theorem is proved $\diamond$

As an immediate consequence of Theorem 8.1 we get the following generalization of the classical Lebesgue theorem:

THEOREM 8.2. (Lebesgue Dominated Convergence Theorem for the Pettis integral) *Let* $f : \Omega \to X$ *be a function satisfying the following two conditions:*

(α) *There exists a sequence of Pettis* μ*-integrable functions* $f_n : \Omega \to X$ *such that* $\lim_n x^*f_n = x^*f$ *in* μ*–measure, for each* $x^* \in X^*$.

(β) *There exists* $h \in L_1(\mu)$ *such that for each* $x^* \in B(X^*)$ *and each* $n \in \mathbf{N}$, *the inequality* $|x^*f_n| \leq h$ *holds* μ*–a.e. (the exceptional set depends on* x^**).*

Then $f \in \mathbf{P}(\mu, X)$ and

$$\lim_n \int_E f_n d\mu = \int_E f d\mu$$

weakly in X, for all $E \in \Sigma$.

Proof. The condition (a) of Theorem 8.1 follows from (β) $\diamond$

EXAMPLE 8.1. As an application of the above theorems we shall prove the λ–integrability of $f : [0,1) \to L_\infty(\lambda)$ defined by $f(t) = \chi_{[0,t)}$.

First of all notice that f is weakly measurable, since each x^*f is of bounded variation.

Let π_n be the partition of [0,1) consisting of the intervals $[(i-1)/2^n, i/2^n)$, $i \leq 2^n$ and let

$$f_n(t) = \chi_{[0,i/2^n)}$$

Clearly, $f_n : [0,1) \to L_\infty(\lambda)$ and $|x^*f_n| \leq \|x^*\|$ for all n and $x^* \in L_\infty(\lambda)$. Each $x^* \in L_\infty(\lambda)$ may be identified with an additive set function $\mu \in ba(\mathcal{L})$ that vanishes on $\mathcal{N}(\lambda)$. Hence,

$$x^*f_n(t) = \mu([0, i/2^n)) \text{ if } t \in [(i-1)/2^n, i/2^n)$$

and

$$x^*f(t) = \mu([0,t))$$

for all $t \in [0, 1)$.

If $E_{t,n} \in \pi_n$ is that element which contains t, then we have $|x^*f(t) - x^*f_n(t)| \leq |\mu|(E_{t,n})$. It follows from the boundedness of μ that

$$\lim_n |\mu|(E_{t,n}) = 0$$

for all but countably many $t \in [0, 1)$.

Thus, $\lim_n x^*f_n = x^*f$ λ–a.e. and $f \in \mathbf{P}(\lambda, L_\infty(\lambda))$ $\diamond$

Also the next Lebesgue type result is a particular case of Theorem 8.1.

THEOREM 8.3. *Let $f : \Omega \to X$ be a function satisfying the following two conditions:*

(α') *There exists a sequence (f_n) of X–valued Pettis μ–integrable functions such that*

$$\lim_n \int_E x^* f_n d\mu = \int_E x^* f d\mu$$

for all $E \in \Sigma$ and $x^* \in X^*$.

(β') *There exists a finitely additive $\kappa : \Sigma \to X$ such that $\kappa(\Sigma)$ is weakly relatively compact and for each $n \in \mathbf{N}$ and $x^* \in X^*$ the inequality*

$$\int_\Omega |x^* f_n| d\mu \leq |x^*\kappa|(\Omega)$$

holds.

Then f is Pettis μ–integrable and

$$\lim_n \int_E f_n d\mu = \int_E f d\mu$$

weakly in X for all $E \in \Sigma$.

Proof. Notice, that for each $E \in \Sigma$ and $x^* \in X^*$

$$|\int_E x^* f d\mu| \leq \overline{\lim_n} \int_E |x^* f_n| d\mu \leq |x^*\kappa|(\Omega)$$

Hence,

$$\int_\Omega |x^* f| d\mu \leq 2\,|x^*\kappa|(\Omega)$$

and the Pettis integrability of f follows from Theorem 6.4. The weak convergence of the integrals follows now from Theorem 8.1 and the comments before it. ◇

It is natural to ask when the condition (α') of Theorem 8.3 is sufficient for the Pettis integrability of a function. If X contains an isomorphic copy of c_0, then (α') is too weak to guarantee the integrability. Indeed, let f be the function considered in Example 4.1. With the same notation, we have

$$x^*f = \sum_{n=1}^{\infty} \alpha_n 2^{-n} \chi_{(2^{-n},2^{-n+1}]}$$

$$f_n(t) = (2\chi_{(2^{-1},1]}(t),\ldots,2^n\chi_{(2^{-n},2^{-n+1}]}(t),0,0,0,\ldots)$$

and

$$\lim_n \int_E x^*f_n d\mu = \int_E x^*f d\mu$$

for all $x^* \in X^*$ and $E \in \Sigma$, but $f \notin \mathbf{P}(\mu,X)$ ◇

It appears however, that c_0 is the only inconvenient Banach space.

THEOREM 8.4. *Let X be without any isomorphic copy of c_0. If $f : \Omega \to X$ is scalarly μ–integrable and there are functions $f_n \in \mathbf{P}(\mu,X)$ such that*

$$\lim_n \int_E x^*f_n d\mu = \int_E x^*f d\mu$$

for all $E \in \Sigma$ and $x^* \in X^*$, then $f \in \mathbf{P}(\mu,X)$ and

$$\lim_n \int_E f_n d\mu \in \int_E f d\mu$$

weakly in X for all $E \in \Sigma$.

Proof. In virtue of Corollary 3.1 the set Ω can be decomposed into pairwise disjoint set $\Omega_n \in \Sigma$, such that for each $n \in \mathbf{N}$ and $x^* \in X^*$ the inequality $|x^*f\chi_{\Omega_n}| \leq n\|x^*\|$ holds μ–a.e. It follows from Theorem

8.1 that f is Pettis integrable on each Ω_n. Now it is sufficient to apply the series characterization of Banach spaces not containing c_0 [B-P] ◇

9. Approximation by simple functions – The compact case.

The fact that each strongly measurable function can be approximated by almost everywhere convergent sequence of simple functions is the starting point for the whole theory of the Bochner integration. It yields in particular the approximation of each Bochner integrable function by a sequence of simple functions convergent in the L_1–norm. Unfortunately, in the case of the Pettis integral such an approximation is impossible (as we could see in Example 3.1, the norm of Pettis integrable function may be even non–measurable). However, still there is a wide class of Pettis integrable functions that can be approximated in a weaker sense.

To prove the first theorem we need a few lemmata.

LEMMA 9.1. *If $\nu : \Sigma \to X$ is a measure and $f : \Omega \to [0,1]$ is a Σ–measurable function, then*

$$\int_\Omega f d\nu \in \overline{conv}\, \nu(\Sigma)$$

(the integral is understood the sense of Bartle–Dunford–Schwartz [B-D-S]: there is $x \in X$ such that $\int_\Omega f\, d(x^*\nu) = x^*(x)$ for each $x^* \in X^*$)

Proof. Assume at the beginning that $f = \sum_{i=1}^m a_i\chi_{E_i}$ with $0 \le a_i \le \dots \le a_m$ and E_i being pairwise disjoint elements of Σ satisfying the equality $\Omega = \cup_{i=1}^m E_i$. Applying the Abel transformation

$$\sum_{i=1}^m a_i b_i = a_m \sum_{j=1}^m b_j - \sum_{i=1}^{m-1}(a_{i+1} - a_i)\sum_{j=1}^i b_j$$

we get

$$\int_\Omega f d\nu = \sum_{i=1}^m a_i\nu(E_i) = a_m \sum_{i=1}^m \nu(E_i) - \sum_{i=1}^{m-1}(a_{i+1} - a_i)\sum_{j=1}^i \mu(E_j) =$$

$$= a_m\nu(\Omega) - \sum_{i=1}^{m-1}(a_{i+1} - a_i)\nu(\bigcup_{j=1}^{i} E_j) \in conv\nu(\Sigma)$$

since $0 \leq a_m - \sum_{i=1}^{m-1}(a_{i+1} - a_i) = a_1 \leq 1$.

The general case follows from the density of simple Σ–measurable functions in $\mathcal{L}_\infty(\Sigma)$ (the space of all bounded real–valued Σ–measurable functions endowed with the uniform norm) $\diamond$

LEMMA 9.2. *Let $\nu : \Sigma \to X$ be a μ–continuous measure. If $T_\nu : X^* \to L_1(\mu)$ is defined by*

$$T_\nu(x^*) = \frac{d(x^*\nu)}{d\mu}$$

then for each $g \in L_\infty(\mu)$ the equality $T_\nu^*(g) = \int_\Omega g d\nu$ holds.

Proof. The continuity of T_ν is a consequence of the closed graph theorem. The rest follows from a direct calculation. $\diamond$

LEMMA 9.3. *Let Π be a directed set, and let $U_\pi : X \to X$ be a bounded continuous operator, for each $\pi \in \Pi$. If $\sup_\pi \|U_\pi\| < \infty$ and $\lim_\pi U_\pi(x) = x$ for each $x \in X$, then the convergence is uniform on each relatively compact subset of X.*

Proof. Let $K \subset X$ be a compact set. Then, for a given $\varepsilon > 0$ there exists an ε–net $\{x_1, \ldots, x_n\} \subset K$ of K. Let π_ε be such that $\|U_\pi(x_i) - x_i\| < \varepsilon$ for each $i \in \{1, \ldots, n\}$ and each $\pi \geq \pi_\varepsilon$. If M is an upper bound of $\{\|U_\pi\| : \pi \in \Pi\}$, then we have for each $x \in K$ and $\pi \geq \pi_\varepsilon$:

$$\|U_\pi x - x\| \leq \inf_{1\leq i\leq n}\{\|U_\pi(x - x_i)\| + \|U_\pi x_i - x_i\| + \|x_i - x\|\} \leq (2 + M)\varepsilon$$

It follows, that $U_\pi x \to x$ uniformly on K. $\diamond$

LEMMA 9.4. *Let $\nu : \Sigma \to X$ be a μ–continuous measure. If $\nu(\Sigma)$ is relatively compact in the norm topology of X, then for each positive ε there exists an X–valued simple function h such that*

$$\sup\{\|\nu(E) - \int_E h d\mu\| : E \in \Sigma\} < \varepsilon$$

Proof. Let T_ν be defined as in the previous lemma. In virtue of Lemmata 9.1 and 9.2, we have

$$T^* B_{L_\infty(\mu)} \subset \overline{conv}(\nu(\Sigma) - \nu(\Sigma))$$

since $f \in B_{L_\infty(\mu)}$ has a representation $f = f_1 - f_2$ with $f_1, f_2 : \Omega \to [0,1]$.

The Mazur theorem [cf [D-U], p. 51) yields the compactness of the set $\overline{conv}(\nu(\Sigma) - \nu(\Sigma))$ and so T_ν^* is a compact operator. The compactness of T_ν is a consequence of Schauder's theorem [cf [D-S], p. 485).

Thus,

$$K = \{\frac{d(x^*\nu)}{d\mu} : x^* \in B(X^*)\} = T(B(X^*))$$

is norm relatively compact.

Consider now for each $\pi \in \Pi_\Sigma$ the operator $U_\pi : L_1(\mu) \to L_1(\mu)$ defined for each $g \in L_1(\mu)$ by

$$U_\pi(g) = g_\pi = \sum_{E\in\pi}(\int_E g d\mu)\mu(E)^{-1}\chi_E$$

and notice that $\lim_\pi U_\pi(g) = g$ in $L_1(\mu)$ for each simple g. Hence, the same holds for all $g \in L_1(\mu)$. In view of Lemma 9.3 the net $(U_\pi(g))$ is convergent to g uniformly on K.

Let f_π^ν be defined by

$$f_\pi^\nu = \sum_{E\in\Sigma} \frac{\nu(E)}{\mu(E)}\chi_E$$

where $\pi \in \Pi_\Sigma$.

Since for each $x^* \in X^*$ the net $\{x^* f_\pi^\nu : \pi \in \Pi_\Sigma\}$ is a martingale convergent in $L_1(\mu)$ to $\frac{d(x^*\nu)}{d\mu}$ (cf [D-U]) and since at the same time $x^* f_\pi^\nu = U_\pi \frac{d(x^*\nu)}{d\mu})$, we see that $\{x^* f_\pi^\nu : \pi \in \Pi_\Sigma\}$ is convergent in $L_1(\mu)$ to $\frac{d(x^*\nu)}{d\mu}$ and that the convergence is uniform on K.

Let us fix $\varepsilon > 0$ and take π_0 such that

$$\int_\Omega |\frac{d(x^*\nu)}{d\mu} - x^* f_\pi^\nu| d\mu < \varepsilon$$

for each $\pi \geq \pi_0$ and each $x^* \in X^*$.

In particular, if $E \in \Sigma$, then

$$|x^*\nu(E) - \int_E x^* f_\pi^\nu d\mu| \leq \int_E |\frac{d(x^*\nu)}{d\mu} - x^* f_\pi^\nu| d\mu < \varepsilon$$

It is enough to put $h = f_{\pi_0}^\nu$. ◇

The first approximation theorem is now an easy corollary of the above lemma.

THEOREM 9.1. *If f is Pettis μ–integrable and $\nu_f(\Sigma)$ is norm relatively compact, then f is a limit of a sequence of X–valued simple functions, in the norm topology of* $\mathbf{P}(\mu, X)$.

REMARK 9.1. A short calculation shows that the converse to the above theorem also holds: If $f \in \mathbf{P}(\mu, X)$, $f = \lim_\pi f_\pi$ in the norm topology of $\mathbf{P}(\mu, X)$ and $\nu_{f_\pi}(\Sigma)$ is norm relatively compact for each $\pi \in \Pi$ (= an arbitrary directed set), then $\nu_f(\Sigma)$ is norm relatively compact too.

A few questions arise in the context of Theorem 9.1.

DEFINITION 9.1. X has the *μ–Pettis compactness property* (μ–PCP) if for each $f \in \mathbf{P}(\mu, X)$ the set $\nu_f(\Sigma)$ is norm relatively compact. X has the PCP if it has the μ–PCP for each finite (Ω, Σ, μ).

PROBLEM 15. To characterize all X possessing the PCP (or the μ–PCP).

PROBLEM 16. Which $C(K)$–spaces have the PCP?

PROBLEM 17. To characterize (Ω, Σ, μ) such that each X has the μ–PCP.

Several partial answers to Problem 15 are known:

(a) Separable spaces have the PCP,

(b) Subspaces of separably complementable spaces have the PCP (X is *separably complementable* if for each separable subspace Y of X there exists a separable space Z such that $Y \subseteq Z \subseteq X$, and, there is a bounded projection of X onto Z). In particular WCG spaces and their subspaces have the PCP.

(c) AL–spaces (because they are separably complementable). In particular $C^*[0,1]$ has the PCP.

(d) [Ta, 1980] If l_∞ is not a quotient of X and we assume that the union of less than the continuuum λ–null sets is λ–null, then X has the λ–PCP. If (MA) is assumed and Σ is the μ–completion of a countably generated σ–algebra, then each X has the μ–PCP.

The following result due to Stegall [F-T] gives a partial answer to Problem 17.

THEOREM 9.2. *If μ is perfect then each X has the μ–PCP.*

Proof. Let f be a μ–Pettis integrable function (as it will be seen from the proof it is enough to assume, that f is Dunford integrable and ν_f is a μ–continuous measure, to get the relative norm compactness $\nu_f(\Sigma)$). In view of Corollary 3.1 we may assume, that f is scalarly bounded. It is obvious, that for each $E \in \Sigma$ the equality $T_f^*\chi_E = \nu_f(\Sigma)$ holds, so in order to prove the norm relative compactness of $\nu_f(\Sigma)$ it is sufficient to show the compactness of T_f. To do it, choose any sequence $(x_n^*) \subseteq B(X^*)$. Since f is weakly measurable, $(x_n^* f)$ has a subsequence $(x_{n_k}^* f)$ that converges a.e. Otherwise, we could apply Fremlin's subsequence theorem [F], to get a subsequence without measurable cluster points, taken in the space of all real–valued functions endowed with the topology of pointwise convergence.

If x^* is a weak* cluster point of $(x_{n_k}^*)$ then $x_{n_k}^* f \to x^* f$ pointwise, and hence in $L_1(\mu)$, because of the Lebesgue theorem.

Thus, T_f is compact, and the assertion is proved. ◇

EXAMPLE 9.1. The function f constructed in [F-T] shows that l_∞ does not have the PCP. In particular, no Banach space with the PCP can contain l_∞ as an isomorphic subspace. However, the following is open:

PROBLEM 18. Assume that X fails the PCP. Must X contain c_0?

PROBLEM 19. Which X have the μ–PCP and the μ–PIP simultaneously?

PROBLEM 20. Which $C(K)$ have the PIP and the PCP at the same time?

10. Approximation by simple functions – The separable case.

The following theorem is the main result of this section:

THEOREM 10.1. *Let X be an arbitrary normed space and let $f : \Omega \to X$ be a Pettis μ–integrable function. Then the following are equivalent*

(i) $\{x^*f : x^* \in B(X^*)\}$ is a separable subset of $L_1(\mu)$,

(ii) There exists a σ–algebra $\Sigma_0 \subseteq \Sigma$ such that $(\Omega, \Sigma_0, \mu|\Sigma_0)$ is separable, and f is weakly measurable with respect to Σ_0.

(iii) There exists a sequence (f_n) of X–valued simple functions, such that for each $x^* \in X^*$ one of the following conditions is satisfied:

(a) $\{x^*f_n : n \in \mathrm{N}\}$ is uniformly integrable and μ–a.e. convergent to x^*f,

(b) $\{x^*f_n : n \in \mathrm{N}\}$ is uniformly integrable and convergent in μ–measure to x^*f,

(c) $(x^*f_n : n \in \mathrm{N}\}$ is convergent to x^*f in $L_1(\mu)$,

(d) $\{x^*f_n : n \in \mathrm{N}\}$ is convergent to x^*f weakly in $L_1(\mu)$.

(iv) $\nu_f(\Sigma)$ is a separable subset of X.

Proof. ($i \Rightarrow ii$) Assume, that the set $\{x^*f : x^* \in B(X^*)\}$ is separable. Then there exists a sequence (x_n^*) in $B(X^*)$, such that $\{x_n^*f : n \in \mathrm{N})\}$ is dense in $\{x^*f : x^* \in B(X^*)\}$. If Σ_0 is the σ–algebra generated by all x_n^*f and by $\mathcal{N}(\mu)$ then, clearly $\mu|\Sigma_0$ is separable and each x^*f is Σ_0–measurable.

(ii) $\Rightarrow$ (iii)a. Assume that f is weakly measurable with respect to a separable $(\Omega, \Sigma_0, \mu|\Sigma_0)$ and let $\tilde{\Sigma} = \sigma(\{E_n : n \in \mathrm{N}\}) \subseteq \Sigma_0$ be a countably generated σ–algebra, that is $\mu|\Sigma_0$–dense in Σ_0. Moreover, let

π_n be the partition of Ω generated by the sets $E_1, \ldots, E_n$. Put for each n

$$f_n = \sum_{E \in \pi_n} \frac{\nu_f(E)}{\mu(E)} \chi_E$$

with the convention $0/0 = 0$.

It is well known, that $\{f_n, \sigma(\pi_n), n \in \mathrm{N}\}$ is an X–valued martingale and $\lim_n x^* f_n = E(x^* f | \tilde{\Sigma})$ in $L_1(\Omega, \tilde{\Sigma}, \mu | \tilde{\Sigma})$ and $\mu|\tilde{\Sigma}$–a.e. (cf [D-U]). Moreover, since the conditional expectation operator is a contraction on $L_1(\mu)$, we have

$$\int |x^* f_n| d\mu \leq \int |x^* f| d\mu$$

for all n.

This yields the uniform integrability of $\{x^* f_n : n \in \mathrm{N}\}$. As by the assumption $\tilde{\Sigma}$ is dense in Σ_0 we have $E(x^* f | \tilde{\Sigma}) = x^* f$ μ–a.e. and so $\lim_n x^* f_n = x^* f$ μ–a.e.

The implications $a \Rightarrow b \Rightarrow c \Rightarrow d$ are obvious, and so it remains to prove that (iii)d yields (iv) and (iv $\Rightarrow$ i).

(iii)d $\Rightarrow$ (iv) The condition (iii)d means that for each $E \in \Sigma$ the sequence $(\nu_{f_n}(E))$ is weakly convergent to $\nu_f(E)$. Hence, $\nu_f(\Sigma)$ is contained in the weak closure of the set $\bigcup_{n=1}^{\infty} \nu_{f_n}(\Sigma)$ and the last set is separable, since the ranges of all ν_{f_n}–s are finite dimensional.

(iv $\Rightarrow$ i) Suppose, that $\{x^* f : x^* \in B(X^*)\}$ is non–separable. We shall prove that $\nu_f(\Sigma)$ is non–separable.

To do it, take an arbitrary $x_1^* \in S(X^*)$ and $h_1 \in L_\infty(\mu)$ such that $\langle h_1, x_1^* f \rangle = 1$. Assume then, that we have already constructed for an ordinal $\beta < \omega_1$ a family $\{(x_\alpha^*, h_\alpha) : \alpha < \beta\}$ with the following properties:

(α) $x_\alpha^* \in S(x^*)$,

(β) $h_\alpha \in L_\infty(\mu)$,

(γ) $x_\gamma^* f \in \overline{\mathrm{lin}}\{x_\alpha^* f : \alpha < \gamma\}$ for each $\gamma < \beta$,

(δ) $\langle h_\gamma, x_\alpha^* f \rangle = \begin{cases} 1 & \text{if } \alpha = \gamma < \beta \\ 0 & \text{if } \alpha < \gamma < \beta \end{cases}$

Since $\{x^* f : x^* \in B(X^*)\}$ is non–separable, one can find $x_\beta^* \in S(X^*)$ such that $x_\beta^* \notin \overline{\mathrm{lin}}\{x_\alpha^* f : \alpha < \beta\}$. Then, applying the Hahn–Banach theorem we get $h_\beta \in L_\infty(\mu)$ such that $\langle h_\beta, x_\beta^* f \rangle = 1$ and $\langle h_\beta, x_\alpha^* f \rangle = 0$ for all $\alpha < \beta$.

Consequently, we get a net $\{(x^*_\alpha, h_\alpha) : \alpha < \omega_1\}$ satisfying $(\alpha) - (\delta)$ for all α, β, γ less than ω_1.

Consider now the operator T_f associated with f. We have

$$\|T^*_f(h_\beta) - T^*_f(h_\alpha)\| \geq 1$$

whenever $\alpha < \beta$, and so the set $T^*_f(L_\infty(\mu)\}$ is non–separable in X^{**}. But $\operatorname{lin}\{\chi_E : E \in \Sigma\}$ is norm dense in $L_\infty(\mu)$ and so $\operatorname{lin}\nu_f(\Sigma)$ is norm dense in $T^*_f(L_\infty(\mu))$. It follows that $\nu_f(E)$ is non–separable. This completes the proof of the whole theorem. ◇

REMARK 10.1. The uniform integrability of the sets $\{x^*f_n : n \in \mathrm{N}\}$ in the conditions (iii)a and (iii)b may be replaced by the uniform integrability of the set $\{x^*f_n : n \in \mathrm{N},\ x^* \in B(X^*)\}$. This follows easily from the proof of (ii $\Rightarrow$ iiia), if one applies the uniform integrability of the set $\{x^*f : x^* \in B(X^*)\}$.

Combining Theorem 8.1 with Theorem 10.1 and Remark 10.1 we get the following characterization of Pettis integrability in the case of separable measure spaces.

THEOREM 10.2. *Let $f : \Omega \to X$ be a function. Then, f is Pettis μ–integrable and $\nu_f(\Sigma)$ is a separable set if and only if there exists a sequence (f_n) of X–valued simple functions such that:*
(j) The family $\{x^*f_n : n \in \mathrm{N}, x^* \in B(X^*)\}$ is uniformly integrable,
(jj) For each $x^* \in X^*$ we have $\lim_n x^*f_n = x^*f$ μ–a.e. .

In the particular case of scalarly bounded functions we get the following:

THEOREM 10.3. *Let $f : \Omega \to X$ be a scalarly bounded function. Then, f is Pettis μ–integrable and weakly measurable with respect to a separable measure space $(\Omega, \Sigma_0, \mu|\Sigma_0)$ if and only if there exists a bounded sequence (f_n) of X–valued simple functions, such that*
$\lim_n x^*f_n = x^*f$ μ–a.e. for all $x^* \in X^*$ (the exceptional sets depend on x^*).

The boundedness of (f_n) in the above theorem means the existence of a positive number M such that for each $n \in \mathbf{N}$ and $x^* \in X^*$ the inequality $|x^* f_n| \leq M\|x^*\|$ holds μ–a.e. and the exceptional set may vary with x^*. However, in this special case, when f_n-s are strongly measurable, it exactly means that there exists $M > 0$, such that $\sup_n \|f_n(\omega\| \leq M$ μ–a.e.

As it has been proven in Theorem 8.2 the range of an indefinite Pettis integral of a function defined on a perfect measure space is norm relatively compact. Hence, it follows from Theorem 10.1, that such a function is weakly measurable with respect to a separable measure space.

The following is open:

PROBLEM 21. Let (Ω, Σ, μ) be a separable perfect measure space and let $f : \Omega \to X$ be Pettis μ–integrable. Does there exist a countably generated σ–algebra $\tilde{\Sigma} \subseteq \Sigma$ such that f is weakly measurable with respect to the $\mu|\tilde{\Sigma}$–completion of $\tilde{\Sigma}$?

Observe, that there is a large difference between the $\mu|\tilde{\Sigma}$–completion of $\tilde{\Sigma}$ and the μ–completion of $\tilde{\Sigma}$.

Without the perfection of μ, the answer is negative (at least if one assumes the validity of Martin's Axiom).

DEFINITION 10.1. X has the *μ–Pettis Separability Property* (μ–PSP) if for each $f \in P(\mu, X)$ the set $\nu_f(\Sigma)$ is separable. Similarly, we define the PSP.

PROBLEM 22. Which X have the μ–PSP (or the PSP)? Which X have the μ–PIP and the μ–PSP?

PROBLEM 23. Which (Ω, Σ, μ) have the property, that each X has the μ–PSP?

PROBLEM 24. Is it true, that if X has the $(\mu$-)PIP then it also has the $(\mu$-)PSP?

PROBLEM 25. Which $C(K)$ have the PSP?

DEFINITION 10.2. X is *μ–Pettis essentially separable* if each element of $\mathbf{P}(\mu, X)$ is weakly equivalent to a strongly measurable function. X is *Pettis essentially separable* if it is μ–PES for each μ.

PROBLEM 26. Which X are (μ–) PES?

PROBLEM 27. Which $C(K)$ are PES?

11. The weak Radon–Nikodym property – The general case.

Throughout the previous sections we have been interested in the following situation. There is a function taking its values in a Banach space. Under, what conditions is the function Pettis integrable? As it has been shown, the integral is always a measure of σ–finite variation.

Now, we shall consider the opposite case: For a given μ–continuous X–valued measure ν of σ–finite variation, find conditions guaranteeing the existence of a Pettis μ–integrable function $f : \Omega \to X$, such that $\nu = \nu_f$.

Several conditions ensuring the existence of a strongly measurable f are known (cf [D-U]), but we shall be interested here mainly in the non–strongly measurable case. Unfortunately, for such formulated problem no satisfactory answer is known, so we shall investigate rather the following: Which X have the property, that each X–valued measure of σ–finite variation is a Pettis integral?

The following theorem is the starting point for the whole theory. Its proof makes use of the lifting, however, in the case of a separable X, it can be done without it. In the case of a measure of finite variation the theorem is a consequence of a representation theorem of A and C. Ionescu–Tulcea ([IT]). Explicitly it was first stated by Dinculeanu [Di]. The σ–finite case was proved by Rybakov [R]. We shall still present a more general formulation.

THEOREM 11.1. *Let $\nu : \Sigma \to X$ be a weak* measure. If $|\nu|$ is a σ–finite measure, such that $\mathcal{N}(\mu) \subseteq \mathcal{N}(|\nu|)$, then there exists a weak* scalarly integrable function $f : \Omega \to X^*$ such that*

$$\langle x, \nu(E) \rangle = \int_E \langle x, f \rangle d\mu$$

for each $x \in X$ and each $E \in \Sigma$.

Proof. Assume first, that ν satisfies the inequality $|\nu|(E) \leq M\mu(E)$ for all $E \in \Sigma$ and, take a lifting ρ on $L_\infty(\mu)$. Denote for $x \in X$ by f_x the Radon–Nikodym derivative of the measure $\langle x, \nu\rangle$ with respect to μ. Clearly, $|f_x| \leq M$ μ–a.e., and so $|\rho(f_x)| \leq M$ everywhere.

Define $f : \Omega \to X^\#$ (the algebraic dual of X) by

$$\langle x, f(\omega)\rangle = \rho(f_x)(\omega)$$

for each $\omega \in \Omega$ and $x \in X$.

It follows, that $|\langle x, f(\omega)\rangle| = |\rho(f_x)(\omega)| \leq M||x||$ for each $\omega \in \Omega$ and so $f : \Omega \to X^*$.

Since $f_x = \langle x, f\rangle$ μ–a.e., we get the equality

$$\int_E \langle x, f\rangle d\mu = \int_E \rho(f_x)\, d\mu = \int_E f_x d\mu = \langle x, \nu(E)\rangle$$

for each $E \in \Sigma$ and each $x \in X$.

Consider now the general case. According to the scalar version of the Radon–Nikodym theorem, there is a measurable function $h : \Omega \to [0, \infty)$ such that

$$|\nu|(A) = \int_A h\, d\mu$$

for all $A \in \Sigma$. It follows, that we can decompose Ω into pairwise disjoint sequence of sets $\Omega_n \in \Sigma$, such that $|\nu|(E \cap \Omega_n) \leq n\mu(E \cap \Omega_n)$, for all $E \in \Sigma$ and $n \in \mathbf{N}$.

As we have just proved for each $n \in \mathbf{N}$ there exists a weak*–measurable $f_n : \Omega_n \to X^*$ such that

$$\langle x, \nu(E)\rangle = \int_E \langle x, f_n\rangle d\mu$$

for each $\Omega_n \supseteq E \in \Sigma$ and $x \in X$. Setting $f = \sum_{n=1}^\infty f_n \chi_{\Omega_n}$ we have for each $x \in X$

$$\int_\Omega |\langle x, f\rangle| d\mu = \sum_{n=1}^\infty \int_{\Omega_n} |\langle x, f\rangle| d\mu = \sum_{n=1}^\infty |\langle x, \nu\rangle|(\Omega_n) =$$

$$= |\langle x, \nu \rangle|(\Omega) < \infty$$

Thus, $\langle x, f \rangle \in L_1(\mu)$ and for each $E \in \Sigma$ the equality

$$\langle x, \nu(E) \rangle = \int_E \langle x, f \rangle d\mu$$

holds. ◊

REMARK 11.1 It is worth to notice, that there are weak*–measures satisfying the assumptions of the above theorem which are not measures in the norm topology of X^*. In fact, ν investigated in Example 4.3 has such a property: for each $A \subseteq \mathbf{N}$ we have $|\nu|(A) = cardA$ if A is finite, and $|\nu|(A) = \infty$ otherwise. $|\nu|$ is plainly a σ–finite measure such that $\mathcal{N}(\mu) \subseteq \mathcal{N}(|\nu|)$.

The property saying that for each μ–continuous X^*–valued measure ν of σ–finite variation there is a weak* measurable $f : \Omega \to X^*$ such that

$$\langle x, \nu(E) \rangle = \int_E \langle x, f \rangle d\mu$$

for each $E \in \Sigma$ and $x \in X$, can be called the μ–Weak* *Radon–Nikodym Property* ($\mu - W^*RNP$). So it is a consequence of Theorem 11.1, that each conjugate Banach space has the W^*RNP (please notice, that we use this name in a different meaning than it is used in [Ta, 1984]). f will be called the weak* *density* (or Radon–Nikodym derivative) of ν with respect to μ.

As a consequence of Theorem 11.1 we obtain the following result [R, 1968]:

THEOREM 11.2. *Let $\nu : \Sigma \to X$ be a μ–continuous measure of σ–finite variation. Then there exists a weak* measurable $f : \Omega \to X^{**}$ such that*

$$\langle x^*, \nu(E) \rangle = \int_E \langle x^*, f \rangle d\mu$$

for each $x^* \in X^*$ and $E \in \Sigma$. ◊

Of particular interest is the case when the X^{**}–valued weak* density of $\nu : \Sigma \to X$ is in $\mathbf{P}(\mu, X^{**})$.

DEFINITION 11.1. X has the $\mu - W^{**}RNP$ if for each X–valued μ–continuous measure of σ–finite variation there exists $f \in \mathbf{P}(\mu, X^{**})$ such that

$$\langle x^{***}, \nu(E)\rangle = \int_E \langle x^{***}, f\rangle d\mu$$

for each $x^{***} \in X^{***}$ and $E \in \Sigma$.

In the obvious way the $W^{**}RNP$ is defined.

DEFINITION 11.2. If the function, in the case described in the previous definition, can be taken from $\mathbf{P}(\mu, X)$ then X is said to have the *μ–Weak Radon-Nikodym Property* (μ–$WRNP$). In a similar way the $WRNP$ is defined.

REMARK 11.2. $L_1[0,1]$ is an example of a Banach space without the W^{**}RNP. Indeed $L_1[0,1]$ is complementable in $L_1^{**}[0,1]$ (cf. [H]), so the W^{**} RNP of $L_1[0,1]$ would imply the RNP of the space, and it is well known, that $L_1[0,1]$ does not enjoy the last property.

It is interesting and useful to know, that the W^{**} RNP and the WRNP are determined by a single measure.

THEOREM 11.3. *If X has the $\lambda - W^{**}$RNP (resp. λ–WRNP), then it has also the W^{**} RNP (resp. WRNP).*

Proof. Let (Ω, Σ, μ) be an arbitrary complete probability measure space and let $\nu : \Sigma \to X$ be a μ–continuous measure of σ–finite variation. Without loss of generality, we may assume, that μ is non–atomic and $||\nu(E)|| \le \mu(E)$ for each $E \in \Sigma$.

(A) Assume first, that Σ is the completion of a countably generated σ–algebra $\tilde{\Sigma} \subseteq \Sigma$ with respect to $\mu|\tilde{\Sigma}$. Let $(E_n) \subseteq \Sigma$ be a sequence generating $\tilde{\Sigma}$ and let $\chi : \Omega \to [0,1]$ be its Marczewski function:

$$\chi(\omega) = 2\sum_{n=1}^{\infty} 3^{-n}\chi_{E_n}(\omega)$$

It can be easily checked, that χ^{-1}: $\mathcal{L} \cap \chi(\Omega) \to \mathcal{P}(\Omega)$ is a Boolean σ–isomorphism of $\mathcal{B}_{[0,1]} \cap \chi(\Omega)$ onto $\tilde{\Sigma}$. Let $\bar{\mu} : \mathcal{L} \to [0,1]$ be the image of μ under χ and, let $\theta : [0,1] \to [0,1]$ be a function defined by $\lambda([0,\theta(t)]) = \bar{\mu}([0,1])$. If $\xi = \theta \circ \chi$, then for each $E \in \mathcal{L}$, we have $\mu[\xi^{-1}(E)] = \lambda(E)$ and the measure algebras of μ and λ are isomorphic.

Let now $\bar{\nu} : \mathcal{L} \to X$ be given by

$$\bar{\nu}(B) = \nu[\xi^{-1}(B)]$$

We have $||\bar{\nu}(B)|| \leq \lambda(B)$ for each $B \in \mathcal{L}$. Hence, by the assumption, there is $f \in \mathbf{P}(\lambda, X^{**})$ (resp. $\mathbf{P}(\lambda, X)$), such that $\bar{\nu}(B) = \int_E f d\lambda$.

It follows, that for each $E \in \Sigma$

$$\nu(E) = P - \int_E f \circ \xi \, d\mu$$

(B) Assume now, that Σ is arbitrary and notice that $\nu(\Sigma)$ is a norm relatively compact subset of X^{**} (resp. X). To see it, consider an arbitrary countably generated σ–algebra $\tilde{\Sigma} \subseteq \Sigma$. With the same notation as in (A), we have $\nu(\tilde{\Sigma}) = \bar{\nu}(\mathcal{L})$. In view of Theorem 9.2 the set $\bar{\nu}(\mathcal{L})$ is norm relatively compact. Denote now by Ξ the collection of all complete measure spaces $(\Omega, \Delta, \mu|\Delta)$ with $\Delta \subseteq \Sigma$ being the completion of a countably generated σ–algebra Σ_Δ with respect to $\mu|\Sigma_\Delta$. We order Ξ upwards by inclusion. In view of (A), for each Δ there is $f_\Delta \in \mathbf{P}(\mu|\Delta, X^{**})$ (resp. $\mathbf{P}(\mu|\Delta, X)$), such that

$$\nu(E) - P - \int_E f_\Delta \, d\mu$$

for each $E \in \Delta$.

We shall prove, that the net (f_Δ) is Cauchy in the norm of $\mathbf{P}(\mu, X^{**})$ (resp. (μ, X)).

To prove it, fix $\varepsilon > 0$ and take a simple function $h_\varepsilon : \Omega \to X$, such that

$$\sup_{E \in \Sigma} ||\nu(E) - \int_E h_\varepsilon d\mu|| \leq \varepsilon$$

(see lemma 9.4).

Now fix $\tilde{\Delta} \in \Xi$ such that h_ε is $\tilde{\Delta}$–measurable. Then, for each $\Delta \geq \tilde{\Delta}$

$$|f_\Delta - h_\varepsilon| \leq 4 \sup_{E\in\Sigma} ||\nu(E) - \int_E h_\varepsilon d\mu|| < \varepsilon$$

It follows, that for $\Delta_1, \Delta_2 \geq \tilde{\Delta}$ the inequality $|f_{\Delta_1} - f_{\Delta_2}| < 2\varepsilon$ holds, and so the net is Cauchy, as required.

But Ξ is countably directed, so there exists Δ_0 such that for each $\Delta \geq \Delta_0$ we have $|f_\Delta - f_{\Delta_0}| = 0$.

It follows, that each such f_Δ is weakly μ–equivalent to f_{Δ_0} and so for each $E \in \Sigma$, we get the equality

$$\nu(E) = P - \int_E f_\Delta d\mu$$

This completes the proof. $\diamond$

REMARK 11.3. The measure λ in Theorem 11.3 can be replaced by any non–atomic perfect measure.

PROBLEM 28. Let κ be the first real–valued measurable cardinal, and let μ be a non–atomic probability measure on $\mathcal{P}(\kappa)$. Suppose, that X has the μ–W^{**} RNP (or the μ–WRNP). Does X have the W^{**} RNP (resp. WRNP)?

PROBLEM 29. Let $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ be a perfect measure space, such that
(a) $\Omega \subseteq \tilde{\Omega}$ and $\Sigma = \Omega \cap \tilde{\Sigma}$.
(b) $\mu(E \cap \Omega) = \tilde{\mu}(E)$ for each $E \in \tilde{E}$.

Assume, that X has the $\mu - W^{**}$ RNP (resp. the μ–WRNP). Does X have the $\tilde{\mu} - W^{**}$ RNP (resp. $\tilde{\mu}$–WRNP)?

The reverse implications always hold.

The above problem is strongly connected with the following one:

PROBLEM 30. Let (Ω, Σ, μ) and $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ be two measure spaces satisfying the above conditions (a) and (b). Suppose, $f \in \mathbf{P}(\mu, X)$ is given. When does there exist $g \in \mathbf{P}(\tilde{\mu}, X)$ such, that $g|\Omega$ is weakly μ–equivalent to f? For which X, each $f \in \mathbf{P}(\mu, X)$ can be extended to an arbitrary (perfect) "superspace" $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ satisfying (a)–(b)? Which (Ω, Σ, μ) are good for all X?

Even if we assume, that $\nu_f(\Sigma)$ is norm relatively compact, the answer may be negative [SSW].

PROBLEM 31. Let $Y \supseteq X$ and (Ω, Σ, μ) be fixed. Assume, that each $\nu : \Sigma \to X$ that is μ–continuous and of σ–finite variation has an Y–valued Pettis μ–integrable density. Does it follow from this, that X has the $\mu - W^{**}$RNP? Suppose, Y has the property for each (Ω, Σ, μ). Does X have the W^{**}RNP?

If $[0,1]$ cannot be covered by less then the continuum closed λ–negligible sets, then the answer is affirmative ([Ta, 1984], p.87).

To prove further results, we need yet new notions.

DEFINITION 11.3. X has the *μ–Compact Range Property* (μ–CRP) if each μ–continuous measure $\nu : \Sigma \to X$ of σ–finite variation has norm relatively compact range. If X has the μ–CRP for each μ – then we say that X has the CRP.

PROPOSITION 11.1. *If X has the W^{**}RNP, then it has the CRP.*

Proof. Let $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ be a perfect measure space satisfying the conditions (a), (b) formulated in Problem 29, and let $\nu : \Sigma \to X$ be a μ-continuous measure of σ-finite variation. Define $\tilde{\nu} : \Sigma \to X$ by setting $\tilde{\nu}(E) = \nu(E \cap \Omega)$. It is clear, that ν is $\tilde{\mu}$–continuous and $\tilde{\nu}(\tilde{\Sigma}) = \nu(\Sigma)$.

Since X has the W^{**}RNP, there exists $f \in \mathbf{P}(\tilde{\mu}, X^{**})$ such that for each $x^* \in X^*$ and $E \in \tilde{\Sigma}$

$$\langle x^*, \tilde{\nu}(E)\rangle = \int_E \langle x^*, f\rangle d\mu$$

In virtue of Theorem 9.2, the range of $\tilde{\nu}$ is norm relatively compact. Hence, the same holds for ν. ◇

PROBLEM 32. Is it possible to replace the W^{**}RNP and the CRP in Proposition 11.1 by the μ–W^{**} RNP and the μ–CRP respectively, if μ is a fixed measure?

It follows from the proof of Proposition 11.1, that the answer is yes, if Problem 29 has the affirmative solution.

PROPOSITION 11.2. c_0 and $L_1(\mu)$ fail the CRP, for a non–purely atomic μ.

Proof. Let (ε_n) be the Bernoulli sequence on (Ω, Σ, μ) (i.e. ε_n take only values +1 and -1 with the same probability and are independent). Then $\varepsilon_n \to 0$ weakly in $L_1(\mu)$ but (ε_n) does not contain any norm convergent subsequence. In particular $\nu(E) = \left(\int_E \varepsilon_n d\mu\right)$ is a c_0–valued measure without norm relatively compact range.

In the case of $L_1(\mu)$, the measure $\nu : \Sigma \to L_1(\mu)$ given by $\nu(E) = \chi_E$ has non–norm relatively compact range. ◇

DEFINITION 11.4. If Σ_0 is a sub-σ-algebra of Σ, $f \in \mathbf{P}(\mu, X)$, and $g \in \mathbf{P}(\Omega, \Sigma_0, \mu|\Sigma_0), X)$, then g is called the *conditional expectation* of f with respect to Σ_0 (we shall write $g = E(f|\Sigma_0))$ if $\int_E f\, d\mu = \int_E g d\mu$ for all $E \in \Sigma_0$.

DEFINITION 11.5. Given a directed set $(\Pi, \leq)$, a family of σ–algebras $\Sigma_\pi \subseteq \Sigma$, and functions $f_\pi \in \mathbf{P}((\Omega, \Sigma_\pi, \mu|\Sigma_\pi); X)$ with $\pi \in \Pi$, the system $\{f_\pi, \Sigma_\pi; \pi \in \Pi\}$ is a *martingale* if $\pi \leq \rho$ yields $\Sigma_\pi \subseteq \Sigma_\rho$ and $E(f_\rho|\Sigma_\pi) = f_\pi$. The *martingale is bounded* if there is $M > 0$ such that for each $x^* \in X^*$ and each $\pi \in \Pi$ the inequality $|\langle x^*, f_\pi\rangle| \leq M||x^*||$ holds μ–a.e. The *martingale is convergent* in $\mathbf{P}(\mu, X)$ if there is $f \in \mathbf{P}(\mu, X)$ such, that $\lim_\pi \| f_\pi - f \| = 0$

The following gives a martingale characterization of the WRNP and the W^{**}RNP.

THEOREM 11.4. *For a Banach space X the following are equivalent:*

(i) X has the WRNP (resp. W^{**} RNP).

(ii) Given any (Ω, Σ, μ) and any bounded martingale $\{f_n, \Sigma_n; n \in \mathbf{N}\}$ of X–valued Pettis μ–integrable (simple) functions, then $\{f_n, \Sigma_n; n \in \mathbf{N}\}$ is convergent in $\mathbf{P}(\mu, X)$ (resp. $\mathbf{P}(\mu, X^{**})$).

Proof. Assume (i) is satisfied and take a bounded martingale

$\{f_n, \Sigma_n; n \in \mathbf{N}\}$ in $\mathbf{P}(\mu, X)$. Assume that $M > 0$ is such that $|\langle x^*, f_n\rangle| \leq M\|x^*\|$ μ–a.e. (the exceptional sets depend on x^*).

Let $\tilde{\Sigma}_0 = \bigcup_{n=1}^{\infty} \Sigma_n$ and let $\tilde{\nu} : \tilde{\Sigma}_0 \to X$ be given by

$$\tilde{\nu}(E) = \lim_n \int_E f_n d\mu$$

for each $E \in \tilde{\Sigma}_0$.

Clearly, $|\langle x^*, \tilde{\nu}(E)\rangle| \leq M\mu(E)\|x^*\|$ for each $x^* \in X^*$ and $E \in \tilde{\Sigma}_0$. Hence, $\|\tilde{\nu}(E)\| \leq M\ \mu(E)$ and $\tilde{\nu}$ extends uniquely to a measure $\nu_1 : \Sigma_0 = \sigma(\tilde{\Sigma}_0) \to X$ satisfying the similar condition for all $E \in \Sigma_0$. Setting for each $E \in \Sigma$

$$\nu(E) = \int_E E(\chi_E|\Sigma_0)\, d\nu_1$$

we get an extension of ν_1 to the whole Σ satisfying for all $E \in \Sigma$ the inequality

$$\|\nu(E)\| \leq M\ \mu(E)$$

Since X has the WRNP (or W^{**}RNP), we get $f \in \mathbf{P}(\mu, X)$ (resp. $\mathbf{P}(\mu, X^{**})$) being the density of ν with respect to μ.

Since X has the W^{**} RNP, it has the CRP (by Prop. 11.1). Thus in a similar way, as it has been done in the proof of Theorem 11.3, we can show, that $\{(f_n, \Sigma_n) : n \in \mathbf{N}\}$ is a Cauchy martingale in $\mathbf{P}(\mu, X^{**})$.

Since $(\nu|\Sigma_n)(E) = \int_E f_n d\mu$ for each $n \in \mathbf{N}$, we have $\langle x^*, f_n\rangle = E(\langle x^*, f\rangle|\Sigma_n)$ for each $x^* \in X^*$ and this gives

$$\lim_n \int_\Omega |E(x^* f|\Sigma_0) - x^* f_n|\, d\mu = 0$$

Together with the Cauchy condition, this yields

$$\lim_n \| f_n - f \| = 0$$

Assume now, that (ii) is satisfied and take a measure $\nu : \Sigma \to X$ satisfying for each $E \in \Sigma$ the inequality

$$\|\nu(E)\| \leq \mu(E)$$

Define for each $\pi \in \Pi_\Sigma$ the function f_π by

$$f_\pi = \sum_{E\in\pi} \frac{\nu(E)}{\mu(E)}\chi_E$$

and let $\Sigma_\pi = \sigma(\pi)$. $\{(f_\pi,\Sigma_\pi); \pi \in \Pi_\Sigma\}$ is a bounded martingale in $\mathbf{P}(\mu,X)$. If $\pi_1 \le \pi_2 \le \ldots$ then, by the assumption, $\{(f_{\pi_n},\Sigma_{\pi_n}); n \in \mathbf{N}\}$ is convergent in $\mathbf{P}(\mu,X)$ (resp. $\mathbf{P}(\mu,X^{**})$). It follows, that the whole martingale is Cauchy, i.e.

$$\forall(\varepsilon > 0)\exists\pi_0\forall(\xi \ge \pi_0)\forall(\eta \ge \pi_0)$$

$$\sup\left\{\int_\Omega |\langle x^*, f_\xi - f_\eta\rangle| d\mu : x^* \in B(X^*)\right\} < \varepsilon$$

Let $f : \Omega \to X^{**}$ be a weak* density of ν with respect to μ:

$$\langle x^*, \nu(E)\rangle = \int_E \langle x^*, f\rangle d\mu$$

for each $x^* \in X^*, E \in \Sigma$.

Since $\nu_{f_\pi} = \nu|\Sigma_\pi$, we have $E(x^*f|\Sigma_\pi) = x^*f_\pi$ for all $x^* \in X^*$. It follows from the martingale convergence theorem, that
$\forall(x^* \in B(X^*))\exists[\pi(x^*) \ge \pi_0]\forall[\xi \ge \pi(x^*)]$

$$\int_\Omega |\langle x^*, f_\xi - f\rangle| d\mu < \varepsilon$$

Hence, for each $\pi \ge \pi_0$ and $x^* \in B(X^*)$

$$\int_\Omega |\langle x^*, f_\pi - f\rangle| d\mu \le \int_\Omega |\langle x^*, f_\pi - f_{\pi(x^*)}\rangle| + \int_\Omega |\langle x^*, f_{\pi(x^*)} - f\rangle| d\mu < 2\varepsilon$$

Equivalently,

$$\lim_\pi \sup\left\{\int_\Omega |\langle x^*, f - f_\pi\rangle| d\mu : x^* \in B(X^*)\right\} = 0$$

It follows, that there exists in Π_Σ a sequence $\pi_1 \le \pi_2 \le \ldots$ such that

$$\lim_n \sup\left\{\int_\Omega |\langle x^*, f - f_{\pi_n}\rangle| d\mu : x^* \in B(X^*)\right\} = 0$$

and so, in particular

$$\lim_n \int_E \langle x^*, f_{\pi_n} \rangle d\mu = \langle x^*, \nu(E) \rangle$$

for each $x^* \in X^*$ and $E \in \Sigma$.

In this manner, if $g = \lim_n f_{\pi_n} \in \mathbf{P}(\mu, X)$ (resp. $\mathbf{P}(\mu, X^{**})$), then for each $E \in \Sigma$, we get the required equality

$$\nu(E) = \int_E g d\mu \qquad \diamondsuit$$

A similar martingale characterization can be given for the CRP [M. 1980]:

THEOREM 11.5. *X has the μ-CRP if and only if each bounded martingale in $\mathbf{P}(\mu, X)$ is Cauchy in the norm topology of $\mathbf{P}(\mu, X)$.*

The following however is open:

PROBLEM 33. For which X and (Ω, Σ, μ) each Cauchy martingale in $\mathbf{P}(\mu, X)$ is convergent?

This can be reformulated equivalently:

PROBLEM 33$'$. Which X have the property, that each μ–continuous X–valued measure of σ–finite variation and with norm relatively compact range has a Pettis μ–integrable density?

PROBLEM 34. Assume X has the property, that each X–valued μ–continuous measure of σ–finite variation and with norm relatively compact range has a Pettis μ–integrable density. Does X possess the μ–WRNP?

PROBLEM 35. Assume, that X has the W^{**}RNP or the CRP. Does there exist Y with the WRNP such that $X \subseteq Y$ isomorphically?

A lot of further problems concerning the WRNP, the W^{**}RNP and the CRP can be formulated yet, but we shall pose rather a different one, that is devoted to the Pettis differentiation of a single Banach space valued measure.

PROBLEM 36. Let X and (Ω, Σ, μ) be fixed. For each $E \in \Sigma_\mu^+$, denote by $\mathcal{A}_\mu(E)$ the average range of ν over E:

$$\mathcal{A}_\mu(E) = \left\{ \frac{\nu(F)}{\mu(F)} : F \subseteq E,\ F \in \Sigma_\mu^+ \right\}$$

What conditions has to satisfy the set $\mathcal{A}_\mu = \{\mathcal{A}_\mu(E) : E \in \Sigma_\mu^+\}$ in order to ensure the existence of a Pettis μ–integrable density of ν with respect to μ?

In the strongly measurable case the solutions are well known. The best reference is [D-U]. In the non–separable case no general solution is known.

We shall finish this section with investigation of measure taking values in a Banach space with Schauder basis.

THEOREM 11.6. *Suppose (e_n) is a basis in X and (e_n^*) is the associated sequence of coefficient functionals. Let $(f_n) \subset L_1(\mu)$ and a measure $\nu : \Sigma \to X$ be such that*

$$e_n^* \nu(E) = \int_E f_n d\mu \quad \text{for} \quad E \in \Sigma \text{ and } n \in \mathbf{N}\ .$$

Then the following three conditions are equivalent:

(i) ν has a Pettis μ–integrable density.

(ii) $\sum_{n=1}^\infty f_n e_n$ converges strongly μ–a.e.

(iii) $\sum_{n=1}^\infty f_n e_n$ converges weakly μ–a.e.

Either of them implies that

$$\nu(E) = P - \int_E \sum_{n=1}^\infty f_n e_n \, d\mu \quad \text{for } E \in \Sigma$$

Proof. Notice first, that if $f : \Omega \to X$ and $\nu : \Sigma \to X$ is a μ–continuous measure then, the set

$$W = \left\{ x^* \in X^* : x^* f \in L_1(\mu) \text{ and } x^* \nu(E) = \int_E x^* f \, d\mu \quad \text{for } E \in \Sigma \right\}$$

is weak* sequentially closed.

Indeed, it follows from the definition of W that

$$\int_E |x^*f|d\mu \leq \|x^*\|\,\|\nu\|(E) \quad \text{for } E \in \Sigma \text{ and } x^* \in W$$

where $\|\nu\|$ denotes the semivariation of ν. This – together with the Banach–Steinhaus theorem – gives the uniform integrability of any sequence (x_n^*f) such that $(x_n^*) \subset W$ and $x_n^*(x) \to x^*(x)$ for each $x \in X$. Hence, Vitali's convergence theorem gives

$$x^*f \in L_1(\mu) \text{ and } \int_E x_n^*f\,d\mu \to \int_E x^*f\,d\mu \text{ for each } E \in \Sigma .$$

This yields $x^* \in W$ proving the weak* sequential closeness of W.

To prove the implication (iii) $\to$ (i) put $f = \sum_{n=1}^{\infty} f_n e_n$ and $V = \{\Sigma_{n=1}^{p} a_n e_n^* : a_n \in \mathbf{R}, p \in \mathbf{N}\}$. Then V is weak* sequentially dense in X^* and $V \subseteq W$. Hence $X^* \subseteq W$.

To establish the implication (i) $\to$ (ii) assume that $\nu(E) = \int_E f d\mu$ for $E \in \Sigma$ where $f : \Omega \to X$. Then, by assumption, $e_n^* f = f_n$ μ–a.e. Hence, according to the definition of basis, $f = \sum_{n=1}^{\infty} f_n e_n$ holds μ–a.e. which yields (ii).

The implication (ii) $\to$ (iii) is clear. ◇

COROLLARY 11.1. (Dunford–Morse) *If X has a boundedly complete basis, then X has the Radon–Nikodym property.*

Theorem 11.6 can be used to get a simple proof of Rieffel's characterization [cf D-U] of measures having Bochner integrable densities [M, 1976 and L-M].

COROLLARY 11.2. (Rieffel). *Let ν be an X–valued μ–continuous measure defined on Σ. If for every $E \in \Sigma_\mu^+$ there exists $F \in \Sigma_\mu^+$ such that $F \subseteq E$ and $A_\mu(F)$ is norm relatively compact, then there exists a strongly measurable $f : \Omega \to X$ such that*

$$\nu(A) = P - \int_A f\,d\mu \quad \text{for each } A \in \Sigma .$$

Proof. Using the method of exhaustion it is easy to see that there is a closed separable subspace of X containing $\mathcal{A}_\mu(\Omega)$. Thus, we may assume that X itself is separable. In view of a Theorem of Banach and Mazur (cf. [H]) X can be considered as a subspace of $C[0,1]$.

Let (e_n) be an orthonormal basis of $C[0,1]$ and let (e_n^*) be the associated biorthogonal sequence. In order to prove the existence of a strongly measurable $f : \Omega \to C[0,1]$ such that

$$\langle x^*, \nu(E)\rangle = \int_E x^* f \, d\mu \quad \text{for } x^* \in C^*[0,1] \quad \text{and} \quad E \in \Sigma$$

it is sufficient, in view of Theorem 11.6, to prove the μ–a.e. convergence of the series $\sum_{n=1}^\infty f_n e_n$ where

$$\langle e_n^*, \nu(E)\rangle = \int_E f_n \, d\mu \quad \text{for } x^* \in C^*[0,1], n \in \mathrm{N} \text{ and } E \in \Sigma$$

To do it consider $F \in \Sigma_\mu^+$ with the norm relatively compact $\mathcal{A}_\mu(F)$. since (e_n) is a basis we see that

$$\frac{\nu(A)}{\mu(A)} = \sum_{n=1}^{\infty} \left(\frac{1}{\mu(A)} \int_A f_n \, d\mu \right) e_n$$

uniformly for $A \in \Sigma_\mu^+$ and $A \subseteq F$.

Hence, given $\varepsilon > 0$ there exists n_0 with

$$\left\| \int_A \Sigma_{n=m}^p f_n e_n \, d\mu \right\| \le \varepsilon \mu(A) \quad \text{for } A \subseteq F, \, A \in \Sigma \quad \text{and} \quad p > m \ge n_0 .$$

This yields $\|\Sigma_{n=m}^p f_n e_n\| \le \varepsilon$ μ–a.e. and so the series $\sum_{n=1}^\infty f_n e_n$ is a.e. convergent in the norm topology of C[0,1]. One can easily show that $f = \sum_{n=1}^\infty f_n e_n$ has almost all its values in X ◊

PROBLEM 37. Let X be a Banach space with basis. What property of the basis is necessary and sufficient for the WRNP (=RNP in this case) of X? Which Banach spaces possessing basis have the WRNP?

PROBLEM 38. Let $\nu : \Sigma \to X$ be a measure of finite variation and let (e_n) be a basis of X. Is there a result similar to that in Theorem 11.6

guaranteeing the norm relative compactness of $\nu(\Sigma)$? What property of (e_n) is equivalent to the relative norm compactness of the range of each X–valued measure of finite variation?

PROBLEM 39. Assume, that X has the property that each of its closed subspaces with basis has the CRP (the WRNP or the W^{**} RNP). Does X itself enjoy the same property?

12. The weak Radon–Nikodym property of conjugate Banach spaces.

In this chapter I present a proof of the following result totally describing the conjugate Banach spaces with the WRNP.

THEOREM 12.1. *X^* has the weak Radon–Nikodym property if and only if X contains no isomorphic copy of l_1.*

The necessity has been proved in [M–R] and the sufficiency in [Ja] and [M, 1979].

We shall begin the proof with the following

PROPOSITION 12.1. l_∞ does not have the WRNP

Proof. Let π_n be the dyadic partition of [0,1] into 2^n intervals and, let $\tilde{\pi}_n$ be the collection of all possible unions of elements taken from π_n. If (A_n) is an enumeration of $\cup_{n=1}^{\infty}\tilde{\pi}_n$, then clearly $\lim_n \lambda(A_n) = 0$. Define a measure $\nu : \mathcal{L} \to c_0 \subset l_\infty$ by setting

$$\nu(E) = (\lambda(E \cap A_n))$$

Then, $\nu(\mathcal{L})$ is a norm relatively compact subset of c_0, $\|\nu(E)\| \leq \lambda(E)$ for each $E \in \mathcal{L}$ and ν is without Pettis λ–integrable derivative in l_∞. Indeed, let $f : [0,1] \to l_\infty = l_1^*$ be a weak* density of ν with respect to λ. It means in particular, that if (e_n^*) is the standard biorthogonal sequence in l_∞, then

$$\lambda(E \cap A_n) = \langle e_n^*, \nu(E)\rangle = \int_E \langle e_n^*, f\rangle d\lambda$$

for each $n \in \mathbf{N}$.

But the sequence (χ_{A_n}) is pointwise dense in $\{0,1\}^{[0,1]}$. Thus, if χ_A is non–λ–measurable cluster point of (χ_{A_n}), then χ_A is λ–a.e. equal to a pointwise cluster point of $(\langle e_n^*, f\rangle)$. Such a point is of the form $\langle x^{***}, f\rangle$ for a functional $x^{***} \in X^{***}$. This means, that f is not weakly measurable and hence it cannot be a Pettis integrable density of ν with respect to λ. $\diamond$

DEFINITION 12.1. A uniformly bounded family $\mathcal{H}$ of real–valued functions on (Ω, Σ, μ) has the *Bourgain property* if for each $E \in \Sigma_\mu^+$ and each pair $a < b$ of reals, there is a finite collection $\mathcal{F} \subseteq \mathcal{P}(E) \cap \Sigma_\mu^+$ such, that for each function $h \in \mathcal{H}$ one can find $F \in \mathcal{F}$ with
$\inf\{h(\omega) : \omega \in F\} \geq a$ or $\sup\{h(\omega) : \omega \in F\} \leq b$.

The utility of this property lies in the following result:

PROPOSITION 12.2. *If $\mathcal{H}$ satisfies the Bourgain property, then each function in $\mathcal{H}$ is measurable and each function in the pointwise closure of $\mathcal{H}$ is the almost everywhere pointwise limit of a sequence from $\mathcal{H}$.*

Proof. It is easy to see that the Bourgain property of $\mathcal{H}$ yields the same property of the pointwise closure of $\mathcal{H}$. In order to prove the Proposition take $f \in \bar{\mathcal{H}}^{\tau_p}$ (τ_p – the topology of pointwise convergence) and an ultrafilter $\mathcal{U}$ on $\mathcal{H}$ which has f as a cluster point. Then, put for $E \in \Sigma_\mu^+$ and $\varepsilon > 0$

$$\mathcal{H}(E,\varepsilon) = \{h \in \mathcal{H} : \sup h(E) - \inf h(E) < \varepsilon\}$$

It follows, that if $E \in \Sigma_\mu^+$ then, there exists $F \in E \cap \Sigma_\mu^+$ with $\mathcal{H}(F,\varepsilon) \in \mathcal{U}$. Using Zorn's Lemma, we can find for each positive ε a maximal family $\mathcal{P}_\varepsilon$ of pairwise disjoint sets in Σ_μ^+ such, that $\mathcal{K}(F,\varepsilon) \in \mathcal{U}$ for each $F \in \mathcal{P}_\varepsilon$. It is obvious, that $\mu(\Omega \setminus \bigcup_{F \in \mathcal{P}_\varepsilon} F) = 0$. Moreover, if $\mathcal{R}_\varepsilon$ is the family of all finite subcollections of $\mathcal{P}_\varepsilon$, then

$$f \in \overline{\bigcap_{\mathcal{R} \in \mathcal{R}_\varepsilon} \bigcap_{E \in \mathcal{R}} \mathcal{H}(E,\varepsilon)}^{\tau_p}$$

and the set on the right hand side is a member of $\mathcal{U}$.

Now let for each $m \in \mathbf{N}$ the sequence $\{A_{m,n} : n \in \mathbf{N}\}$ be an enumeration of $\mathcal{P}_{1/m}$ and let

$$B = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{m,n}$$

We have $\mu(\Omega \backslash B) = 0$ and for an arbitrary $\omega_{m,n} \in A_{m,n}$, the sequence (h_m) defined by

$$h_m = \sum_{n=1}^{\infty} f(\omega_{m,n}) \chi_{A_{m,n}}$$

is on B uniformly convergent to f. Taking for each $m \in \mathbf{N}$ a function

$$f_m \in \bigcap_{k=1}^{m} \bigcap_{n=1}^{m} \mathcal{H}(A_{k,n}; 1/k)$$

such that

$$|f_m(\omega_{k,n}) - f(\omega_{k,n})| < 1/k$$

for each $1 \leq k, n \leq m$, we get a sequence $(f_m) \subseteq \mathcal{H}$ that is μ–a.e. convergent to f. This gives the required measurability and approximation.

$\diamond$

DEFINITION 12.2. Let S be a topological space and let η be a positive finite measure defined on a σ–algebra $\mathcal{B}$ containing all the Borel subsets of S. η is said to be *hereditary supported*, if for each $B \in \mathcal{B}_\eta^+$ there exists $A \in \mathcal{P}(B) \cap \mathcal{B}_\eta^+$, such that
(*) for each open U, we have $U \cap A = \emptyset$ or $U \cap A \in \mathcal{B}_\eta^+$.
We shall write $A = \operatorname{supp}(\eta|A)$.

It is easily seen, that if η is hereditary supported, then for each $B \in \mathcal{B}_\eta^+$ there exists $A \in \mathcal{P}(B) \cap \mathcal{B}_\eta^+$ such that (*) is satisfied and $\eta(A) = \eta(B)$.

The following are two examples of hereditary supported measures:
1) Each Radon measure.
2) Let (Ω, Σ, μ) be complete and let ρ be a lifting on Σ. Let $\mathcal{C}_\rho$ be the topology defined by taking as its basis, the family

$\{\rho(A)\backslash N : A \in \Sigma, N \in \mathcal{N}(\mu)\}$ (cf [I-T], p. 59). Then, μ is hereditary supported with respect to $\mathcal{C}_\rho$.

PROPOSITION 12.2. *Let S be a topological space and let $\mathcal{H}$ be a uniformly bounded family of real-valued continuous functions on S. Moreover, let η be a hereditary supported measure on a σ-algebra $\mathcal{B} \supseteq \mathcal{B}o(S)$. Then, if $\mathcal{H}$ does not contain any sequence equivalent to the standard unit vector basis of l_1, then $\mathcal{H}$ has the Bourgain property.*

Proof. Suppose, that $\mathcal{H}$ does not have the Bourgain property. This means, that there exists $T \in \mathcal{B}_\eta^+$ such that $T = \mathrm{supp}(\eta|T)$ and $a < b$ such, that for each finite collection $\mathcal{R} \subseteq \mathcal{P}(T) \cap \mathcal{B}_\eta^+$ there is $f \in \mathcal{H}$ with $\inf f(R) < a$ and $\sup f(R) > b$, for every $R \in \mathcal{R}$.

Put $\mathcal{R} = \{T\}$. It follows, that there is $f_1 \in \mathcal{H}$ such that if we define $A_{11}, A_{12} \in \mathcal{B}$ by

$$A_{11} = \{s \in T : f_1(s) < a\}$$

$$A_{12} = \{s \in T : f_1(s) > b\}$$

then $A_{11}, A_{12} \in \mathcal{B}_\eta^+$.

We shall now construct inductively a collection $\{A_{n,m} : m = 1, \ldots, 2^n;\ n \in \mathbb{N}\}$ of sets from $\mathcal{B}_\eta^+$ and a sequence $(f_n) \subseteq \mathcal{H}$ satisfying the following properties:

$$A_{n+1,2m-1} \cup A_{n+1,2m} \subseteq A_{n,m}$$

$$f_{n+1}(s) < a \quad \text{if } s \in A_{n+1,2m-1}$$

$$f_{n+1}(s) > b \text{ if } s \in A_{n+1,2m}$$

Assume, we have already constructed $\{f_m : m = 1, \ldots, k\}$ and $\{A_{n,m} : m = 1, \ldots, 2^n; n = 1, \ldots, k\}$. By the assumption, we can find for each $m \in \{1, \ldots, 2^k\}$ a set $T_{k,m} \in \mathcal{B}_\eta^+ \cap \mathcal{P}(A_{k,m})$ such, that $T_{k,m} = \mathrm{supp}(T_{k,m})$. Moreover, there is $f_{k+1} \in \mathcal{H}$ with

$$\inf f_{k+1}(T_{k,m}) < a \quad \text{and} \quad \sup f_{k+1}(T_{k,m}) > b$$

for every $m \in \{1, \ldots, 2^k\}$.

Put now

$$A_{k+1,2m-1} = \{s \in T_{k,m} : f_{k+1}(s) < a\}$$

$$A_{k+1,2m} = \{s \in T_{k,m} : f_{k+1}(s) > b\}$$

It follows that $A_{k+1,m} \in \mathcal{B}_\eta^+$ for every $m \in \{1,\ldots,2^{k+1}\}$. Rosenthal's argument [Ro] shows, that the sequence (f_n) is equivalent to the standard basis of l_1 in the sup norm. ◇

As particular cases of the above proposition we get the following results:

COROLLARY 12.1. *Let ρ be a lifting on $L_\infty(\Omega,\Sigma,\mu)$ and let $\mathcal{H}$ be a uniformly bounded family of real-valued functions defined on Ω and such, that $\mathcal{H} = \rho(\mathcal{H})$. If $\mathcal{H}$ does not contain any sequence equivalent to the standard basis of l_1, then $\mathcal{H}$ has the Bourgain property.*

COROLLARY 12.2. *Let X be a Banach space containing no isomorphic copy of l_1. If μ is a complete finite Radon measure on $B(X^*)$ equipped with the weak* topology, then the family $\mathcal{H} = B(X)$ has the Bourgain property.*

Now we are ready to present a proof of Theorem 12.1.

The proof of the sufficiency:
Let $\nu : \Sigma \to X$ be a measure satisfying the condition

$$||\nu(E)|| \leq \mu(E)$$

for all $E \in \Sigma$, and let ρ be a lifting on $L_\infty(\mu)$.

By the weak* Radon–Nikodym property of X^* there exists $f : \Omega \to X^*$ such that

$$\rho(\langle x, f\rangle) = \langle x, f\rangle \quad \text{for each } x \in X, \text{and}$$

$$\langle x, \nu(E)\rangle = \int_E \langle x, f\rangle d\mu \quad \text{for each } x \in X \text{ and } E \in \Sigma\ .$$

Fix $\varepsilon > 0, E \in \Sigma$ and $x^{**} \in B(X^{**})$. Then put

$$\mathcal{H} = \{\langle x, f\rangle : x \in B(X),\ |\langle x^{**} - x, \nu(E)\rangle| < \varepsilon\}$$

Since X contains no isomorphic copy of l_1, it follows that $\mathcal{H}$ contains no sequence equivalent to the standard basis of l_1, in the sup norm. In view of Corollary 12.1, $\mathcal{H}$ has the Bourgain property.

It follows from the Goldstine theorem, that $\langle x^{**}, f\rangle$ is in the pointwise closure of $\mathcal{H}$ and so there are $x_n \in X$, $n \in \mathbf{N}$, such that $\langle x_n, f\rangle \in \mathcal{H}$ for each n and $\lim_n \langle x_n, f\rangle = \langle x^{**}, f\rangle$ μ–a.e.

Hence we get

$$|\langle x^{**}, \nu(E)\rangle - \int_E \langle x^{**}, f\rangle d\mu| \leq 2\,\varepsilon$$

by the Lebesgue Convergence Theorem, and this proves the WRNP of X^*.

The proof of the necessity:

Assume, that X contains a subspace Y that is isomorphic to l_1. Then, l_∞ is isomorphic to $X^*/Y^\perp$. Denote the isomorphism by R. Moreover, let T be an isomorphic embedding of $L_1[0,1]$ into l_∞. Then $RT : L_1[0,1] \to X^*/Y^\perp$ is an isomorphism.

By the lifting property of $L_1[0,1]$, there is an operator $S : L_1[0,1] \to X^*$ such that $RT = Q \circ S$, where Q is the quotient mapping of X^* onto $X^*/Y^\perp$.

Let $\nu : \mathcal{L} \to l_\infty$ be the measure constructed in the proof of Proposition 12.1. Then $\kappa : \mathcal{L} \to X^*$ given by $\kappa(E) = S\chi_E$ for each $E \in \mathcal{L}$ is an X^{**}–valued λ–continuous measure of finite variation satisfying for each $E \in \mathcal{L}$ the equality $Q\kappa(E) = R\nu(E)$. If $g : [0,1] \to X^*$ were such that

$$\langle x^{**}, \kappa(E)\rangle = \int_E \langle x^{**}, g\rangle d\lambda$$

for each $E \in \mathcal{L}$, $x^{**} \in X^{**}$, then $R^{-1}Qg$ would be the Pettis integrable derivative of ν, what is impossible by Proposition 12.1. ◇

In a similar way we can prove the following result of Haydon [Ha] (which in fact is a particular case of Theorem 12.1):

THEOREM 12.2. *If X does not contain any isomorphic copy of l_1, then each element of X^{**} is measurable as a function on $B(X^*)$ with respect to any complete weak* Radon measure μ defined on $\mathcal{B}o(B(X^*), weak^*)$, and the identity function on $B(X^*)$ is Pettis μ–integrable.*

Proof. According to Proposition 4.1 for each μ–measurable $E \subseteq B(X^*)$ there is $\nu(E) \in B(X^*)$ such that

$$\langle x, \nu(E) \rangle = \int_E \langle x^*, x \rangle d\mu$$

for every $x \in X$.

From now on we can copy the proof of Theorem 12.1, applying Corollary 12.2 instead of Corollary 12.1. ◇

Since the canonical projection of X^{***} onto X^* is weak*–continuous, we get the following:

PROPOSITION 12.3. *A dual Banach space has the WRNP if and only if it has the W**RNP.*

We shall apply now Theorem 12.1 to present a condition that is sufficient for the Pettis integrability at least for a certain class of Banach spaces.

DEFINITION 12.3. A set $W \subseteq X$ is weakly precompact if each bounded sequence in W has a weakly Cauchy subsequence.

LEMMA 12.1. *Let X be a separable Banach space and let $f : \Omega \to X^*$ be a weak*-scalarly μ–bounded and weak*–scalarly μ–measurable function. Assume, that for each $\delta > 0$ there is $E \in \Sigma$ with $\mu(\Omega \backslash E) < \delta$ and such that the set $\{\langle f, x \rangle \chi_E : \|x\| \le 1\}$ is weakly precompact in $L_\infty(\mu)$. Then $f \in \mathbf{P}(\mu, X^*)$.*

Proof. Fix $\delta > 0$ and $E \in \Sigma$ as above. Define $T : X \to L_\infty(\mu)$ by $Tx = \langle f, x \rangle \chi_E$ and observe that the assumption guarantees the weak precompactness of $T(B(X))$. According to [DFJP] T factors through a Banach space without any copy of l_1. It is now a consequence of Theorem

12.1, that $T^* : L_\infty(\mu)^* \rightarrow X^*$ factors through a space possessing the WRNP. In particular, the same holds for $T^* : L_1(\mu) \rightarrow X^*$. Hence there is $g \in \mathbf{P}(\mu, X^*)$ such that

$$T^*\chi_F = P - \int_F g \, d\mu$$

for each $F \in \Sigma$.

In particular,

$$\langle \chi_F, Tx \rangle = \int_F \langle g, x \rangle d\mu$$

and so for each $x \in X$

$$\langle f, x \rangle \chi_E = \langle g, x \rangle \quad \mu - a.e.$$

The separability of X yields $f\chi_E = g$ μ–a.e. and so $f\chi_E \in \mathbf{P}(\mu, X^*)$. The boundedness of f implies $f \in \mathbf{P}(\mu, X^*)$. ◇

DEFINITION 12.4. Let K be a compact space. A function $f : K \rightarrow \mathbf{R}$ is said to be *universally measurable* if it is measurable with respect to the completion of each Radon measure defined on Borel subsets of K.

THEOREM 12.3. *Let X be a separable Banach space and let K be a compact space. If $f : K \rightarrow X^*$ is scalarly bounded and scalarly universally measurable then $f \in \mathbf{P}(\mu, X^*)$ for each Radon μ.*

Proof. Let $\delta > 0$ and μ be a Radon probability on K. Since X is separable, there exists a compact set $L \subseteq K$ such that $\mu(K \backslash L) < \delta$ and $\langle f, x \rangle$ is continuous on L for each $x \in X$. Let

$$A = \{\langle f, x \rangle|_L : ||x|| \leq 1\}$$

and $M_r(L)$ be the set of all real–valued measurable functions on L equipped with the pointwise convergence topology. As f is universally measurable, the set A is relatively compact in $M_r(L)$. According to a theorem of Bourgain–Fremlim–Talagrand [BFT, Theorem 2F], every sequence in A has a pointwise convergent subsequence and so, it is weakly precompact in

$C(L)$. A direct application of Rosenthal's theorem [Ro] says, that A contains no copy of the standard unit vector basis of l_1. Since, the canonical embedding of $C(L)$ into $L_\infty(K,\mu)$ is a contraction, the set $\{\langle f,x\rangle\chi_L : \|x\| \leq 1\}$ contains no copy of the l_1–basis in the $L_\infty(K,\mu)$–norm either. Thus, it is weakly precompact and Lemma 12.1 completes the proof. ◇

PROBLEM 40. Which non–separable Banach spaces have the property, that given compact K and a scalarly bounded and scalarly universally measurable $f : K \to X^*$, the function f is μ–Pettis integrable for every Radon μ?

The function constructed in Example 7.1 shows that in general even the weak measurability with respect to the Borel sets is not sufficient for the Pettis integrability.

We shall finish the considerations with proving that the function $f : [0,1] \to L_\infty(\lambda)$ given by $f(t) = \chi_{[0,t)}$ (and considered already in Ex. 7.1) is Pettis integrable with respect to any Borel measure on [0,1]. Indeed, if $\eta \in L^*_\infty(\mu)$ then $\eta = \eta^+ - \eta^-$, where η^+, η^- are taken from the Jordan decomposition of η.

If $t \in [0,1]$ then,

$$\langle \eta, f(t)\rangle = \int_0^1 f(t)\,d\eta = \int_0^1 f(t)\,d\eta^+ - \int_0^1 f(t)\,d\eta^- =$$

$$= \int_0^1 \chi_{[0,t)}\,d\eta^+ - \int_0^1 \chi_{[0,t)}\,d\eta^- = \eta^+([0,t)) - \eta^-([0,t))$$

Thus, $\langle \eta, f\rangle$ is a difference of two monotonic functions of t, and so it is Borel measurable. The conclusion follows from Theorem 12.3 ◇

A similar result holds for the function presented in Example 3.3.

Comments

The examples are mainly taken from [D-U, 1977] and [Ta, 1984]. 40 problems are formulated. Some of them are taken from the literature (mainly [E] and [Ta, 1984]) but some are formulated for the first time (at least I never read about them). The bibliography is far of being compete. In fact only papers directly connected with investigated topics are quoted. The same can be said about the whole notes. Some important topics have not been touched even (e.g. local theory of sets with the WRNP, and several geometric properties of Banach spaces).

Chapter 2. Theorem 2.2 is due to Tortrat [T].

Chapter 3. Proposition 3.1 is taken from [M, 1979]. It is a consequence of a folk result concerning the existence of a dominating function of a pointwise bounded family of real–valued functions. Theorem 3.3 comes from [E].

Chapter 4. The Pettis integral was introduced in [P] where also its basic properties were established. The σ–finiteness of ν_f in Th. 4.1 is due to Rybakov [R, 1971]. Theorem 4.2 is a variation of a result from [D-F].

Chapter 5. Theorem 5.1 is due to Brooks [Br]. Theorem 5.3 was proved by Uhl [U]. Theorem 5.4 was proved independently by Diestel [D, 1973] and Dimitrov [Dm].

Chapter 6. The notion of the core is due to Geitz [G]. He also proved Theorem 6.1 for perfect μ. Its final form was proved by Talagrand [Ta, 1984]. Also from [Ta, 1984] theorems 6.2 and 6.3 are taken. Lemma 6.5 and Theorem 6.4 were independently proved by Drewnowski and Musiał. Proposition 6.3 comes from Huff [Hu].

Chapter 7. Example 7.1 is taken from [Ph]. The notion of the PIP was introduced and investigated by Edgar [E]. Also Prop. 7.2 comes from [E].

Chapter 8. Theorems 8.1 and 8.2 were first proved in [G] for perfect measures, and then, they were generalized (independently and with different proofs) by Talagrand [Ta, 1984] and Musiał [M, 1985] to the case of arbitrary measures. The proofs are taken from [M, 1985]. Theorem 8.4 can be found in [M, 1987].

Chapter 9. Theorem 9.1 is taken from [M, 1980] Theorem 9.2 is due to Stegall [F-T]

Chapter 10. Theorem 10.1 is taken from [M, 1985] but it was also independently proved in [Ta, 1984]. The PES property was defined in [D-U, 1983].

Chapter 11. The weak Radon–Nikodym property was introduced by Musiał [M, 1979]. The W^{**} RNP was introduced by Janicka [M, 1980]. Theorem 11.3 was proved by Musiał in [M, 1982]. The CRP was introduced in [M, 1980], where also Proposition 11.1. Theorem 11.4 and Theorem 11.5 can be found.

Chapter 12. Theorem 12.3 was proved in [RSU]. The Bourgain property was introduced by Bourgain (unpublished) (cf. [RS]). Its basic properties were also proved by Bourgain.

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