RANDOM FRACTALS (*)

by SIEGFRIED GRAF (**)

Contents

(*) Lectures presented at the School on Measure Theory and Real Analysis.
Grado (Italy), October 14–25, 1991.

(**) Address of the Author: Fakultät für Mathematik und Informatik, Universität Passau, Innstr. 33, D-W-8390 Passau (Germany).

1. Introduction.

It is a purpose of these lectures to provide an introduction to the theory of random fractals. The basic theme connecting the different sections is that of statistical self-similarity.

In Section 2 we will consider a randomization of the construction of the classical Cantor set.

Section 3 describes Mandelbrot's percolation process, a randomized Cantor type construction in the square. It was invented by Mandelbrot to model certain aspects of turbulence.

Section 4 investigates a general notion of statistically self-similar sets, generalizing the random sets constructed in the first two sections.

Section 5 reviews results connected with random sets derived from Brownian motion.

Section 6 contains hints at the literature about other types of random fractals. That some of the interesting contributions to the field (for instance by Dekking, Falconer, Grimmett and Peyriére) are not mentioned is due to the limited amount of time which was available for the lectures and for the preparation of this manuscript.

2. Random Cantor sets.

In this section we will study a specific statistically self-similar construction. It resembles that of the classical middle third Cantor set, except that the lengths of the intervals at each stage are random.

2.1 CONSTRUCTION.

Let $\Delta = \{(t_1,t_2) \in [0,1]^2 | t_1 + t_2 \leq 1, t_1 > 0, t_2 > 0\}$ and let ν be a probability measure on the Borel field $\mathcal{F}$ of Δ.

The construction proceeds as follows: Choose a point $(t_1,t_2) \in \Delta$ at random with respect to ν. Set $J_1 = [0,t_1]$, $J_2 = [1-t_2,1]$. Next choose two points (t_1^1,t_2^1) and (t_1^2,t_2^2) at random with respect to ν and independently from each other and independently of (t_1,t_2). Set $J_{11} = [0,t_1^1 t_1]$, $J_{12} = [(1-t_2^1)t_1,t_1]$ and $J_{21} = [(1-t_2),(1-t_2)+t_1^2 t_2]$, $J_{22} = [1-t_2^2 t_2,1]$. If we continue this process we obtain a family $(J_\sigma)_\sigma$ of intervals, where σ runs through the set $\{1,2\}^*$ of all finite sequences of 1s and 2s. Then

$$\bigcap_{n\in\mathbf{N}} \bigcup_{\sigma \text{ has length } n} J_\sigma$$

is a typical realization of the random object we want to consider.

Now we will give a formal description of our construction.

2.1.1 DEFINITION.

Let $N \in \mathbf{N}, N \geq 1$ be given. $\{1,\ldots,N\}^o$ stands for the set containing the empty sequence $\emptyset$ as its only element. Define

$$\{1,\ldots,N\}^* = \bigcup_{n\in\mathbf{N}} \{1,\ldots,N\}^n .$$

If $\sigma = (\sigma_1,\ldots,\sigma_n)$ and $\tau = (\tau_1,\ldots,\tau_m)$ are in $\{1,\ldots,N\}^*$ then $|\sigma| = n$ is the length of σ and $\sigma * \tau = (\sigma_1,\ldots,\sigma_n,\tau_1,\ldots,\tau_m)$ is the *juxtaposition* of σ and τ ($\emptyset * \sigma = \sigma$ and $\sigma * \emptyset = \sigma$).

Let $\{1,\ldots,N\}^{\mathbf{N}}$ carry the product of the discrete topology on $\{1,\ldots,N\}$.

For $\sigma \in \{1,\ldots,N\}^* \cup \{1,\ldots,N\}^{\mathbf{N}}$ and $n \in \mathbf{N}$ with $n \leq |\sigma|$ if σ is finite let

$$\sigma_{|n} = (\sigma_1,\ldots,\sigma_n)$$

be the restriction of σ to its first n entries. For $\sigma \in \{1,\ldots,N\}^*$ and $\tau \in \{1,\ldots,N\}^* \cup \{1,\ldots,N\}^{\mathbf{N}}$ we say that σ is preceding τ and write $\sigma < \tau$ if $|\sigma| \leq |\tau|$ for τ finite and

$$\sigma = (\sigma_1,\ldots,\sigma_{|\sigma|}) = (\tau_1,\ldots,\tau_{|\sigma|}) .$$

In this section D will denote $\{1,2\}^*$. We consider the product topology and the product σ–field (= Borel σ–field) $\mathcal{F}^D$ on the space Δ^D. By ν^D we denote the product measure on Δ^D. Then $(\Delta^D, \mathcal{F}^D, \nu^D)$ is a probability space. The elements of Δ^D can be considered as binary trees whose branching points are labelled with points from Δ. The elements of Δ^D will be denoted by $\omega = (t^\sigma)_{\sigma\in D}$, where $t^\sigma = (t_1^\sigma, t_2^\sigma) \in \Delta$.

2.1.2 The Compact Set Corresponding to a Tree From Δ^D.

Given an element $\omega = (t^\sigma)_{\sigma\in D}$ of Δ^D, define

$$J_\emptyset(\omega) = [0,1]$$

and if $J_\sigma(\omega) = [a,b]$ has been defined then let

$$J_{\sigma*1}(\omega) = [a, a + t_1^\sigma(b-a)] ,$$

$$J_{\sigma*2}(\omega) = [a + (1 - t_2^\sigma)(b-a), b] .$$

Set

$$K_n(\omega) = \bigcup_{\sigma\in\{1,2\}^n} J_\sigma(\omega)$$

and

$$K(\omega) = \bigcap_{n\in\mathbb{N}} K_n(\omega) .$$

Then $K(\omega)$ is obviously compact and non–empty. It is called *the compact set corresponding to the tree* ω.

2.1.3 Proposition.

Let $\mathcal{K}([0,1])$ be the space of all non–empty compact subsets of $[0,1]$ with the Borel field induced by the Hausdorff metric. Then the map ψ : $\Delta^D \to \mathcal{K}([0,1]), \omega \to K(\omega)$ is measurable, i.e. a random variable.

The proof will be left as an exercise.

2.1.4 LEMMA.

For $\omega = (t^\sigma)_{\sigma\in D} \in \Delta^D$ *and* $\sigma \in D$ *let* $l_\sigma(\omega) = t^{\emptyset}_{\sigma_1} \cdot t^{\sigma_1}_{\sigma_2} \cdot t^{\sigma_1\sigma_2}_{\sigma_3} \cdot \ldots t^{\sigma|(|\sigma|-1)}_{|\sigma|}$. *Set*

$$\Omega_\infty = \{\omega \in \Delta^D | \lim_{n\to\infty} \max_{\sigma\in\{1,2\}^n} l_\sigma(\omega) = 0\} .$$

Then $\Omega_\infty \in \mathcal{F}^D$ *with* $\nu^D(\Omega_\infty) = 1$.

Proof. We will omit the proof that $\Omega_\infty \in \mathcal{F}^D$. Since $(\max_{\sigma\in\{1,2\}^n} l_\sigma(\omega))_{n\in\mathrm{N}}$ is non–increasing we have $\Omega\backslash\Omega_\infty = \bigcup_{k\in\mathrm{N}} \bigcap_{n\in\mathrm{N}} \Omega_{k,n}$ with

$$\Omega_{k,n} = \{\omega \in \Delta^D | \max_{\sigma\in\{1,2\}^n} l_\sigma(\omega) \geq \frac{1}{k}\} ,$$

hence $\Omega_{k,n+1} \subset \Omega_{k,n}$.

By Čebyshev's inequality

$$\nu^D(\Omega_{k,n}) = \nu^D(\{\omega \in \Delta^D | \max_{\sigma\in\{1,2\}^n} l_\sigma(\omega) \geq \frac{1}{k}\}$$
$$\leq k \int \max_{\sigma\in\{1,2\}^n} l_\sigma(\omega) d\nu^D(\omega) .$$

The last integral can be estimated as follows

$$\int \max_{\sigma\in\{1,2\}^n} l_\sigma(\omega) d\nu^D(\omega) \leq \int \Sigma_{\sigma\in\{1,2\}^n} l_\sigma(\omega) d\nu^D(\omega)$$
$$= (\int (t_1 + t_2) d\nu(t_1, t_2))^n .$$

The last equality can be proved by induction on n. Since $0 < \int (t_1 + t_2) d\nu(t_1, t_2) < 1$ it follows that $\nu^D(\bigcap_{n\in\mathrm{N}} \Omega_{k,n}) = 0$. Therefore, the lemma is proved.

2.1.5 THEOREM.

For ν^D*-a.e.* $\omega \in \Delta^D$ *the set* $K(\omega)$ *is homeomorphic to* $\{1,2\}^{\mathrm{N}}$, *i.e. a topological Cantor set.*

Proof. Let Ω_∞ be as in Lemma 2.1.4. For $\omega \in \Omega_\infty$ define $\pi_\omega : \{1,2\}^{\mathbf{N}} \to K(\omega)$ by $\{\pi_\omega(\eta)\} = \bigcap_{n\in\mathbf{N}} J_{\eta|n}(\omega)$. By the definition of Ω_∞, $\lim_{n\to\infty} \operatorname{diam}(J_{\eta|n}(\omega)) = \lim_{n\to\infty} l_{\eta|n}(\omega) = 0$, hence $\bigcap_{n\in\mathbf{N}} J_{\eta|n}(\omega)$ is a singleton. It is also easy to check that π_ω is a continuous bijection, hence a homeomorphism. The details are left as an excercise. Thus, Lemma 2.1.4 implies the theorem.

2.1.6 DEFINITION.

The random variable $\psi : \Delta^D \to \mathcal{K}([0,1]), \omega \to K(\omega)$ together with a probability ν on Δ is called a *ν–random Cantor set.* For every outcome ω the set $K(\omega)$ is called the *realization* of the ν–random Cantor set corresponding to ω. The image measure $\nu^D \circ \psi^{-1}$ on $\mathcal{K}([0,1])$ is called the distribution of the ν–random Cantor set w.r.t. ν^D and is denoted by P_ν.

2.1.7 EXAMPLES.

a) Let ν be the Dirac measure $\varepsilon_{(\frac{1}{3},\frac{1}{3})}$ in the point $(\frac{1}{3},\frac{1}{3}) \in \Delta$. Then P_ν–a.e. K equals the classical middle third Cantor set.
b) For ν the normalized Lebesgue measure on Δ the corresponding "typical" realization will be investigated later.
c) Let $f : \Delta \to \mathbf{R}_+$ be defined by

$$f(t_1,t_2) = \frac{1}{2\pi} 1_{[0,\frac{1}{2}]\times[0,\frac{1}{2}]}(t_1,t_2)(1/(t_1t_2(1-t_1-t_2)^3)^{1/2}) .$$

Let $\nu = f \cdot \lambda^2$, where λ^2 is two dimensional Lebesgue measure. The ν–random Cantor set is known to probabilists as the zero–set of Brownian bridge.

2.2 STATISTICAL SELF–SIMILARITY.

2.2.1 DEFINITION.

Let ν be a probability on Δ. A probability measure P on $\mathcal{K}([0,1])$ is called *ν–(statistically) self–similar* if P is the image of $\nu \otimes P \otimes P$ with respect to the map $T : \Delta \times \mathcal{K}([0,1] \times \mathcal{K}([0,1]) \to \mathcal{K}([0,1])$, where $T((t_1,t_2),K_1,K_2) = t_1K_1 \cup [(1-t_2) + t_2K_2]$.

2.2.2 THEOREM.

P_ν is the unique ν–self–similar probability on $\mathcal{K}([0,1])$.

Proof. We will omit the uniqueness part of the proof. To prove that P_ν is ν–self–similar define $j : \Delta \times \Delta^D \to \Delta^D$ by $j(t,(t^\sigma)_{\sigma\in D},(s^\sigma)_{\sigma\in D}) = (u^\sigma)_{\sigma\in D}$, where

$$u^\emptyset = t\,,$$
$$u^{1*\sigma} = t^\sigma\,,$$
$$u^{2*\sigma} = s^\sigma\,,$$

for every $\sigma \in \{1,2\}^*$, and note that $\nu \otimes \nu^D \otimes \nu^D \circ j^{-1} = \nu^D$. Moreover observe that the following diagram commutes:

$$\begin{array}{ccc} \Delta \times \Delta^D \times \Delta^D & \xrightarrow{j} & \Delta^D \\ \Big\downarrow id_\Delta \times \psi \times \psi & & \Big\downarrow \psi \\ \Delta \times \mathcal{K}([0,1] \times \mathcal{K}([0,1]) & \xrightarrow[T]{} & \mathcal{K}([0,1]) \end{array} .$$

Since $\nu^D \circ \psi^{-1} = P_\nu$, $\nu \otimes \nu^D \otimes \nu^D \circ (id_\Delta \times \psi \times \psi)^{-1} = \nu \otimes P_\nu \otimes P_\nu$ this proves

$$\nu \otimes P_\nu \otimes P_\nu \circ T^{-1} = P_\nu\,.$$

2.2.3 LEMMA.

There exists a unique $\alpha \in (0,1)$ such that

$$\int (t_1^\alpha + t_2^\alpha)\,d\nu(t_1,t_2) = 1\,.$$

Proof. The map $g : \mathbf{R}_+ \to \mathbf{R}$, $\beta \to \int (t_1^\beta + t_2^\beta)\,d\nu(t_1,t_2)$ is continuous and (strictly) decreasing. Since $g(0) = 2$ and $g(1) < 1$ the lemma follows from the intermediate value theorem.

2.2.4 DEFINITION.

The α in the above lemma is called the *similarity dimension* of the ν–random Cantor set.

2.2.5 EXAMPLES.

a) In the case of the classical Cantor set, i.e. for $\nu = \varepsilon_{(\frac{1}{3},\frac{1}{3})}$, we have

$$1 = \int (t_1^\alpha + t_2^\alpha)\, d\nu(t_1,t_2) = (\frac{1}{3})^\alpha + (\frac{1}{3})^\alpha ,$$

hence $\alpha = \frac{\log 2}{\log 3}$.

b) If ν is normalized Lebesgue measure then

$$\begin{aligned}
1 &= \int (t_1^\alpha + t_2^\alpha)\, d\nu(t_1,t_2) \\
&= 2 \int_0^1 \int_0^{1-t_1} (t_1^\alpha + t_2^\alpha)\, dt_2\, dt_1 \\
&= 2 \int_0^1 [(1-t_1)t_1^\alpha + \frac{1}{\alpha+1} t_2^{\alpha+1} |_0^{1-t} dt_1 , \\
&= 2 \int_0^1 (t_1^\alpha - t_1^{\alpha+1} + \frac{1}{\alpha+1}(1-t_1)^{\alpha+1})\, dt_1 \\
&= 2[\frac{1}{\alpha+1} - \frac{1}{\alpha+2} + \frac{1}{(\alpha+1)(\alpha+2)}] \\
&= 2 \frac{\alpha+2-(\alpha+1)+1}{(\alpha+1)(\alpha+2)}
\end{aligned}$$

implies

$$(\alpha+1)(\alpha+2) = 4 ,$$

i.e. $\alpha^2 + 3\alpha + 2 = 4$, or $(\alpha + \frac{3}{2})^2 = \frac{17}{4}$, hence $\alpha = \frac{1}{2}(\sqrt{17} - 3)$.

c) In the case of the zero–set of Brownian bridge, i.e. if ν is as in Example 2.1.7c), we will show that $\alpha = \frac{1}{2}$. We have

$$1 = \frac{1}{2\pi} \int_0^{1/2} \int_0^{1/2} (t_1^\alpha + t_2^\alpha)(1/(t_1 t_2 (1 - t_1 - t_2)^3)^{1/2} dt_1 dt_2 .$$

By symmetry the right hand side equals

$$\frac{1}{\pi}\int_0^{1/2}\int_0^{1/2} t_1^{\alpha}/(t_1t_2(1-t_1-t_2)^3)^{1/2}dt_1dt_2 .$$

For $\alpha = \frac{1}{2}$ this last expression turns to

$$\frac{1}{\pi}\int_0^{1/2}\int_0^{1/2} 1/(t_2(1-t_1-t_2)^3)^{1/2}dt_1dt_2$$
$$(u = 1-(t_1+t_2),\ t = t_2)$$
$$= \frac{1}{\pi}\int_0^{1/2}\int_{\frac{1}{2}-t}^{1-t} 1/(t^{1/2}u^{3/2})du\,dt$$
$$= \frac{2}{\pi}\int_0^{1/2} t^{-1/2}((\frac{1}{2}-t)^{-1/2}-(1-t)^{-1/2})dt$$
$$= \frac{2}{\pi}(\int_0^{1/2} t^{-1/2}(\frac{1}{2}-t)^{-1/2}dt - \int_0^{1/2} t^{-1/2}(1-t)^{-1/2}dt)$$
$$(s = 1-2t)$$
$$= \frac{2}{\pi}(\int_0^{1}(1-s)^{-1/2}s^{-1/2}ds - \int_0^{1/2} t^{-1/2}(1-t)^{-1/2}dt)$$
$$= \frac{2}{\pi}\int_{1/2}^{1}(s(1-s))^{-1/2}ds$$
$$(t = 2s-1)$$
$$= \frac{2}{\pi}\int_0^{1}(t^2-1)^{-1/2}dt$$
$$= 1$$

2.3 A Natural Random Measure Corresponding to a ν–Random Cantor Set.

As before ν is a probability on Δ and α is the similarity dimension of the ν–random Cantor set. For $\sigma \in \{1,2\}^*$ the random variable l_σ is as defined in Lemma 2.1.4.

2.3.1 DEFINITION.

For $n \in \mathbf{N}$ let $X_n : \Delta^D \to \mathbf{R}$ be defined by

$$X_n(\omega) = \sum_{\sigma \in \{1,2\}^n} l_\sigma^\alpha(\omega)$$

and let $\mathcal{F}_0 = \{\emptyset, \Delta^D\}$, and, for $n \geq 1$, let $\mathcal{F}_n$ be the σ–field generated by the canonical projections $g_\sigma : \Delta^D \to \Delta$, where $|\sigma| \leq n-1$.

2.3.2 THEOREM (Mauldin–Williams [20]).

$(X_n)_{n \in \mathbf{N}}$ is an L^p–bounded martingale w.r.t. $(\mathcal{F}_n)_{n \in \mathbf{N}}$ for every $p \in [1, \infty)$.

Proof. See, for instance, Graf [13], p. 378/379.

2.3.3 REMARKS AND DEFINITION.

Let $L = \{\omega \in \Delta^D | \lim_{n \to \infty} X_n(\omega) \text{ exists in } \mathbf{R}\}$. By the martingale convergence theorem we have $\nu^D(L) = 1$. Let $X : \Delta^D \to \mathbf{R}$ be a random variable with $X(\omega) = \lim_{n \to \infty} X_n(\omega)$ for every $\omega \in L$. For $\sigma \in D$ define $\varphi_\sigma : \Delta^D \to \Delta^D$ by

$$\varphi_\sigma((t^\tau)_{\tau \in D}) = (t^{\sigma * \tau})_{\tau \in D} .$$

Let $L_\sigma = \varphi_\sigma^{-1}(L)$ and $M = \bigcap_{\sigma \in D} L_\sigma$. Then $\nu^D(M) = 1$. Define $X_\sigma := X \circ \varphi_\sigma$. For every $\omega \in M$ we have, for every $n \in \mathbf{N}$,

$$X(\omega) = \sum_{\sigma \in \{1,2\}^n} l_\sigma^\alpha(\omega) X_\sigma(\omega) .$$

For $\omega = (t^\sigma)_{\sigma \in D} \in M$ we will define a finite measure ν_ω on the Borel field of $\{1,2\}^{\mathbf{N}}$ as follows:

For $\tau \in D$ let $A(\tau) = \{\eta \in \{1,2\}^{\mathbf{N}} | \eta > \tau\}$.

Define

$$\nu_\omega(A(\tau)) = l_\tau^\alpha(\omega) X_\tau(\omega) .$$

Then ν_ω can be extended to a unique finite measure on the Borel field of $\{1,2\}^{\mathbf{N}}$ (The details are left as an exercise.).

Let $\pi_\omega : \{1,2\}^{\mathbf{N}} \to K(\omega)$ be as defined in the proof of Theorem 2.1.5. We set

$$\mu_\omega = \nu_\omega \circ \pi_\omega^{-1} .$$

Then μ_ω is a finite measure on $\mathbf{R}$ whose support equals $K(\omega)$ (The measures μ_ω have first been constructed by Mauldin–Williams [20]). Let $\mathcal{M}_+([0,1])$ be the space of all (non–negative) finite measures on the Borel field of [0,1] and let $\mathcal{M}_+([0,1])$ be equipped with the topology of weak (narrow) convergence.

The map $M \cap \Omega_\infty \to \mathcal{M}_+([0,1]),\ \omega \to \mu_\omega$ is measurable. Its distribution w.r.t. ν^D is denoted by Q_ν.

2.3.4 DEFINITION.

Let $T : \Delta \times \mathcal{M}_+([0,1]) \times \mathcal{M}_+([0,1]) \to \mathcal{M}_+([0,1])$ be defined by $T((t_1,t_2),\mu_1,\mu_2)(A) =$

$$t_1^\alpha \mu_1(\{s \in [0,1] : t_1 s \in A\}) + t_2^\alpha \mu_2(\{s \in [0,1] : (1-t_2)+t_2 s \in A\}) .$$

A probability P on $\mathcal{M}_+([0,1])$ is called *ν–self–similar* if $\nu \otimes P \otimes P \circ T^{-1} = P$.

2.3.5 THEOREM (Arbeiter [1]).

The probability Q_ν is the unique ν–self–similar probability P on $\mathcal{M}_+([0,1])$ with $\int \rho([0,1])dP(\rho) = 1$.

Proof. Again we will not prove uniqueness. That Q_ν is ν–self–similar follows from a "commuting diagram" argument similar to that given in the proof of 2.2.3.

2.4 THE HAUSDORFF DIMENSION OF THE ν–RANDOM CANTOR SET.

Let $a > 0$ and let $h : [0,a) \to \mathbf{R}_+$ be increasing with $h(0) = 0$. For $\delta > 0$ and $E \subseteq \mathbf{R}^m$ let

$$\mathcal{H}_\delta^h(E) = \inf\{\sum_{i\in I} h(\mathrm{diam}(U_i)) \,|\, E \subset \cup_{i\in I} U_i,\ U_i \text{ open},\ \mathrm{diam}(U_i) \le \delta\}$$

$$\bar{\mathcal{H}}_\delta^h(E) = \inf\{\sum_{i\in I} h(\mathrm{diam}(A_i)) | E \subset \cup_{i\in I} A_i, \mathrm{diam}(A_i) \leq \delta\}.$$

Then $\mathcal{H}_\delta^h(E) \uparrow \sup_{\delta>0} \mathcal{H}_\delta^h(E) = \sup_{\delta>0} \bar{\mathcal{H}}_\delta^h(E) \uparrow \bar{\mathcal{H}}_\delta^h(E)$. Let $\mathcal{H}^h(E) = \sup_{\delta>0} \mathcal{H}_\delta^h(E)$.

If $\beta \geq 0$ and $h(t) = t^\beta$ then $\mathcal{H}_\delta^h$, $\bar{\mathcal{H}}_\delta^h$, $\mathcal{H}^h$ are denoted by $\mathcal{H}_\delta^\beta$, $\bar{\mathcal{H}}_\delta^\beta$, $\mathcal{H}^\beta$, respectively. Call H–dim$(E) = \inf\{\beta | \mathcal{H}^\beta(E) = 0\}$ the *Hausdorff dimension* of E.

2.4.1 THE SCALING PROPERTY OF HAUSDORFF MEASURES.

Let $E, E' \subset \mathbf{R}^m$ be sets and let $f : E \to E'$ be Lipschitz continuous (a similarity) with Lipschitz (similarity) constant c (i.e. $d(f(x), f(y)) \leq (=) c d(x, y)$ for all $x, y \in E$ and c is the smallest constant with the property). Let $\delta > 0$ and $\beta \geq 0$ be given. Then

$$\bar{\mathcal{H}}_{c\delta}^\beta(f(E)) \leq (=) c^\beta \bar{\mathcal{H}}_\delta^\beta(E).$$

Passing to the limit yields

$$\mathcal{H}^\beta(f(E)) \leq (=) c^\beta \mathcal{H}^\beta(E).$$

If f is one–to–one and $f^{-1} : f(E) \to E$ is Lipschitz continuous then

$$0 = \mathcal{H}^\beta(f(E)) \Leftrightarrow 0 = \mathcal{H}^\beta(E).$$

Let ν be a probability on Δ and P_ν the ν–self–similar probability on $\mathcal{K}([0,1])$. Let α be the similarity dimension of the ν–random Cantor set.

2.4.2 THEOREM.

For every $\beta \geq 0$,

a) $P_\nu(\{K \in \mathcal{K}([0,1]) | \mathcal{H}^\beta(K) = 0\}) = 0$ *or* $= 1$.

b) $P_\nu(\{K \in \mathcal{K}([0,1]) | \mathcal{H}^\beta(K) = \infty\}) = 0$ *or* $= 1$.

Proof. Using ν–self–similarity of P_ν, the subadditivity of $\mathcal{H}^\beta$, and

2.4.1 we obtain

$$\begin{aligned}P_\nu(\{K|\mathcal{H}^\beta(K)=0\}) &= \nu\otimes P_\nu\otimes P_\nu(\{(t_1,t_2),K_1,K_2):\\ &\quad \mathcal{H}^\beta(t_1K_1\cup((1-t_2)+t_2K_2)))=0\}\\ &= \nu\otimes P_\nu\otimes P_\nu(\{((t_1,t_2),K_1,K_2): t_1^\beta\mathcal{H}^\beta(K_1)=0 \text{ and }\\ &\quad t_2^\beta\mathcal{H}^\beta(K_2)=0\})\\ &= P_\nu(\{K|\mathcal{H}^\beta(K)=0\})^2\end{aligned}$$

This implies statement a).
b) is proved similarly.

2.4.3 COROLLARY.

There exists a constant $c \geq 0$ *such that* $H - \dim(K) = c$ *for* P_ν*–a.e.* K.

2.4.4 THEOREM (Falconer [10], Mauldin–Williams [20]).

For P_ν*–a.e.* $K \in \mathcal{K}([0,1])$,

$$\mathcal{H}^\alpha(K) < \infty\,.$$

Proof. Since $P_\nu = \nu^D \circ \psi^{-1}$ it is enough to show that $\mathcal{H}^\alpha(K(\omega)) < \infty$ for ν^D- a.e. ω. Let $\omega \in \Omega_\infty \cap M$ be given and let $\delta > 0$ be arbitrary. Since $\omega \in \Omega_\infty$ there exists an $n \in \mathbb{N}$ with $\max_{\sigma\in\{1,2\}^n} l_\sigma < \delta$. Hence

$$\bar{\mathcal{H}}^\alpha_\delta(K(\omega)) \leq \sum_{\sigma\in\{1,2\}^n} \text{diam}\,(J_\sigma(\omega))^\alpha = \sum_{\sigma\in\{1,2\}^n} l^\alpha_\sigma(\omega) = X_n(\omega)$$

which implies

$$\lim_{\delta\downarrow 0} \bar{\mathcal{H}}^\alpha_\delta(K(\omega)) \leq \lim_{n\to\infty} X_n(\omega) = X(\omega) < \infty\,.$$

The last equality holds since $\omega \in M$. Since $\nu^D(M \cap \Omega_\infty) = 1$ the statement of the theorem follows.

2.4.5 DEFINITION AND REMARKS.

A subset $\Gamma \subset \{1,\ldots,N\}^*$ is called a *covering* if, for each $\eta \in \{1,\ldots,N\}^{\mathbf{N}}$, there exists a $\tau \in \Gamma$ with $\tau < \eta$. A minimal covering Γ is characterized by the fact that $\tau \in \Gamma$ with $\tau < \eta$ is uniquely determined. Let Min denote the collection of all minimal coverings. We say that $\Gamma_1 \in$ Min is a refinement of $\Gamma_2 \in$ Min and write $\Gamma_2 < \Gamma_1$ if, for every $\tau \in \Gamma_1$, there is a (unique) $\sigma \in \Gamma_2$ with $\sigma < \tau$.

For $\omega = (t^\tau)_{\tau\in D} \in \Omega$ and $\sigma = (\sigma_1,\ldots,\sigma_n) \in D$ let $t_\sigma(\omega) = t_{\sigma_n}^{\sigma_1\sigma_2\ldots\sigma_{n-1}}$.

2.4.6 LEMMA.

Let $\omega = (t^\sigma)_{\sigma\in D} \in \Omega_\infty$ *be given. Then*

$$\frac{1}{3}\sup_{n\in\mathbf{N}}(\inf\{\sum_{\sigma\in\Gamma} t_\sigma(\omega)l_\sigma^\beta(\omega) | \Gamma \in \text{Min}, \Gamma > \{1,2\}^n\}) \le \mathcal{H}^\beta(K(\omega)) .$$

Proof. Due to the compactness of $K(\omega)$ it suffices to consider finite open coverings of $K(\omega)$ in the calculation of its Hausdorff measure. For $n \in \mathbf{N}$ let $\delta_n = \min_{\sigma\in\{1,2\}^n} l_\sigma(\omega)$. Let $U_1,\ldots,U_m$ be an arbitrary finite open cover of $K(\omega)$ with $\text{diam}(U_i) \le \delta_n$. For $i \in \{1,\ldots,m\}$ let

$$\Gamma_i = \{\sigma \in D | l_{\sigma|(|\sigma|-1)}(\omega) \ge \text{diam}(U_i) > l_\sigma(\omega), J_\sigma(\omega) \cap U_i \ne \emptyset\} .$$

For each $\sigma \in \Gamma_i$ we have $J_\sigma \subset \{s \in \mathbf{R} | d(s,U_i) < \text{diam}(U_i)\}$.

The Lebesgue measure of this last set is less than or equal to $3 \times$ diam (U_i), hence

$$\sum_{\sigma\in\Gamma_1} l_\sigma(\omega) = \lambda(\cup_{\sigma\in\Gamma_i} J_\sigma) \le 3 \times \text{diam}(U_i) ,$$

where λ denotes Lebesgue measure.

By the definition of Γ_i we deduce

$$3 \times \text{diam}(U_i) \ge \sum_{\sigma\in\Gamma_i} l_\sigma(\omega) \ge \text{diam}(U_i) \sum_{\sigma\in\Gamma_i} t_\sigma(\omega) \text{ and hence}$$

$$\sum_{\sigma\in\Gamma_i} t_\sigma(\omega) \leq 3 \,.$$

Using this estimate and the fact that

$$\max_{\sigma\in\Gamma_i} l_\sigma^\beta(\omega) \geq \left(\sum_{\sigma\in\Gamma_i} t_\sigma(\omega)\right)^{-1} \sum_{\sigma\in\Gamma_i} t_\sigma(\omega) l_\sigma^\beta(\omega)$$

we obtain

$$\begin{aligned}
\sum_{i=1}^{m} \operatorname{diam}(U_i)^\beta &\geq \sum_{i=1}^{m} \max_{\sigma\in\Gamma_i} l_\sigma^\beta \\
&\geq \sum_{i=1}^{m} \left(\sum_{\sigma\in\Gamma_i} t_\sigma(\omega)\right)^{-1} \sum_{\sigma\in\Gamma_i} t_\sigma(\omega) l_\sigma^\beta(\omega) \\
&\geq \frac{1}{3} \sum_{\nu=1}^{m} \sum_{\sigma\in\Gamma_i} t_\sigma(\omega) l_\sigma^\beta(\omega) \\
&\geq \frac{1}{3} \sum_{\sigma\in\cup\Gamma_i} t_\sigma(\omega) l_\sigma^\beta(\omega) \,.
\end{aligned}$$

Since $\cup_{i=1}^m \Gamma_i$ is a covering there exists a minimal covering $\Gamma \subset \cup_{i=1}^m \Gamma_i$. We deduce

$$\sum_{i=1}^{m} \operatorname{diam}(U_i)^\beta \geq \frac{1}{3} \sum_{\sigma\in\Gamma} t_\sigma(\omega) l_\sigma^\beta(\omega) \,.$$

For $\sigma \in \Gamma$ we have $l_\sigma(\omega) < \delta_n$ hence

$$\Gamma > \{1,2\}^n$$

and

$$\sum_{i=1}^{m} \operatorname{diam}(U_i)^\beta \geq \underbrace{\frac{1}{3} \inf \{\sum_{\sigma\in\Gamma} t_\sigma(\omega) l_\sigma^\beta(\omega) | \Gamma > \{1,2\}^n\}}_{=A_n} \,.$$

Passing to the infimum on the left–hand side yields

$$\mathcal{H}_{\delta_n}^\beta(K(\omega)) \geq \frac{1}{3} A_n \,.$$

Since $\lim_{n\to\infty} \delta_n = 0$ we obtain

$$\mathcal{H}^{\beta}(K(\omega)) = \lim_{n\to\infty} \mathcal{H}^{\beta}_{\delta_n}(K(\omega)) > \frac{1}{3} \lim_{n\to\infty} A_n = \frac{1}{3} \sup_{n\in\mathbb{N}} A_n$$

which is the claim of the lemma.

2.4.7 LEMMA.

Let $\beta < \alpha$. The set of all $\omega = (t^{\sigma})_{\sigma\in D}$ from Δ^D with

$$\sup_{n\in\mathbb{N}} \inf \{\sum_{\sigma\in\Gamma} t_{\sigma}(\omega) l^{\beta}_{\sigma}(\omega) | \Gamma \in \mathrm{Min},\ \Gamma > \{1,2\}^n\} > 0$$

has positive ν^D–measure.

Proof. Since $\beta < \alpha$ we have

$$\int (t_1^{\beta} + t_2^{\beta})\, d\nu(t_1, t_2) > 1\ .$$

Hence there exists a constant $c > 0$ with

$$\int_{\{t_1 \geq c, t_2 \geq c\}} (t_1^{\beta} + t_2^{\beta})\, d\nu(t_1, t_2) > 1\ .$$

As in the proof of Lemma 2.2.4 we deduce that there is a unique γ with

$$\int_{\{t_1 \geq c, t_2 \geq c\}} (t_1^{\gamma} + t_2^{\gamma})\, d\nu(t_1, t_2) = 1$$

and that $\alpha \geq \gamma > \beta$.

Define

$$\bar{l}_{\sigma}(\omega) = \begin{cases} l_{\sigma}(\omega), & \text{if } t^{\sigma|\nu}_{\sigma_{\nu+1}} \geq c, \text{for } \nu = 0, \ldots, |\sigma| - 1 \\ 0, & \text{elsewhere} \end{cases}$$

and

$$\bar{X}_n(\omega) = \sum_{\sigma\in\{1,2\}^n} \bar{l}_{\sigma}(\omega)\ .$$

Again one can show that $(\bar{X}_n)_{n\in\mathbf{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbf{N}}$ and that $\int \bar{X}_n^p d\nu^D < \infty$ for every $p \in [1,+\infty)$.

Due to the martingale convergence theorem there exists a random variable $\bar{X} : \Delta^D \to \mathbf{R}$ such that

$$\lim_{n\to\infty} \bar{X}_n = \bar{X}$$

for ν^D–a.e. $\omega \in \Delta^D$. Let φ_σ be defined as in 2.3.3.

Set $\bar{X}_\sigma = \bar{X} \circ \varphi_\sigma$.

CLAIM: For ν^D–a.e. $\omega = (t^\sigma)_{\sigma\in D}$ there exists an $m \in \mathbf{N}$ such that for every $\sigma \in D$ with $|\sigma| \geq m$

$$\bar{l}_\sigma^\gamma \bar{X}_\sigma \leq \bar{l}_\sigma^\beta .$$

Proof. Choose $p \in \mathbf{N}$ such that $(\gamma - \beta)p > \gamma$.

For every $\sigma \in D$ Čebyshev's inequality implies

$$\begin{aligned}\nu^D(\{\omega | \bar{l}_\sigma^{\gamma-\beta}\bar{X}_\sigma > 1\}) &\leq \int (\bar{l}_\sigma^{\gamma-\beta}\bar{X}_\sigma)^p d\nu^D \\ &= \int \bar{l}_\sigma^{p(\gamma-\beta)}\, d\nu^D \int \bar{X}_\sigma^p d\nu^D \\ &= \int \bar{l}_\sigma^{p(\gamma-\beta)}\, d\nu^D \int \bar{X}^p d\nu^D .\end{aligned}$$

The first equality holds because $\bar{l}_\sigma$ and $\bar{X}_\sigma$ are independent. Taking the unions of the sets on the left–hand side when σ runs through $\{1,2\}^n$ yields

$$\begin{aligned}&\nu^D(\{\omega | \exists\sigma \in \{1,2\}^n : \bar{l}_\sigma{}^{\gamma-\beta}(\omega)\bar{X}_\sigma(\omega) > 1\}) \\ &\leq \int \sum_{\sigma\in\{1,2\}^n} \bar{l}_\sigma{}^{(\gamma-\beta)p} d\nu^D \int \bar{X}^p d\nu^D .\end{aligned}$$

Using independence we have

$$\int \sum_{\sigma\in\{1,2\}^n} \bar{l}_\sigma^{(\gamma-\beta)p} d\nu^D = \left[\int_{\{t_1\geq c, t_2\geq c\}} (t_1^{(\gamma-\beta)p} + t_2^{(\gamma-\beta)p})\, d\nu(t_1,t_2)\right]^n$$

and, since $(\gamma-\beta)p > \gamma$,

$$\int_{\{t_1\geq c, t_2\geq c\}} (t_1^{(\gamma-\beta)p} + t_2^{(\gamma-\beta)p})\, d\nu(t_1,t_2) < 1 .$$

Since $\int \bar{X}^p d\nu^D < \infty$ we deduce

$$\sum_{n\in\mathbb{N}} \nu^D(\{\omega \mid \exists \sigma \in \{1,2\}^n : \bar{l}_\sigma^{\gamma-\beta}(\omega)\bar{X}_\sigma(\omega) > 1\}) < \infty .$$

By the Borel Cantelli Lemma this implies

$$\nu^D(\cap_{q\in\mathbb{N}} \cup_{n\geq q} \{\omega \mid \exists \sigma \in \{1,2\}^n : \bar{l}_\sigma^{\gamma-\beta}(\omega)\bar{X}_\sigma(\omega) > 1\}) = 0 .$$

This identity immediately implies the claim. Standard considerations show that for ν^D–a.e. $\omega \in \Delta^D$ and for every $\Gamma \in$ Min,

$$\sum_{\sigma\in\Gamma} \bar{l}_\sigma^{\gamma}(\omega)\bar{X}_\sigma(\omega) = \bar{X}(\omega) .$$

Let $\omega = (t^\sigma)_{\sigma\in D} \in \Delta^D$ be such that this last property and the claim for $m \in \mathbb{N}$ is satisfied. If $\Gamma \in$ Min is a refinement of $\{1,2\}^m$ then we get

$$\begin{aligned}\sum_{\sigma\in\Gamma} t_\sigma(\omega) l_\sigma^\beta(\omega) &\geq \sum_{\sigma\in\Gamma} t_\sigma(\omega)\bar{l}_\sigma^\beta(\omega) \\ &\leq \sum_{\sigma\in\Gamma} t_\sigma(\omega)\bar{l}_\sigma^\gamma(\omega)\bar{X}_\sigma(\omega) \\ &\geq c\sum_{\sigma\in\Gamma} \bar{l}_\sigma^\gamma(\omega)\bar{X}_\sigma(\omega) = c\bar{X}(\omega) .\end{aligned}$$

Since $\int \bar{X} d\nu^D = 1$ we deduce $\nu^D(\bar{X} > 0) > 0$ and the lemma is proved.

2.4.8 THEOREM (Falconer [10], Mauldin–Williams [20]).

If α is the similarity dimension of the ν–random Cantor set then, for P_ν–a.e. $K \in \mathcal{K}([0,1])$,
(i) $\mathcal{H}^\alpha(K) < \infty$
(ii) H-dim$(K) = \alpha$.

Proof. By Theorem 2.4.4 statement (i) holds. This implies H–dim$(K) \leq \alpha$ for P_ν–a.e. K. It remains to show that, for every $\beta < \alpha$, $\mathcal{H}^\beta(K) > 0$ for P_ν–a.e. K. By Theorem 2.4.2 a) this is shown if, for every $\beta < \alpha$, $P_\nu(\{K | \mathcal{H}^\beta(K) > 0\}) > 0$. From Lemmas 2.4.6 and 2.4.7 this last statement follows.

2.4.9 THEOREM (Graf ([13]).

If $\nu(\{(t_1,t_2) \in \Delta | t_1^\alpha + t_2^\alpha \neq 1\} > 0$ *then* $\mathcal{H}^\alpha(K) = 0$ *for* P_ν*–a.e.* $K \in \mathcal{K}([0,1])$.

Proof. First note that, for every $\gamma > 1$ and every $E \subset [0,1]$, we have $\bar{\mathcal{H}}_\gamma^\alpha(E) = \bar{\mathcal{H}}_1^\alpha(E)$.

CLAIM. $\bar{\mathcal{H}}_1^\alpha(K) = 0$ for P_ν–a.e. $K \in \mathcal{K}([0,1])$.

Proof. For ν–a.e. $(t_1,t_2) \in \Delta$ and $P_\nu \otimes P_\nu$–a.e. (K_1,K_2) we have

$$
\begin{aligned}
(*) \quad \bar{\mathcal{H}}_1^\alpha(t_1K_1 \cup (1-t_2) + t_2K_2)) &\leq \bar{\mathcal{H}}_1^\alpha(t_1K_1) + \bar{\mathcal{H}}_1^\alpha(t_2K_2) \\
&\underset{2.4.1}{=} t_1^\alpha \bar{\mathcal{H}}_{(1/t_1)}^\alpha(K_1) + t_2^\alpha \bar{\mathcal{H}}_{(1/t_2)}(K_2) \\
&= t_1^\alpha \bar{\mathcal{H}}_1^\alpha(K_1) + t_2^\alpha \bar{\mathcal{H}}_1^\alpha(K_2) .
\end{aligned}
$$

Using the fact that P_ν is ν–self–similar we deduce

$$
\begin{aligned}
\int \bar{\mathcal{H}}_1^\alpha(K)dP_\nu(K) &= \int\int\int \bar{\mathcal{H}}_1^\alpha(t_1K_1 \cup (1-t_2) \\
&\qquad + t_2K_2))dP_\nu(K_1)dP_\nu(K_2)d\nu(t_1,t_2) \\
&\leq \int [t_1^\alpha \int \bar{\mathcal{H}}_1^\alpha(K_1)dP_\nu(K_1) \\
&\qquad + t_2^\alpha \int \bar{\mathcal{H}}_1^\alpha(K_2)dP_\nu(K_2)d\nu(t_1,t_2)] \\
&= \int \bar{\mathcal{H}}_1^\alpha(K)dP_\nu(K) .
\end{aligned}
$$

Thus all integrals are equal and therefore the inequality $(*)$ is an equality almost everywhere.

Let $c = \underset{K \in \mathcal{K}([0,1])}{\text{ess sup}} \bar{\mathcal{H}}_1^\alpha(K)$.

Then we get

$$c \geq t_1^\alpha c + t_2^\alpha c$$

for ν–a.e. $(t_1, t_2) \in \Delta$.

If $c > 0$ this would imply $t_1^\alpha + t_2^\alpha = 1$ ν–a.e. which contradicts our assumption. Thus we have proved $\bar{\mathcal{H}}_1^\alpha(K) = 0$ for P_ν–a.e. K.

Now we will show that $\mathcal{H}^\alpha(K) = 0$ for P_ν–a.e. K, or, equivalently, that $\mathcal{H}^\alpha(K(\omega)) = 0$ for ν^D–a.e. ω. For $\omega = (t^\sigma) = ((t_1^\sigma, t_2^\sigma))$ let $S_1^\sigma, S_2^\sigma : \mathbf{R} \to \mathbf{R}$ be defined by $S_1^\sigma(s) = t_1^\sigma s$ and $S_2^\sigma(s) = (1 - t_2^\sigma) + t_2^\sigma s$. Set $S_\sigma = S_{\sigma_1}^\emptyset \circ S_{\sigma_2}^{\sigma|1} \circ \ldots \circ S_{|\sigma|}^{\sigma|(|\sigma|-1)}$ for $|\sigma| \geq 1$ and $S_\emptyset = id_{[0,1]}$ for $\sigma = \emptyset$. Then S_σ is a similarity with similarity constant $l_\sigma(\omega)$.

Moreover it is easy to check that, for every $n \in \mathbf{N}$,

$$\binom{*}{*} \quad K(\omega) = \cup_{\sigma \in \{1,2\}^n} S_\sigma(K(\varphi_\sigma(\omega))) \ .$$

Noting that $(*)$ is an a.e. equality and using recursion one can show that, for ν^D–a.e. ω and for every $n \in \mathbf{N}$,

$$\bar{\mathcal{H}}_1^\alpha(K(\omega)) = \sum_{\sigma \in \{1,2\}^n} \bar{\mathcal{H}}_1^\alpha(S_\sigma(K(\varphi_\sigma(\omega))) \ .$$

Let $\omega = (t^\sigma) \in \Delta^D$ be such that this last property and, in addition, $\bar{\mathcal{H}}_1^\alpha(K(\omega)) = 0$, and $\lim_{n\to\infty} \max_{\sigma \in \{1,2\}^n} l_\sigma(\omega) = 0$ are satisfied (These properties hold for ν^D–a.e. ω). Let $\delta > 0$ be arbitrary. Choose $n \in \mathbf{N}$ with $l_\sigma(\omega) \leq \delta$ for all $\sigma \in \{1, 2\}^n$.

Then we deduce

$$0 = \bar{\mathcal{H}}_1^\alpha(K(\omega)) = \sum_{\sigma \in \{1,2\}^n} \bar{\mathcal{H}}_1^\alpha(S_\sigma(K(\varphi_\sigma(\omega))) \ .$$

Since $\operatorname{diam}(S_\sigma(K(\varphi_\sigma(\omega))) \leq l_\sigma(\omega) \leq 1$ we have $\bar{\mathcal{H}}_1^\alpha(S_\sigma(K(\varphi_\sigma(\omega)))) = \bar{\mathcal{H}}_{l_\sigma}^\alpha(S_\sigma(K(\varphi_\sigma(\omega)))) = 0$.

Since $l_\sigma(\omega) \leq \delta$ this implies $\mathcal{H}_\delta^\alpha(S_\sigma(K(\varphi_\sigma(\omega)))) = 0$. From $\binom{*}{*}$ we deduce

$$\mathcal{H}_\delta^\alpha(K(\omega)) = 0 \ .$$

Taking the limit for $\delta \to 0$ yields the theorem.

2.4.10 THEOREM (Graf [13]).

Suppose that there exists a $\delta > 0$ *such that* $t_1 \geq \delta$, $t_2 \geq \delta$ *and* $t_1^\alpha + t_2^\alpha = 1$ *for* ν*–a.e.* $(t_1, t_2) \in \Delta$. *Then* $0 < \mathcal{H}^\alpha(K) < \infty$ *for* ν*–a.e.* $K \in \mathcal{K}([0,1])$.

Proof. Under the assumptions of the theorem we have, for ν^D–a.e. $\omega \in \Delta^D$ and every $\Gamma \in$ Min,

$$\sum_{\sigma \in \Gamma} l_\sigma^\alpha(\omega) = 1 .$$

From Lemma 2.4.6 we know that

$$\frac{1}{3} \sup_{n \in \mathbb{N}} \inf \{ \sum_{\sigma \in \Gamma} t_\sigma(\omega) l_\sigma^\alpha(\omega) | \Gamma > \{1,2\}^n \} \leq \mathcal{H}^\alpha(K(\omega))$$

for ν^D–a.e. ω, hence

$$0 < \frac{1}{3}\delta \leq \mathcal{H}^\alpha(K(\omega))$$

for ν^D–a.e. ω.

2.4.11 PROBLEM.

Is the δ–condition in the preceding theorem essential?

2.4.12 THEOREM (Graf–Mauldin–Williams [15]).

Suppose that $\nu(\{(t_1,t_2) \in \Delta | t_1^\alpha + t_2^\alpha \neq 1\}) > 0$ *and that there exists a* $\xi > 0$ *with* $\int \max(t_1^{-\xi}, t_2^{-\xi}) d\nu(t_1,t_2) < \infty$. *Suppose further that there is a* $\delta \in (0,1)$ *and* $p \in \mathbb{N}$ *such that*

$$\int_{\{t_1 \leq 1-\delta, t_2 \leq 1-\delta\}} (t_1 + t_2)^s d\nu(t_1,t_2) \geq s^{-p}$$

for all large s.

Then, for P_ν*–a.e.* $K \in \mathcal{K}([0,1])$,

$$0 < \mathcal{H}^h(K) < \infty$$

where $h(x) = x^{\alpha}(\log|\log x|)^{1-\alpha}$.

Proof. see Graf–Mauldin–Williams [15].

2.4.13 EXAMPLES.

a) In the case of the classical Cantor set, i.e. for $\nu = \epsilon_{(\frac{1}{3},\frac{1}{3})}$, Theorem 2.4.10 yields

$$0 < \mathcal{H}^{\alpha}(K) < \infty,$$

where $\alpha = \frac{\log 2}{\log 3}$.

b) If ν is normalized Lebesgue measure on Δ then H–dim $(K) = \alpha = \frac{1}{2}(\sqrt{17}-3)$, $\mathcal{H}^{\alpha}(K) = 0$, and $0 < \mathcal{H}^{h}(K) < \infty$ for P_ν–a.e. $K \in \mathcal{K}([0,1])$, where $h(x) = x^{\alpha}(\log|\log x|)^{1-\alpha}$. The last property is derived from Theorem 2.4.12 by noting that, obviously,

$$\nu(t_1^{\alpha} + t_2^{\alpha} \neq 1) > 0,$$

$\int \max(t_1^{-\xi}, t_2^{-\xi})\, d\nu(t_1 + t_2) < \infty$ for every $\xi \in (0,1)$ and

$$\begin{aligned}
&\int_{\{t_1 \le \frac{1}{2}, t_2 \le \frac{1}{2}\}} (t_1 + t_2)^s d\nu(t_1, t_2)\\
&= 2\int_0^{1/2}\int_0^{1/2} (t_1 + t_2)^s dt_2\, dt_1\\
&= 2\int_0^{1/2}\int^{1/2+t_1} t^s\, dt\, dt_1\\
&= 2\int_0^{1/2} \frac{1}{s+1} t^{s+1}\Big|_{t_1}^{(1/2)+t_1} dt_1\\
&= 2\int_0^{1/2} \frac{1}{s+1}\Big(\frac{1}{2} + t_1\Big)^{s+1} - t_1^{s+1})\, dt_1\\
&= 2\frac{1}{s+1}\left[\frac{1}{s+2}\Big(\frac{1}{2} + t_1\Big)^{s+2} - \frac{1}{s+2} t_1^{s+2}\right]\Bigg|_{t_1=0}^{t_1=\frac{1}{2}}\\
&= 2\frac{1}{s+1}\frac{1}{s+2}\left[1 - \Big(\frac{1}{2}\Big)^{s+2} - \Big(\frac{1}{2}\Big)^{s+2}\right].\\
&\ge \frac{1}{s+1}\frac{1}{s+2} \ge s^{-3}
\end{aligned}$$

for all large s.

c) In the case of the zero–set of Brownian bridge, i.e. if ν is as in Example 2.1.7 c), we have H-dim$(K) = \alpha = \frac{1}{2}$, $\mathcal{H}^\alpha(K) = 0$, and $0 < \mathcal{H}^h(K) < \infty$ for P_ν–a.e. K, where $h(x) = x^{1/2}(\log|\log(x)|)^{1/2}$. The last property follows from Theorem 2.4.12 if we observe that

(i) $\nu(t_1^\alpha + t_2^\alpha \neq 1) > 0$

(ii)

$$\int \max(t_1^{-\xi}, t_2^{-\xi})\, d\nu(t_1,t_2)$$
$$= \frac{1}{2\pi}\int_0^{1/2}\int_0^{1/2} \max(t_1^{-\xi}, t_2^{-\xi})(t_1t_2(1-t_1-t_2)^3)^{-1/2}dt_1dt_2$$
$$\leq \frac{1}{2\pi}\int_0^{1/2}\int_0^{1/2} (t_1t_2)^{-\xi}(t_1t_2(1-t_1-t_2)^3)^{-1/2}dt_1dt_2$$
$$\leq \frac{1}{2\pi}\int_0^{1/4}\int_0^{1/2} (t_1t_2)^{-(\xi+\frac{1}{2})}(\frac{1}{4})^{-\frac{3}{2}}dt_1dt_2$$
$$+ \frac{1}{2\pi}\int_{1/4}^{1/2}\int_0^{1/4} (t_1t_2)^{-(\xi+\frac{1}{2})}(\frac{1}{4})^{-\frac{3}{2}}dt_1dt_2$$
$$+ \frac{1}{2\pi}\int_{1/4}^{1/2}\int_{1/4}^{1/2} 16^{-(\xi+\frac{1}{2})}(1-t_1-t_2)^{-\frac{3}{2}}dt_1dt_2 < \infty$$

for $\xi \in (0, \frac{1}{2})$

(iii)

$$\int_{[0,\frac{1}{2}]^2} (t_1+t_2)^s d\nu(t_1,t_2)$$
$$= \frac{1}{2\pi}\int_0^{1/2}\int_0^{1/2} (t_1+t_2)^s(t_1t_2(1-t_1-t_2)^3)^{-\frac{1}{2}}dt_2dt_1$$
$$\geq \frac{1}{2\pi}\int_{1/4}^{1/2}\int_{1/4}^{1/2} (t_1+t_2)^s(\frac{1}{4}(1-\frac{1}{4}-\frac{1}{4})^3)^{-\frac{1}{2}}dt_2dt_1$$
$$\geq \frac{1}{2\pi}2^{\frac{5}{2}}\int_{1/4}^{1/2}\int_{1/4}^{1/2} (t_1+t_2)^s dt_2dt_1 \geq s^{-3}$$

for large s.

(The results summarized above were first obtained by Taylor [26] and Taylor–Wendel [29]).

2.4.14 REMARKS (Other notions of dimension).

a) Box–counting dimension

Let F be a bounded subset of $\mathbf{R}^m$. For $\epsilon > 0$ a subset F' of F is called an ϵ–net if $d(x, y) \geq \varepsilon$ for all $x, y \in F'$ with $x \neq y$. By $N_\epsilon(F)$ we denote the maximum cardinality of an ε–net in F. The numbers

$$B\text{-}\underline{\dim}(F) = \liminf_{\epsilon \to 0} \frac{\log N_\epsilon(F)}{-\log \epsilon}$$

and

$$B\text{-}\overline{\dim}(F) = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon(F)}{-\log \epsilon}$$

are called the *lower* and *upper box–counting dimension* of F. If the two numbers agree their common value is called the *box(–counting) dimension* of F and denoted by B–$\dim(F)$. Other names for box–counting dimension are Kolmogorov entropy, entropy dimension, capacity dimension, metric dimension, logarithmic dimension, and information dimension.

b) Modified box–counting dimension

Let F be subset of $\mathbf{R}^m$. Then

$$MB\text{-}\dim(F) = \inf\{\sup_{i \in I}(B\text{-}\underline{\dim}(F_i)) \mid F \subset \cup_{i \in I} F_i,\ I \text{ countable}\}$$

and

$$MB\text{-}\overline{\dim}(F) = \inf\{\sup_{i \in I}(B - \overline{\dim}(F_i)) \mid F \subset \cup_{i \in I} F_i,\ I \text{ countable}\}$$

are called the *lower* and *upper modified box–counting dimension.* MB-$\overline{\dim}(F)$ agrees with the so–called “packing dimension”, which is defined in a different way. One has the following inequalities:

$$MB\text{-}\underline{\dim}(F) \leq B\text{-}\underline{\dim}(F),\ MB\text{-}\overline{\dim}(F) \leq B\text{-}\overline{\dim}(F),$$

H–$\dim(F) \leq MB$–$\underline{\dim}(F) \leq MB$-$\overline{\dim}(F) \leq B$-$\overline{\dim}(F) \leq m$ (see Falconer [11], p. 46).

It can be shown that for ν–random Cantor sets the box–dimension exists a.e. and equals the Hausdorff dimension, i.e.

$$H\text{-}\dim(K) = B\text{-}\dim(K)$$

for P_ν–a.e. $K \in \mathcal{K}([0,1])$ (see Falconer [11], Theorem 15.2). This shows that almost all realizations of ν–random Cantor sets are fractals in the sense of Taylor [28]. Taylor calls a subset of $\mathbf{R}^m$ a fractal if its Hausdorff dimension agrees with its upper modified box–counting dimension.

c) Equidistribution

Let F be a compact subset of $\mathbf{R}^m$. A probability measure μ on F is called the *equidistribution* on F if $(\mu_\delta)_{\delta>0}$ tends to μ in the weak* topology as $\delta \to 0$ whenever $\mu_\delta = \frac{1}{N_\delta(F)} \sum_{x \in F_\delta} \epsilon_x$ for some δ–net F_δ of maximum cardinality $N_\delta(F)$.

(An equidistribution need not exist, but if it exists, it is uniquely determined.) In the case of the classical Cantor set C the natural measure on C is the equidistribution on C. It is an open problem whether for all ν–random Cantor sets the natural random measure $(\mu_\omega)_{\omega \in \Delta^D}$ has the property that μ_ω is a scalar multiple of the equidistribution on $K(\omega)$ (and whether such an equidistribution exists) for ν^D–a.e. ω.

3. Mandelbrot's percolation process.

Here we will investigate a random Cantor type construction in the unit square. It was introduced by Mandelbrot [19] to model certain aspects of turbulence. He called it canonical curdling.

3.1 CONSTRUCTION.

Let $p \in (0,1)$ and $k \in \mathbf{N}$ with $k \geq 2$ be given. Set $N = k^2$. Divide the unit square into N squares of side length k^{-1} in the obvious way. We select a subset of these squares to form K_1 in such a way that each square has independent probability p of being selected. Similarly, each square of K_1 is divided into N squares of side length k^{-2} and each of these has independent probability p of being chosen to be a square of K_2. We continue in this way, so that K_n is a random collection of squares of side length k^{-n}. Then $\bigcap_{n \in \mathbf{N}} K_n$ is a typical realization of the random object we want to consider.

Next we will give a formal description of the construction.

Let $E = \{(i,j) \in \mathbf{N}^2 | 1 \leq i,\ j \leq k\}$ and define $\varphi : E \to$

$\{1,\ldots,N\}$ by $\varphi(i,j) = k(i-1)+j$. Then φ is a bijection. Set

$$B_{\varphi(i,j)} = [\frac{i-1}{k}, \frac{i}{k}] \times [\frac{j-1}{k}, \frac{j}{k}] .$$

Let $\nu = \nu_p$ be the probability measure on $I = \{0,1\}^N$ defined by

$$\nu(\{(t_1,\ldots,t_N)\}) = p^{\mathrm{card}\{i|t_i=1\}}(1-p)^{\mathrm{card}\{i|t_i=0\}} .$$

Set $D = \{1,\ldots,N\}^*$ and let ν^D be the product measure on I^D. The elements of I^D will be denoted by $(t^\sigma)_{\sigma\in D}$ where $t^\sigma = (t_1^\sigma,\ldots,t_N^\sigma\} \in I$.

3.1.1 THE COMPACT SET CORRESPONDING TO A TREE FROM I^D.

Let $\omega = (t^\sigma)_{\sigma\in D} \in I^D$ be given. Define

$$J_\emptyset(\omega) = [0,1]^2 .$$

If $J_\sigma(\omega) = [a,b] \times [c,d]$ set

$$J_{\sigma*\rho}(\omega) = \begin{cases} [a+\frac{i-1}{k}(b-a),\ a+\frac{i}{k}(b-a)] \times [c+\frac{j-1}{k}(d-c)+\frac{j}{k}(d-c)] & \text{if } t_\rho^\sigma = 1 \text{ and } \varphi^{-1}(\rho) = (i,j) \\ \emptyset, & \text{if } t_\rho^\sigma = 0 . \end{cases}$$

If $J_\sigma(\omega) = \emptyset$, then $J_{\sigma*\rho}(\omega) = \emptyset$.

Set

$$K_n(\omega) = \cup_{\sigma\in\{1,\ldots,N\}^n} J_\sigma(\omega)$$

and

$$K(\omega) = \cap_{n\in\mathbb{N}} K_n(\omega) .$$

Then $K(\omega)$ is a compact set which may be empty. It is called the compact set corresponding to ω.

3.1.2 PROPOSITION.

Let $\mathcal{K}_o([0,1]^2) = \mathcal{K}([0,1]^2) \cup \{\emptyset\}$, *where* $\mathcal{K}([0,1]^2)$ *carries the topology induced by the Hausdorff metric and* $\emptyset$ *is added as an isolated* *point. Then the map* $\psi : I^D \to \mathcal{K}_o([0,1])^2), \omega \to K(\omega)$ *is measurable, i.e. a random variable.*

Proof. Exercise!

3.1.3 DEFINITION.

The image measure of ν^D with respect to ψ will be denoted by P_ν.

3.2 STATISTICAL SELF-SIMILARITY.

3.2.1 DEFINITION.

For $(i,j) \in E$ let $S_{\varphi(i,j)} : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $S_{\varphi(i,j)}(x) = \frac{1}{k}x + (\frac{i-1}{k}, \frac{j-1}{k})$. Define $T : I \times \mathcal{K}_o([0,1]^2)^N \to \mathcal{K}_o([0,1])^2)$ by

$$T((t_1,\ldots,t_N), K_1,\ldots,K_N) = \cup_{\substack{\rho=1 \\ t_\rho=1}}^{N} S_\rho(K_\rho)\ .$$

A probability measure P on $\mathcal{K}_o([0,1])^2)$ is called *ν–self–similar* if $(\nu \otimes P^N) \circ T^{-1} = P$, where P^N is the product measure on $\mathcal{K}_o([0,1]^2)^N$.

3.2.2 PROPOSITION.

P_ν is ν–self–similar.

Proof. The proposition follows from a "commuting diagram" argument similar to the one used in the proof of Theorem 2.2.2.

3.2.3 THEOREM.

Let q be the smallest nonnegative solution of the equation

$$(*) \quad x = (px + 1 - p)^N\ .$$

Then $P_{\nu_p}(\{\emptyset\}) = q$.

Proof. Since P_ν is ν–self–similar we have

$$\begin{aligned} P_\nu(\{\emptyset\}) &= (\nu \otimes P_\nu^N) \circ T^{-1}(\{\emptyset\}) \\ &= (\nu \otimes P_\nu^N)(\{((t_1,\ldots,t_N),K_1,\ldots,K_N) : \forall i \\ &\quad \in \{1,\ldots,N\} : t_i = 1 \Rightarrow S_i(K_i) = \emptyset)\}) \\ &= (\nu \otimes P_\nu^N)(\{((t_1,\ldots,t_N),K_1,\ldots,K_N) : \forall i \\ &\quad \in \{1,\ldots,N\} : t_i = 1 \Rightarrow K_i = \emptyset)\}) \\ &= \int P_\nu(\{\emptyset\})^{\operatorname{card}\{i|t_i=1\}} d\nu(t_1,\ldots,t_N) \\ &= \sum_{k=0}^{N} P_\nu(\{\emptyset\})^k \binom{N}{k} p^k (1-p)^{N-k} = (P_\nu(\{\emptyset\})p + 1 - p)^N . \end{aligned}$$

Thus $P_\nu(\{\emptyset\})$ is a root of equation $(*)$

By induction we will show that

$$\nu^D(\{\omega | \cup_{\sigma\in\{1,\ldots,N\}^n} J_\sigma(\omega) = \emptyset\}) \leq q$$

for all $n \in \mathbf{N}$.

For $n = 1$ we have

$$\begin{aligned} \nu^D(\{\omega | J_1(\omega) \cup \ldots \cup J_N(\omega) = \emptyset\}) &= \nu(\{(0,\ldots,0)\}) \\ &= (1-p)^N \\ &\leq (pq + 1 - p)^N = q . \end{aligned}$$

Assume that

$$s_n = \nu^D(\{\omega | \cup_{\sigma\in\{1,\ldots,N\}^n} J_\sigma(\omega) = \emptyset\}) \leq q .$$

Then we get

$$\begin{aligned} s_{n+1} &= \nu^D(\{\omega | \cup_{\sigma\in\{1,\ldots,N\}^{n+1}} J_\sigma(\omega) = \emptyset\}) \\ &= \nu^D(\{\omega | \cup_{i=1}^N \cup_{\tau\in\{1,\ldots,N\}^n} J_{i*\tau}(\omega) = \emptyset\}) \\ &= \nu^D(\{\omega | \forall i \in \{1,\ldots, N\} : J_i(\omega) = \emptyset \text{ or } [\, J_i(\omega) \neq \emptyset \text{ and} \\ &\qquad \cup_{\tau\in\{1,\ldots,N\}^n} J_{i*\tau}(\omega) = \emptyset]\}) \\ &= \sum_{k=0}^{N} \binom{N}{k} (ps_n)^k (1-p)^{N-k} \leq (pq + 1 - q)^N = q . \end{aligned}$$

Since $K(\omega) = \emptyset$ if and only if there is an $n \in \mathbf{N}$ with $J_\sigma(\omega) = \emptyset$ for all $\sigma \in \{1, \dots, N\}^n$ it follows that

$$P_\nu(\{\emptyset\}) = \nu^D(\{\omega | K(\omega) = \emptyset\}) = \nu^D(\cup_{n\in\mathbf{N}} \{\omega : J_\sigma(\omega) = \emptyset$$
$$\text{for all } \sigma \in \{1, \dots, N\}^n\}) = \lim_{n\to\infty} s_n \leq q \;.$$

Since $P_\nu(\{\emptyset\})$ is a non negative root of $(*)$ and q is the smallest such root we deduce

$$q = P_\nu(\{\emptyset\}) \;.$$

3.2.4 REMARKS.

a) 1 is always a root of equation $(*)$. Consider the function $f : \mathbf{R}_+ \to \mathbf{R}$ defined by $f(s) = (sp + 1 - p)^N - s$. Then $f'(s) = N(sp + 1 - p)^{N-1}p - 1$. For $p \leq \frac{1}{N}$ we have $f'(s) < 0$ for $s < 1$. Since $f(1) = 0$ this implies $f(s) > 0$ for all $s < 1$. Hence 1 is the smallest non negative root of $(*)$. By the preceding theorem this implies $P_\nu(\{\emptyset\}) = 1$, i.e. $K(\omega) = \emptyset$ for ν^D–a.e. $\omega \in I^D$. For $p > \frac{1}{N}$ we have $f'(1) = Np - 1 > 0$. Hence f is increasing in a neighborhood of 1. Since $f(1) = 0$ we have $f(s_o) < 0$ for some $s_o \in (0,1)$. Since $f(0) = (1-p)^N > 0$ there exists an $s \in (0, s_o)$ with $f(s) = 0$. Hence $P_\nu(\{\emptyset\}) < 1$, i.e. $K \neq \emptyset$ with positive P_ν–probability.

b) It can be shown that P_ν is the unique ν–self–similar probability measure on $\mathcal{K}_o([0,1]^2)$ with $P_\nu(\{\emptyset\})$ being the smallest non negative root q of of $(*)$. If $q < 1$ then $\epsilon_\emptyset$ is a different ν–self–similar measure on $\mathcal{K}_o([0,1]^2)$. There are no other ν–self–similar probabilities on $\mathcal{K}_o([0,1]^2)$, since 1 and q are the only roots of $(*)$ in [0,1]. (The derivative f' attains 0 at at most one point in $[0, +\infty)$, so that $\{f = 0\} \cap \mathbf{R}_+$ contains at most two points.)

3.2.5 DEFINITION.

The *similarity dimension* of the random compact set with distribution P_{ν_p} is defined to be the unique $\alpha = \alpha_p \geq 0$ such that

$$\int \sum_{i=1}^{N} t_i (\frac{1}{k})^\alpha d\nu(t_1, \dots, t_N) = 1 \;.$$

3.2.6 THEOREM.

The similarity dimension of the random compact set with distribution P_{ν_p} *equals*

$$\alpha = \alpha_p = 2 + \frac{\log p}{\log k}\,.$$

Proof.

$$1 = \int \sum_{i=1}^{N} t_i \Big(\frac{1}{k}\Big)^{\alpha} d\nu(t_1, \dots, t_N) = \Big(\frac{1}{k}\Big)^{\alpha} \int \sum_{i=1}^{N} t_i d\nu(t_1, \dots, t_N)$$

$$= \Big(\frac{1}{k}\Big)^{\alpha} \sum_{l=0}^{N} \binom{N}{l} l p^l (1-p)^{N-l} = (Np)\Big(\frac{1}{k}\Big)^{\alpha}\,.$$

Hence we get

$$\alpha = \frac{\log Np}{\log k} = \frac{2\log k + \log p}{\log k} = 2 + \frac{\log p}{\log k}\,.$$

3.3 TOPOLOGICAL PROPERTIES.

For technical reasons we assume $k \geq 3$ in this section.

3.3.1 DEFINITION.

Let $R = [a, b] \times [c, d]$ with $a < b$ and $c < d$. A subset A of R is said to *horizontally* (resp. vertically) *percolate* in R if there exists a connected component C of A with

$$(\{a\} \times [c, d]) \cap C \neq \emptyset \text{ and } (\{b\} \times [c, d]) \cap C \neq \emptyset$$

resp. $([a, b] \times \{c\} \cap C \neq \emptyset$ and $([a, b] \times \{d\}) \cap C \neq \emptyset)$.

Let

$$\Delta_h = \{K \in \mathcal{K}([0, 1]^2) \,|\, K \text{ horizontally percolates in } [0, 1]^2\}$$

and

$$\Delta_v = \{K \in \mathcal{K}([0,1]^2) | K \text{ vertically percolates in} [0,1]^2\}.$$

Define $\theta_h(p) = P_{\nu_p}(\Delta_h), \theta_v(p) = P_{\nu_p}(\Delta_v)$.

3.3.2 REMARK AND DEFINITION.

It follows from the symmetry in the construction that $\theta_h(p) = \theta_v(p)$ for all $p \in (0,1)$. We denote the common value by $\theta(p)$.

3.3.1 LEMMA.

Let $\varphi : \mathcal{K}_o([0,1]^2) \times \mathcal{K}_o([0,1]^2) \to \mathcal{K}_o([0,1] \times [0,2])$ *be defined by*

$$\varphi(K_1, K_2) = K_1 \cup (e_2 + K_2)$$

where e_2 *is the vector* $(0,1)$.

Let Q_p *be the image measure of* $P_{\nu_p} \otimes P_{\nu_p}$ *w.r.t.* φ *and let* $\Delta_{h,2} = \{K \in \mathcal{K}([0,1] \times [0,2]) : K$ *horizontally percolates in* $[0,1] \times [0,2]$*). Then* $Q_p(\Delta_{h,2}) > 0$ *implies* $\theta(p) > 0$.

Proof. see Dekking–Meester [8].

3.3.2 THEOREM (Dekking–Meester [8]).

If $P_{\nu_p}(\{K \in \mathcal{K}([0,1]^2) | K$ *is not totally disconnected*$\}) > 0$ *then* $\theta(p) > 0$.

Sketch of proof. If $K \in \mathcal{K}([0,1]^2)$ is not totally disconnected then there exist two different points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ lying in a connected component of K. If $y_1 \neq x_1$ then for n large there exists a column C_n in $(J_\sigma)_{\sigma \in \{1,\dots,N\}^n}$ consisting of the squares $Q_1, \dots, Q_k n$ (where the squares are counted from top to bottom) such that a connected component C of $K \cap C_n$ crosses the column C_n. For such a C there are two possibilities

(1) C has a non-empty intersection with at least three (consecutive) squares in C_n, say Q_{i-1}, Q_i, Q_{i+1}.

(2) C has a nonempty intersection with at most two (consecutive) squares in C_n, say Q_{j-1} and Q_{j+1}.

Thus we deduce that either

(3) $P_{\nu_p}(\{K \in \mathcal{K}([0,1]^2) | \exists n \, \exists \sigma \in \{1,\ldots,N\}^n : K \cap J_\sigma$ vertically percolates in $J_\sigma\}) > 0$

or

(4) $P_{\nu_p}(\{K \in \mathcal{K}([0,1]^2) | \exists n \, \exists$ two squares $Q, \tilde{Q}$ from $(J_\sigma)_{\sigma \in \{1,\ldots,N\}^n}$ one on top of the other such that $K \cap (Q \cup \tilde{Q})$ horizontally percolates in $Q \cup \tilde{Q}\}) > 0$.

From (3) it follows that there is a $\sigma \in \{1,\ldots,N\}^*$ with

$$0 < P_{\nu_p}(\{K \in \mathcal{K}([0,1]^2) | K \cap J_\sigma$$

vertically percolates in $J_\sigma\}$

$$= p^{|\sigma|}\theta(p) .$$

Hence $\theta(p) > 0$ in this case.

From (4) it follows that $Q_p(\Delta_{h,2}) > 0$.

Thus Lemma 3.3.1 implies $\theta(p) > 0$.

3.3.3 LEMMA.

Set $p_c = \inf\{p | \theta(p) > 0\}$. Then $\theta(p_c) > 0$.

Proof. See Dekking–Meester [8]

3.3.4 LEMMA.

If p_c is as above then $\frac{1}{N} < p_c < 1$.

Proof. See Falconer [11], p. 233.

Combining the results of the last three lemmas yields the following theorem

3.3.5 THEOREM (Chayes–Chayes–Durrett [7]).

There exists a $p_c \in (\frac{1}{N}, 1)$ *such that, for* $p < p_c$, P_{ν_p}*–a.e.* K *is empty or totally–disconnected and, for* $p \geq p_c$,

$$P_{\nu_p}(\{K \in \mathcal{K}([0,1])^2) | K \text{ horizontally percolates in } [0,1]^2\}) > 0 \ .$$

3.3.6 REMARKS.

a) There have been efforts to calculate p_c. By oral communication I have learnt that J.T. Chayes–L. Chayes–R. Durrett as well as M. Dekking have calculated p_c.
b) R. Meester [21] has shown that $\theta(p) = P_{\nu_p}(A)$, where A is the set of all $K \in \mathcal{K}([0,1]^2)$ such that there exists a continuous map $\gamma : [0,1] \to [0,1]^2$ with
   (i) $\gamma(0) \in \{0\} \times [0,1]$, $\gamma(1) \in \{1\} \times [0,1]$
   (ii) $\gamma(t) \in K$ for all $t \in [0,1]$
   (iii) $\gamma(t) \neq \gamma(s)$ for all $s, t \in [0,1]$ with $s \neq t$.
c) This section describes a topological "phase transition" in the behaviour of the random set as p increases: For $0 \leq p \leq \frac{1}{N}$ the random set is empty a.s. Then, for $\frac{1}{N} < p < p_c$, it is either empty or, with positive probability, a dustlike set and then, for $p_c \leq p < 1$, it percolates with positive probability.

3.4 THE HAUSDORFF DIMENSION.

In this section $\alpha = \alpha_p$ is the similarity dimension of the random set from 3.1, i.e. $\alpha = 2 + \frac{\log p}{\log k}$.

3.4.1 THEOREM (Chayes–Chayes–Durrett [7]).

For P_{ν_p}*–a.e.* $K \in \mathcal{K}_o([0,1]^2)$,

$$K \neq \emptyset \Rightarrow \mathcal{H}^\alpha(K) < \infty \text{ and } H\text{-dim}(K) = \alpha \ .$$

Proof. The result is proved with methods similar to those used in the corresponding proof for random Cantor sets.

3.4.2 THEOREM (Graf–Mauldin–Williams [15]).

For P_{ν_p}–a.e. $K \in \mathcal{K}_o([0,1]^2)$,

$$K \neq \emptyset \Rightarrow 0 < \mathcal{H}^h(K) < \infty ,$$

where $h(t) = t^\alpha(\log|\log t|)^{1-\frac{\alpha}{2}}$.

Proof. See [15].

4. Statistically self–similar sets.

It is the purpose of this section to introduce a notion of random sets which at the same time generalizes the concepts presented in the preceding sections and the deterministic concept of self–similarity as introduced by Hutchinson [16].

4.1 CONSTRUCTION.

4.1.1 DEFINITION.

Let J be a subset of $\mathbf{R}^m$. A map $S : J \to J$ is called a *contraction*, if there exists a $c \in (0,1)$ with

$$d(Sx, Sy) \leq cd(x,y)$$

for all $x, y \in J$.

(As always d denotes the euclidean metric.) The smallest c with the above property is called the *Lipschitz* or *contraction–constant* of S and will be denoted by Lip(S). By Con(J) we denote the space of all contractions of J into itself with the topology of pointwise convergence. Con_o(J) is this space with the empty relation $\emptyset$ added as an isolated point.

Let $N \geq 1$ be given and let μ be a probability measure on the Borel field $\mathcal{F}$ of $(Con_o(J))^N$.

In this section we set $D = \{1,\ldots,N\}^*$, μ^D is the product measure on $\Omega = ((Con_o(J))^N)^D$. The elements ω of Ω are trees $(S^\sigma)_{\sigma \in D}$, where $S^\sigma = (S_1^\sigma,\ldots,S_N^\sigma) \in (Con_o(J))^N$. We define

$$S_\sigma(\omega) = S_{\sigma_1}^{\emptyset} \circ S_{\sigma_2}^{\sigma_1} \circ \ldots \circ S_{\sigma_{|\sigma|}}^{\sigma|(|\sigma|-1)}$$

for $\sigma \neq \emptyset$ and $S_\emptyset = id_J$. Moreover, let $l_\sigma(\omega) = Lip(S^{\emptyset}_{\sigma_1}) \cdot Lip(S^{\sigma_1}_{\sigma_2}) \cdot \ldots Lip(S^{\sigma|(|\sigma|-1)}_{\sigma_{|\sigma|}})$ for $\sigma \neq \emptyset$ and $l_\emptyset(\omega) = 1$. Here $Lip(\emptyset)$ is defined to be 0. Then we have

$$Lip(S_\sigma(\omega)) \leq l_\sigma(\omega)$$

for all $\sigma \in \{1, \ldots, N\}^*$.

4.1.2 THE COMPACT SET CORRESPONDING TO A TREE FROM Ω.

Let J be compact.

Given a tree $\omega \in \Omega$ and $\sigma \in \{1, \ldots, N\}^*$ define

$$J_\sigma(\omega) = [S_\sigma(\omega)](J) .$$

Set $K_n(\omega) = \bigcup_{\sigma \in \{1,\ldots,N\}^n} J_\sigma(\omega)$ and $K(\omega) = \bigcap_{n \in \mathbb{N}} K_n(\omega)$.

Then $K(\omega)$ is compact. It is called the compact set corresponding to the tree ω.

The map $\psi : \Omega \to \mathcal{K}_o(J), \omega \to K(\omega)$ is Borel measurable. The pair (ψ, μ) is called the *μ–random set.* $P_\mu = \mu \circ \psi^{-1}$ is called the *distribution of the μ–random set.* Let $\Omega_\infty = \{\omega \in \Omega : \lim_{n \to \infty} \max_{|\sigma|=n} l_\sigma(\omega) = 0\}$. Then $\mu^D(\Omega_\infty) = 1$. For each $\omega \in \Omega_\infty$ let $E_\omega = \{\eta \in \{1, \ldots, N\}^{\mathbb{N}} : \bigcap_{n \in \mathbb{N}} J_{\eta|n} \neq \emptyset\}$ and let $\pi_\omega : E_\omega \to K(\omega)$ be defined by

$$\{\pi_\omega(\eta)\} = \bigcap_{n \in \mathbb{N}} J_{\eta|n}(\omega) .$$

Then π_ω is continuous and onto.

4.1.3 EXAMPLES.

a) *Random Cantor sets*

Let ν be a probability on Δ. For $(t_1, t_2) \in \Delta$ define

$$S_{t_1} : [0,1] \to [0,1] \text{ by } S_{t_1}(x) = t_1 x$$

and

$$S_{t_2} : [0,1] \to [0,1] \text{ by } S_{t_2}(x) = (1 - t_2) + t_2 x .$$

Let μ be the image measure of ν w.r.t. $\Delta \to (Con([0,1]))^2, t_1, t_2) \to (S_{t_1}, S_{t_2})$. Then the ν–random Cantor set and the μ–random set have the same distribution.

b) *Mandelbrot's percolation process*
Let ν_p be as in 3.1 and let $S_1, \dots, S_N$ be as in 3.2.1. Let μ_p be the distribution of $\{0,1\}^N \to Con_0([0,1]^2)^N$,
$(t_1, \dots, t_N) \to (T_1, \dots, T_N)$, where

$$T_i = \begin{cases} \emptyset, & t_i = 0 \\ S_i, & t_i = 1 \end{cases}$$

w.r.t. ν_p. Then the random set constructed in 3.1 and the μ–random set have the same distribution.

c) *Deterministic self–similar sets*
Let $f_1, \dots, f_N \in Con(\mathbf{R}^m)$. Then there exists a compact subset $J \subset \mathbf{R}^m$ with $f_i(J) \subset J$ for all $i = 1, \dots, N$. Let $\mu = \epsilon_{(f_1, \dots, f_N)}$ be a probability on $(Con_0(J))^N$. Then the μ–random Cantor set has the distribution ϵ_K, where K is the unique non–empty compact set with $K = f_1(K) \cup \dots \cup f_N(K)$.

d) *A random von Koch curve (Falconer [11])*
Let $J = \{(t_1, t_2) \in \mathbf{R}^2 | -\frac{1}{\sqrt{3}} t_1 \le t_2 \le \frac{1}{\sqrt{3}} t_1, -\frac{1}{\sqrt{3}}(1 - t_1) \le t_2 \le \frac{1}{\sqrt{3}}(1 - t_1)\}$ and let $S_1, S_2, S_3, S_4, S_2', S_3' : J \to J$ be defined as follows
$S_1(x) = \frac{1}{3}x$,
S_2 is multiplying by $\frac{1}{3}$, rotating by $60°$ and then translating by $(\frac{1}{3}, 0)$
S_3 is multiplying by $\frac{1}{3}$, rotating by $-60°$ and then translating by $(\frac{1}{2}, \frac{1}{6} \times \sqrt{3})$,
$S_4(x) = \frac{1}{3}x + (\frac{2}{3}, 0)$
S_2' is first applying S_2 and then reflecting at the x_1–axis
S_3' is first applying S_3 and then reflecting at the x_1–axis.
Let $\mu = \frac{1}{2}\epsilon_{(S_1, S_2, S_3, S_4)} + \frac{1}{2}\epsilon_{(S_1, S_2', S_3', S_4)}$. Then the corresponding μ–random set is a "random von Koch curve".

e) A random Sierpinski carpet (Mauldin–Williams [20]). Let $J = [0,1]^2$ and let ν be normalized Lebesgue measure on $(\frac{1}{3}, \frac{1}{2})^4$. For $t = (t_1, t_2, t_3, t_4) \in (\frac{1}{3}, \frac{1}{2})^4$ define $S_1^t, \dots, S_8^t : J \to J$ in the following way

$$
\begin{aligned}
S_1^t(x_1,x_2) &= (t_1x_1, t_1, x_2)\\
S_2^t(x_1,x_2) &= ((1-t_2)+t_2x_1, t_2x_2)\\
S_3^t(x_1,x_2) &= (t_3x_1, (1-t_3)+t_3x_2)\\
S_4^t(x_1,x_2) &= ((1-t_4)+t_4x_1, (1-t_4)+t_4x_2)\\
S_5^t(x_1,x_2) &= (t_1+(1-(t_1+t_2))x_1, (1-(t_1+t_2))x_2)\\
S_6^t(x_1,x_2) &= ((1-(t_1+t_3))x_1, t_1+(1-(t_1+t_3))x_2)\\
S_7^t(x_1,x_2) &= (1-(t_2+t_4)+(1-(t_2+t_4))x_1, t_2\\
&\quad +(1-(t_2+t_4))x_2)\\
S_8^t(x_1,x_2) &= (t_3+(1-(t_3+t_4))x_1,\ 1-(t_3+t_4)\\
&\quad +(1-(t_3+t_4))x_2)
\end{aligned}
$$

Let μ be the image of ν with respect to the map $(\frac{1}{3},\frac{1}{2})^4 \to Con(J)$, $t \to (S_1^t,\ldots,S_8^t)$. The μ–random set is a random Sierpinski carpet.

f) Random homeomorphisms
Let $J = [0,1]^2$ and let ν be a probability measure on $(0,1)^2$. For $t=(t_1,t_2)\in(0,1)^2$ define

$$S_1^t, S_2^t : J \to J$$

by

$$S_1^t(x_1,x_2) = (t_1x_1, t_2x_2)\ ,$$

$$S_2^t(x_1,x_2) = (t_1+(1-t_1)x_1,\ t_2+(1-t_2)x_2)\ ,$$

Let μ be the image measure of ν with respect to the map $(0,1)^2 \to Con(J)^2, t \to (S_1^t, S_2^t)$. Then P_μ–a.e. $K \in \mathcal{K}_o(J)$ is the graph of a homeomorphism of $[0,1]$ onto itself which preserves the end–points (see Dubins–Freedman [9] and Graf–Mauldin–Williams [14]).

4.2 Statistical Self–Similarity.

4.2.1 Definition.

Let μ be a probability on $Con_o(J)$. A probability P on $\mathcal{K}_o(J)$ is called μ *(–statistically)–self–similar* if P is the image of $\mu \otimes P^N$ w.r.t. the map $T : (Con_o(J))^N \times (\mathcal{K}_o(J))^N \to \mathcal{K}_0(J)^N$ defined by $T(S_1, \ldots, S_N, K_1, \ldots, K_N) = S_1(K_1) \cup \ldots \cup S_N(K_N)$.

4.2.2 THEOREM.

P_μ is μ–self–similar.

Proof. Similar to the proof of Theorem 2.2.2.

4.2.3 THEOREM.

Let q be the smallest non–negative solution of the equation

$$(*) \quad x = \sum_{j=0}^{N} \mu(\{(S_1, \ldots, S_N) \in Con_o(J)^N | \text{card}\,\{i | S_i \neq \emptyset\} = j\} x^j .$$

Then

$$P_\mu(\{\emptyset\}) = q .$$

Proof. Similar to the proof of Theorem 3.2.3

4.2.4 REMARKS.

a) 1 is always a root of equation $(*)$. Consider the function $f : \mathbf{R}_+ \to \mathbf{R}$ defined by

$$f(s) = \sum_{j=0}^{N} \mu(\{(S_1, \ldots, S_N) | \text{card}\,\{i | S_i \neq \emptyset\} = j\}) s^j - s .$$

Then $f'(s) = \sum_{j=1}^{N} j\, \mu(\{(S_1, \ldots, S_N) | \text{card}\,\{i|S_i \neq \emptyset\} = j\}) s^{j-1} - 1$. We have $f(1) = 0$ and $f'(1) = \int \text{card}\,\{i|S_i \neq \emptyset\} d\mu(S_1, \ldots, S_N) - 1$. Thus, if $\int \text{card}\,\{i|S_i \neq \emptyset\} d\mu(S_1, \ldots, S_N) < 1$, or if $\int \text{card}\,\{i|S_i \neq \emptyset\} d\mu(S_1, \ldots, S_N) = 1$ and $\mu(\{(S_1, \ldots, S_N) : \text{card}\,\{i|S_i \neq \emptyset\} = 1\}) < 1$, $f'(s) < 0$ for all $s \in (0,1)$. This implies $f(s) > 0$ for all $s \in (0,1)$. Hence 1 is the smallest non–negative root of $(*)$. By the preceding theorem we, therefore, deduce $P_\mu(\{\emptyset\}) = 1$, i.e.

$K(\omega) = \emptyset$ for μ^D–a.e. ω. If $\int \operatorname{card}\{i|S_i \neq \emptyset\}d\mu(S_1,\ldots,S_N) > 1$ then $f'(1) > 0$. Hence there is an $s_o \in (0,1)$ with $f(s_o) < 0$. Since $f(0) \geq 0$ this implies that the smallest non–negative root q of $(*)$ is less than 1. Hence $P_\mu(\{\emptyset\}) < 1$, i.e. $K \neq \emptyset$ with positive P_μ–probability. If $\operatorname{card}\{i|S_i \neq \emptyset\} = 1$ μ–a.e. then $f(s) = 0$ for all $s \in [0,1]$, hence $q = 0$ and $P_\mu(\{\emptyset\}) = 0$, i.e. $K \neq \emptyset$ for P_μ–a.e. K.

b) It can be shown that P_μ is the unique μ–self–similar measure on $\mathcal{K}_o(J)$ with $P_\mu(\{\emptyset\})$ being the smallest non–negative root q of $(*)$. If $q < 1$ then $\epsilon_\emptyset$ is a different μ–self–similar measure on $\mathcal{K}_o(J)$. There are no other μ–self–similar probability measures on $\mathcal{K}_o(J)$.

4.2.5 LEMMA.

If $\int \operatorname{card}\{i|S_i \neq \emptyset\}d\mu(S_1,\ldots,S_N) > 1$ *then there exists a unique* $\alpha > 0$ *with*

$$\int \sum_{i=1}^{N} \operatorname{Lip}(S_i)^\alpha d\mu(S_1,\ldots,S_N) = 1\ .$$

Proof. Similar to the proof of 2.2.3.

4.2.6 DEFINITION.

The α in the above lemma is called the *similarity dimension* of the μ–random set.

4.2.7 EXAMPLES.

a) For ν–random Cantor sets and for Mandelbrot's percolation process the similarity dimension defined above agrees with the similarity dimensions in Sections 2 and 3, respectively.

b) For deterministic self–similar sets given by $f_1,\ldots,f_N \in \operatorname{Con}(\mathbf{R}^m)$ the similarity dimension is the unique α with

$$\sum_{i=1}^{N} \operatorname{Lip}(f_i)^\alpha = 1\ .$$

c) For the random von Koch curve defined in 4.1.3 d) we get

$$
\begin{aligned}
1 &= \int \sum_{i=1}^{4} \mathrm{Lip}(T_i)^\alpha d(\frac{1}{2}\epsilon_{(S_1,S_2,S_3,S_4)} + \frac{1}{2}\epsilon_{(S_1,S_2',S_3',S_4)})(T_1,\dots,T_4) \\
&= \frac{1}{2}\sum_{i=1}^{4} \mathrm{Lip}(S_i)^\alpha + \frac{1}{2}(\mathrm{Lip}(S_1)^\alpha + \mathrm{Lip}(S_2')^\alpha \\
&\quad + \mathrm{Lip}(S_3')^\alpha + \mathrm{Lip}(S_4)^\alpha) \\
&= 4(\frac{1}{3})^\alpha ,
\end{aligned}
$$

hence $\alpha = \frac{\log 4}{\log 3}$.

d) For the random Sierpinski carpet defined in 4.1.3 e) we get

$$
\begin{aligned}
1 &= 6^4 \iiiint_{(\frac{1}{3},\frac{1}{2})^4} \sum_{i=1}^{8} Lip(S_i^t)^\alpha dt_1 dt_2 dt_3 dt_4 = 6^4 \iiiint_{(\frac{1}{3},\frac{1}{2})^4} t_3^\alpha + t_4^\alpha \\
&\quad + (1-(t_1+t_2))^\alpha + (1-(t_1+t_3))^\alpha + (1-(t_2+t_4))^\alpha \\
&\quad + (1-(t_3+t_4))^\alpha dt_1 dt_2 dt_3 dt_4 \\
&= 24\int_{1/3}^{1/2} t^\alpha dt + 144 \int_{1/3}^{1/2}\int_{1/3}^{1/2} (1-(s+t))^\alpha ds dt \\
&= \frac{24}{\alpha+1}(\frac{1}{2^{\alpha+1}} - \frac{1}{3^{\alpha+1}} + \frac{2}{\alpha+2}(\frac{1}{3^{\alpha+1}} - \frac{1}{6^{\alpha+1}})) .
\end{aligned}
$$

It can be shown that $\alpha \simeq 1.8947$.

e) Let $(0,1)^2 \to \mathrm{Con}([0,1]^2)^2, t \to (S_1^t, S_2^t)$, ν, μ as in 4.1.3 f). Then $\mathrm{Lip}(S_1^t) = \max(t_1,t_2)$ and $\mathrm{Lip}(S_2^t) = \max(1-t_1, 1-t_2)$. Thus the similarity dimension of the μ–random set is equal to $\alpha \geq 0$ with

$$
1 = \int_{(0,1)^2} \max(t_1,t_2)^\alpha + \max(1-t_1,1-t_2)^\alpha d\nu(t_1,t_2) .
$$

This α equals 1 if and only if $\max(t_1,t_2) + \max(1-t_1,1-t_2) = 1$ for ν–a.e. (t_1,t_2), i.e. $t_1 = t_2$ for ν-a.e. Hence $\alpha = 1$ if and only if ν is concentrated on the diagonal of $(0,1)^2$. In this case

$$
P_\mu = \epsilon_K, \text{ where } K = \{(x,x) | x \in [0,1]\} .
$$

In all other cases we have $\alpha > 1$.

4.3 A Natural Random Measure Corresponding to a μ–Random Set.

As before $J \subset \mathbf{R}^m$ is compact, μ is a probability on $Con_o(J)^N$, $D = \{1,\ldots,N\}^*$, $\Omega = (\mathrm{Con}_o(J)^N)^D$, and μ^D is the product measure on Ω. We will always assume that $\int \mathrm{card}\,\{i | S_i \neq \emptyset\} d\mu(S_1,\ldots,S_N) > 1$ and that α is the similarity dimension of the μ–random set.

4.3.1 Definition.

For $n \in \mathbf{N}$ let $X_n : \Omega \to \mathbf{R}$ be defined by

$$X_n(\omega) = \sum_{\sigma\in\{1,\ldots,N\}^n} l_\sigma^\alpha(\omega) \ .$$

Let $\mathcal{F}_o = \{\emptyset, \Omega\}$ and let $\mathcal{F}_n$ be the σ–field generated by the canonical projections $g_\sigma : \Omega \to \mathrm{Con}_o(J)^N$ with $|\sigma| \leq n-1$.

4.3.2 Theorem (Mauldin–Williams [20]).

$(X_n)_{n\in\mathbf{N}}$ is an L^p–bounded martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbf{N}}$ for every $p \in$ $[1,\infty)$.

Proof. See, for instance, Graf [13], p. 378/379.

4.3.3 Remarks and Definition.

Let $L = \{\omega \in \Omega \,|\, \lim_{n\to\infty} X_n(\omega)$ exists in $\mathbf{R}\}$. By the martingale convergence theorem we have $\mu^D(L) = 1$. Let $X : \Omega \to \mathbf{R}$ be a random variable with $X(\omega) = \lim_{n\to\infty} X_n(\omega)$ for every $\omega \in L$. For $\sigma \in D$ define $\varphi_\sigma : \Omega \to \Omega$ by

$$\varphi_\sigma((S^\tau)_{\tau\in D}) = (S^{\sigma*\tau}))_{\tau\in D} \ ,$$

$X_\sigma = X \circ \varphi_\sigma$.

Set $M = \bigcap_{\sigma\in D}\varphi_\sigma^{-1}(L)$. Then $\mu^D(M) = 1$, and, for every $\omega \in M$ we have

$$X(\omega) = \sum_{\sigma\in\Gamma} l_\sigma^\alpha(\omega) X_\sigma(\omega)$$

for all $\Gamma \in$ Min.

For $\omega = (S^\sigma)_{\sigma \in D} \in M$ we will define a finite measure ν_ω on the Borel field of $\{1, \ldots, N\}^{\mathbf{N}}$ as follows:

For $\tau \in D$ let $A(\tau) = \{\eta \in \{1, \ldots, N\}^{\mathbf{N}} | \eta > \tau\}$. Define

$$\nu_\omega(A(\tau)) = l_\tau^\alpha(\omega) X_\tau(\omega) .$$

Then ν_ω can be extended to a unique finite measure on the Borel field of $\{1, \ldots, N\}^{\mathbf{N}}$. Let $E_\omega \subset \{1, \ldots, N\}^{\mathbf{N}}$ and $\pi_\omega : E_\omega \to K(\omega)$ be defined as in 4.1.2. It can be shown that $\nu_\omega(\{1, \ldots, N)^{\mathbf{N}} \backslash E_\omega) = 0$. We set $\mu_\omega = \nu_\omega \circ \pi_\omega^{-1}$. Then μ_ω is a finite measure on $\mathbf{R}^m$ whose support equals $K(\omega)$. The map $\Omega \to \mathcal{M}_+(\mathbf{R}^m)$, $\omega \to \mu_\omega$ is measurable. Its distribution w.r.t. μ^D is denoted by Q_μ.

4.3.4 DEFINITION.

Let $T : (Con_o(J))^N \times \mathcal{M}_+(J)^N \to \mathcal{M}_+(J)$ be defined by

$$T((S_1, \ldots, S_N), \nu_1, \ldots, \nu_N) = \sum_{i=1}^{N} \mathrm{Lip}(S_i)^\alpha \nu_i \circ S_i^{-1} .$$

A probability P on $\mathcal{M}_+(J)$ is called *μ–self–similar* if $\mu \otimes P^N \circ T^{-1} = P$.

4.3.5 THEOREM (Arbeiter [1]).

The probability Q_μ is the unique μ–self–similar probability P on $\mathcal{M}_+(J)$ which satisfies

$$\int \nu(J)\, dP(\nu) = 1 .$$

Proof. That Q_μ is μ–self–similar follows from a "commuting diagram" argument similar to that given in 2.2.3. The proofs of the other parts of the theorem will be omitted.

4.4 HAUSDORFF MEASURE AND DIMENSION.

Again we will always assume that

$$\int \text{card}\,\{i|S_i \neq \emptyset\}d\mu(S_1,\ldots,S_N) > 1\ .$$

Moreover J is compact with $\overset{\circ}{J} \neq \emptyset$. Let α be the similarity dimension of the μ–random set.

4.4.1 THEOREM.

Suppose that, for μ–a.e. $(S_1,\ldots,S_N) \in Con_o(J)^N$ and every $i \in \{1,\ldots,N\}$ with $S_i \neq \emptyset$, there exists a $c > 0$ such that $d(S_ix, S_iy) \geq cd(x,y)$ for all $x, y \in J$. Then, for every $\beta \geq 0$,

(i) $P_\mu(\{K \in \mathcal{K}_o(J)|K \neq \emptyset \text{ and } \mathcal{H}^\beta(K) = 0\}) = 0$ or $= P_\mu(\mathcal{K}(J))$

(ii) $P_\mu(\{K \in \mathcal{K}_o(J)|\mathcal{H}^\beta(K) = \infty\}) = 0$ or $= P_\mu(\mathcal{K}(J))$.

Proof. Similar to the proof of Theorem 2.4.2.

4.4.2 COROLLARY.

Let the assumptions of Theorem 4.4.1 be satisfied. Then there exists a constant $c \geq 0$ such that, for P_μ–a.e. $K \in \mathcal{K}_o(X)$, $K \neq \emptyset$ implies H–dim(K)= c.

4.4.3 THEOREM (Falconer [10], Mauldin–Williams [20]).

For P_μ–a.e. $K \in \mathcal{K}_o$ ([J]),

$$\mathcal{H}^\alpha(K) < \infty\ .$$

Proof. Similar to the proof of Theorem 2.4.4.

4.4.4 THEOREM (Falconer [10], Mauldin–Williams [20]).

Let μ be such that, for μ–a.e. $(S_1,\ldots,S_N) \in Con_o(J)^N$, the following conditions are satisfied:

(i) If $S_i \neq \emptyset$ *then* S_i *is a similarity (i.e., there is an* $r > 0$ *with* $d(S_i x, S_i y) = r d(x, y)$ *for all* $x, y \in J$*).*
(ii) If $i \neq j$ *then* $S_i(\mathring{J}) \cap S_j(\mathring{J}) = \emptyset$.
Then, for P_μ*-a.e.* $K \in \mathcal{K}_o(J)$,

$$K \neq \emptyset \Rightarrow H\text{-dim}(K) = \alpha .$$

Proof. Similar to that of Theorem 2.4.8.

4.4.5 THEOREM (Graf [13]).

Let μ *satisfy the assumption of Theorem 4.4.4. Moreover, let* $\mu(\{(S_1, \ldots, S_N) \mid \sum_{i=1}^N \mathrm{Lip}(S_i)^\alpha \neq 1\} > 0$. *Then* $\mathcal{K}^\alpha(K) = 0$ *for* P_μ*-a.e.* $K \in \mathcal{K}_o(J)$.

Proof. Similar to that of Theorem 2.4.9.

4.4.6 THEOREM (Graf [13]).

Let μ *satisfy the assumptions of Theorem 4.4.4. Moreover, let there be a* $\delta > 0$ *such that, for* μ*-a.e.* $(S_1, \ldots, S_N) \in Con_o(J)$, $S_i \neq \emptyset$ *implies* $Lip(S_i) \geq \delta$ $(i = 1, \ldots, N)$. *Then the following statements are equivalent:*
(i) $\sum_{i=1}^N \mathrm{Lip}(S_i)^\alpha = 1$ *for* μ*-a.e.* $(S_1, \ldots, S_N) \in Con_o(J)$
(ii) $\mathcal{H}^\alpha(K) > 0$ *for* P_μ*-a.e.* $K \in \mathcal{K}_o([0,1])$ *with* $K \neq \emptyset$.

Proof. See, for instance, Graf [13], Theorem 5.8.

4.4.7 REMARK.

An example in [15] shows, that the δ–condition in the preceding theorem is essential.

4.4.8 THEOREM (Graf–Mauldin–Williams [15]).

Let the assumptions of Theorem 4.4.5 be satisfied. Suppose that there *exists a* $\xi > 0$ *with*

$$\int (\min\{\mathrm{Lip}(S_i) \,|\, i \in \{1,\dots,N\} \textit{ with } \mathrm{Lip}\,(S_i) \neq 0\}^{-\xi} d\mu(S_1,\dots,S_N) < \infty$$

Moreover, suppose that there is a $\delta \in (0,1)$ *and* $p \in \mathbf{N}$ *with*

$$\int_{\{(S_1,\dots,S_N)|\mathrm{Lip}(S_i) \leq 1-\delta \textit{ for } i=1,\dots,N\}} (\sum_{i=1}^{N} \mathrm{Lip}(S_i)^m)^t \, d\mu(S_1,\dots,S_N) \geq t^{-p}$$

for all large t and that J is a finite union of convex sets.

Then, for P_μ*–a.e.* $K \in \mathcal{K}(J)$,

$$0 < \mathcal{H}^h(K) < \infty$$

where $h(x) = x^\alpha(\log|\log x|)^{1-(\alpha/m)}$.

Proof. See Graf–Mauldin–Williams [15].

4.4.9 EXAMPLES.

a) For the random von Koch curve of Example 4.1.3 d) Theorems 4.4.3 and 4.4.6 can be applied and yield that for $\alpha = \frac{\log 4}{\log 3}$ we have

$$0 < \mathcal{H}^\alpha(K) < \infty\,,$$

hence H–dim$(K) = \alpha$ for P_μ–a.e. K.

b) For the random Sierpinski carpet of Example 4.1.3 e) the assumptions of Theorems 4.4.3, 4.4.4, 4.4.5 and 4.4.8 are satisfied. Thus we obtain for $\alpha \simeq 1.8947$ that H–dim (K)= α, $\mathcal{H}^\alpha(K) = 0$ and

$$0 < \mathcal{H}^h(K) < \infty$$

for P_μ–a.e. K, where $h(t) = t^\alpha(\log|\log t|)^{1-\alpha/2}$. The details are left as an exercise (see also Graf–Mauldin–Williams [15], Example 6.7)

c) Let ν, μ be as in Example 4.1.3f. If ν is not concentrated on the diagonal of $(0,1) \times (0,1)$ then the corresponding μ–random set has similarity dimension larger than 1. On the other hand, the Hausdorff dimension of every graph of a homeomorphisms of [0,1] onto itself is equal to 1. Thus the random homeomorphisms exhibit an example where the similarity dimension and the Hausdorff dimension do not agree. The reason is that, for ν not on the diagonal of $(0,1)^2$, the corresponding μ is not concentrated on the set of pairs of similarity maps.

4.4.10 LEMMA.

Let $h(t) = t^\beta \log|\log t|^\theta$, *where* $\beta, \theta \in \mathbf{R}_+$. *Let* $E \subset \mathbf{R}^m$ *and let* $S : E \to \mathbf{R}^m$ *be a similarity.*

Then

$$\mathcal{H}^h(S(E)) = \mathrm{Lip}(S)^\beta \mathcal{H}^h(E) .$$

Proof. Exercise!

4.4.11 THEOREM (Graf–Mauldin–Williams [15]).

Let μ *satisfy all assumptions in Theorem 4.4.8. Let* $(\mu_\omega)_{\omega \in \Omega}$ *be the random measure as defined in 4.3.3. Then there exists a constant* $c > 0$ *such that, for* μ^D*–a.e.* $\omega \in \Omega$,

$$\mu_\omega = c\mathcal{H}^h_{|K(\omega)} ,$$

where $h(t) = t^\alpha (\log|\log t|)^{1-\alpha/m}$.

Sketch of proof. Let $\varphi_\sigma : \Omega \to \Omega$ be defined as in 4.3.3. Then $\mu^D = \mu^D \circ \varphi_\sigma^{-1}$.

We have

$$K(\omega) = \cup_{\sigma \in \{1,\dots,N\}^n} S_\sigma(\omega)(K(\varphi_\sigma(\omega))$$

hence

$$\mathcal{H}^h(K(\omega)) \le \sum_{\sigma \in \{1,\dots,N\}^n} \mathcal{H}^h(S(\omega)(K(\varphi_\sigma(\omega))))$$

$$\underset{4.4.10}{\overset{\uparrow}{=}} \sum_{\sigma \in \{1,\dots,N\}^n} \mathrm{Lip}(S(\omega))^\alpha \mathcal{H}^h(K(\varphi_\sigma)(\omega)) .$$

Since all maps under consideration are similarities we have

$$(\mathrm{Lip}(S(\omega))^{\alpha} = l_{\sigma}^{\alpha}(\omega) \ .$$

Since l_{σ}^{α} and $\mathcal{H}^h(K(\varphi_\sigma(\ )))$ are independent we obtain

$$\begin{aligned}
&\int \mathcal{H}^h(K(\omega))\, d\mu^D(\omega) \\
&\quad \le \sum_{\sigma\in\{1,\dots,N\}^n} \int l_\sigma(\omega)^\alpha d\mu^D(\omega) \int \mathcal{H}^h(K(\varphi_\sigma(\omega)))\, d\mu^D(\omega) \\
&\quad = \int \mathcal{H}^h(K(\omega))\, d\mu^D(\omega) \underbrace{\sum_{\sigma\in\{1,\dots,N)^n} \int l_\sigma(\omega)^\alpha d\mu^D(\omega)}_{=1} \ .
\end{aligned}$$

This implies equality throughout and hence, for μ a.e. ω,

$$\mathcal{H}^h(K(\omega)) = \sum_{\sigma\in\{1,\dots,N\}^n} l_\sigma^\alpha(\omega)\mathcal{H}^h(K(\varphi_\sigma(\omega))$$

as well as $\mathcal{H}^h(S_\sigma(\omega)(K(\varphi_\sigma(\omega))) \cap S_\tau(\omega)(K(\varphi_\tau(\omega)))) = 0$ for $|\sigma| = |\tau| = n$ and $\sigma \neq \tau$. If $E(\ \ |\mathcal{F}_n)$ denotes the conditional expectation with respect to $\mathcal{F}_n$ we deduce

$$\begin{aligned}
E(\mathcal{H}^h(K(\ ))|\mathcal{F}_n) &= \int \mathcal{H}^h(K(\omega))\, d\mu^D(\omega) \sum_{\sigma\in\{1,\dots,N\}^n} 1_\sigma^\alpha \\
&= \int \mathcal{H}^h(K(\omega))\, d\mu^D(\omega) X_n \ . \quad \mu^D - \text{a.e.}
\end{aligned}$$

By the martingale convergence theorem the left hand side converges to $\mathcal{H}^h(K(\ ))$ and the right hand side converges to $\int \mathcal{H}^h(K(\omega))\, d\mu^D(\omega) \cdot X$, hence

$$(*) \quad \mathcal{H}^h(K(\omega)) = c^{-1} X(\omega)$$

for μ^D-a.e. $\omega \in \Omega$, where $c = (\int \mathcal{H}^h(K)\, dP_\mu(K))^{-1}$.

It follows from the definition of μ_ω that, for $A \subset \mathbf{R}^m$ compact,

$$\mu_\omega(A) = \inf_{k\in\mathrm{N}} \sum_{\substack{\sigma\in\{1,\dots,N\}^k \\ J_\sigma(\omega)\cap A\neq\emptyset}} l_\sigma^\alpha(\omega) X_\sigma(\omega) \ .$$

From (*) we deduce

$$X_\sigma(\omega) = c\mathcal{H}^h(K|(\varphi_\sigma(\omega)) ,$$

hence

$$\begin{aligned}\mu_\omega(A) &= c \inf_{k\in\mathbf{N}} \sum_{\substack{\sigma\in\{1,\ldots,N\}^k \\ J_\sigma(\omega)\cap A\neq\emptyset}} l_\sigma^\alpha(\omega)\mathcal{H}^h(K(\varphi_\sigma(\omega)) \\ &\underset{4.4.10}{=} c \inf_{k\in\mathbf{N}} \sum_{\substack{\sigma\in\{1,\ldots,N\}^k \\ J_\sigma(\omega)\cap A\neq\emptyset}} \mathcal{H}^h(S_\sigma(\omega)(K(\varphi_\sigma(\omega))) \\ &= c \inf_{k\in\mathbf{N}} \mathcal{H}^h(\cup_{\substack{\sigma\in\{1,\ldots,N\}^k \\ J_\sigma(\omega)\cap A\neq\emptyset}} S_\sigma(\omega)(K(\varphi_\sigma(\omega))) .\end{aligned}$$

Since $\cup_{\substack{\sigma\in\{1,\ldots,N\}^k \\ J_\sigma(\omega)\cap A\neq\emptyset}} S_\sigma(\omega)(K(\varphi_\sigma(\omega))) \downarrow A\cap K(\omega)$ for $k \uparrow \infty$ we obtain

$$\inf_{k\in\mathbf{N}} \mathcal{H}^k(\cup_{\substack{\sigma\in\{1,\ldots,N\}^k \\ J_\sigma(\omega)\cap A\neq\emptyset}} S_\sigma(\omega)(K(\varphi_\sigma(\omega))) = \mathcal{H}^h(A\cap K(\omega)) .$$

Thus μ_ω and $c\mathcal{H}^h|_{K(\omega)}$ agree on all compact subsets of $\mathbf{R}^m$. This immediately extends to Borel sets. Hence the theorem is proved.

4.4.12 REMARKS.

a) It can be shown that, under the assumptions of 4.4.4, the Hausdorff– and box–dimension of P_μ–a.e. K agree.
b) It is an open problem whether the natural measure on $K(\omega)$ equals a multiple of the equidistribution on $K(\omega)$ for μ^D–a.e. ω.

4.5 THE SECOND ORDER DENSITY.

Let the general assumptions be as in Section 4.4.

4.5.1 DEFINITION.

Let ν be a (finite) Borel measure on $\mathbf{R}^m$. The *carrying dimension* of ν is the unique $\alpha \geq 0$ (if it exists) such that there exists a Borel set $E \subset \mathbf{R}^m$ with $\nu(\mathbf{R}^m\backslash E) = 0$ and

$$H\text{-dim}(E) \leq \alpha$$

and, moreover, for all B Borel with $\nu(B) > 0$, $H\text{-dim}(B) \geq \alpha$.

4.5.2 REMARKS.

a) It can be shown that, under the assumptions of Theorem 4.4.4, P_μ–a.e. K with $K \neq \emptyset$ has carrying dimension α, where α is the similarity dimension of the μ–random set.
b) For measures ν with carrying dimension α one considers the *upper density* at x, i.e.

$$\overline{\lim}_{r-0} \frac{\nu(B(x,r))}{r^\alpha}$$

and the *lower density* at x, i.e.

$$\underline{\lim}_{r\to 0} \frac{\nu(B(x,r))}{r^\alpha} \, .$$

In general these two numbers do not agree. But if ν is m–dimensional Lebesgue measure restricted to a Borel set $A \subset \mathbf{R}^m$ of positive ν–measure then $\alpha = m$ and the two numbers agree and are equal to 1 for ν–a.e. $x \in A$ (Lebesgue density theroem). To retain part of the conclusion of the Lebesgue density theorem in more general situations Bedford and Fisher [5] introduced a generalized notion of density.

4.5.3 DEFINITION.

Let ν on $\mathbf{R}^m$ have carrying dimension α. Then

$$\bar{D}(\nu, x) := \overline{\lim}_{T\to\infty} \frac{1}{T} \int_0^T \frac{\nu(B(x, e^{-t}))}{e^{-\alpha t}} \, dt$$

and

$$\underline{D}(\nu, x) := \underline{\lim}_{T\to\infty} \frac{1}{T} \int_0^T \frac{\nu(B(x, e^{-t}))}{e^{-\alpha t}} \, dt$$

the *upper–* and *lower second order density* of ν at $x \in \mathbf{R}^m$. If $\overline{D}(\nu, x) = \underline{D}(\nu, x)$ then this number is denoted by $D(\nu, x)$ and called the *second order density* of ν at x.

4.5.4 THEOREM (Patzschke–M. Zähle, preprint).

Let the assumptions of Theorem 4.4.6 be satisfied. Then there exists a $c > 0$ *such that, for* μ^D*–a.e.* $\omega \in \Omega$ *and for* μ_ω*–a.e.* $x \in \mathbf{R}^m$,

$$D(\mu_\omega, x) = c \, .$$

Proof. Not published yet.

4.5.5 REMARK.

a) Patzschke–M. Zähle proved a version of the above theorem in a different, more general context.
b) It is an open problem whether the δ–condition in the assumptions of Theorem 4.5.4 can be removed.

4.6 PROBLEMS.

4.6.1 GENERALIZATION OF THE CONSTRUCTION OF μ–RANDOM SETS.

Can the assumption, that J is compact in the construction of a μ–random set, be replaced by a weaker probabilistic condition?

4.6.2 GENERALIZATION OF THE OPEN SET CONDITION.

Can condition (ii) in Theorem 4.5.4 (the so called "open set condition") be replaced by a weaker probabilistic condition?

4.6.3 TOPOLOGICAL PROPERTIES OF μ–RANDOM SETS.

Find the topological properties of μ–random sets for special μ. In particular, if $(\mu_p)_{0\leq p\leq 1}$ is a family of probabilities on $Con_o(J)^N$ with $K = \emptyset$ for P_{μ_o}–a.e. K and $K = J$ for P_{μ_1}–a.e. K, what kinds of topological phase transitions can occur?

To what extent do two typical realizations of a μ–random set agree?

4.6.4 RELAXATION OF INDEPENDENCE CONDITIONS.

Can the product measure μ^D on Ω be replaced by others measures?

5. Brownian motion and fractional Brownian motion.

As the botanist R. Brown already observed in 1827 a small particle suspended in a liquid moves on a highly irregular path. This movement is caused by the bombardment of the particle by molecules.

In 1923 Wiener gave a mathematical model for Brownian motion.

First we will describe Wiener's model.

5.1 DEFINITION OF BROWNIAN MOTION.

5.1.1 DEFINITION.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family $X = (X_t)_{t \in \mathbf{R}_+}$ of random variables $X_t : \Omega \to \mathbf{R}$ is called a *stochastic process* (or a random function on $\mathbf{R}_+$). For each $\omega \in \Omega$ the function $t \to X_t(\omega)$ is called the *realization* (or trajectory) of the process corresponding to the outcome ω.

A *Brownian motion* (or Wiener process) is a stochastic process $X = (X_t)_{t \in \mathbf{R}}$ with the following properties

(i) $P(\{\omega \in \Omega \,|\, X_o(\omega) = 0 \text{ and } t \to X_t(\omega) \text{ continuous}\}) = 1$

(ii) For every $t \geq 0$ and every $h > 0$ the increment $X_{t+h} - X_t$ has a normal distribution which mean 0 and variance h, i.e.

$$P(X_{t+h} - X_t \leq x) = \frac{1}{\sqrt{2\pi h}} \int_{-\infty}^{x} \exp(-u^2/2h)\, du\,.$$

(iii) If $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ then the increments $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

5.1.2 REMARKS.

a) X_t has a normal distribution with mean 0 and variance t.

b) For $h > 0$ the distribution of $X_{t+h} - X_t$ is independent of t.

5.1.3 THEOREM.

There exists a probability space $(\Omega, \mathcal{F}, P\}$ *and a Brownian motion* $X = (X_t)_{t \in \mathbf{R}_+}$ *on* $(\Omega, \mathcal{F}, P)$.

Proof. See, for instance, Adler [3]

5.1.4 REMARK.

In the above theorem Ω can be chosen to be $C_o(\mathbf{R}_+) = \{f : \mathbf{R}_+ \to \mathbf{R} \,|\, f \text{ continuous}, f(0) = 0\}$, $\mathcal{F}$ can be chosen to be the Borel field $\mathcal{B}(C_o(\mathbf{R}_+))$ generated by the topology of uniform convergence on compact sets and the process $(X_t)_{t \in \mathbf{R}_+}$ can be defined by $X_t(f) = f(t)$. The corresponding probability measure $P = W$ is called the *Wiener measure* on $C_o(\mathbf{R}_+)$. It can be shown that W is the unique probability on $C_o(\mathbf{R}_+)$ which satisfies (ii) and (iii) in 5.1.1.

5.1.5 Two Methods for the Construction of Brownian Trajectories.

a) Approximation by random walks

Let $\tau > 0$ be given. Let ν be the probability measure on $\{-1, 1\}$ defined by $\nu = \frac{1}{2}(\epsilon_1 + \epsilon_{-1})$. Let ν^{N} be the product measure on $\{-1, 1\}^{\mathrm{N}}$. Let $Y_n : \{-1, 1\}^{\mathrm{N}} \to \{-1, 1\}$ be the projection onto the n-th coordinate. Define

$$X_\tau(t) = \sqrt{\tau}(Y_1 + \ldots + Y_{[t/\tau]}) ,$$

where $[t/\tau]$ denotes the greatest integer less than or equal to t/τ.

By the central limit theorem, if τ is small compared to t, then the distribution of $X_\tau(t)$ is approximately normal with mean 0 and variance t, since the Y_i have mean 0 and variance 1. Thus for τ small compared to t the values $X_\tau(t)$ are a good approximation to a Brownian trajectory.

b) Random midpoint displacement

Here we will consider Brownian motion as a random function $X : [0, 1] \to \mathbf{R}$. We define the values $X(k2^{-j})$ with $0 \leq k \leq 2^j$ by induction on j. We set $X(0) = 0$ and choose $X(1)$ at random with respect to the normal distribution with mean 0 and variance 1. Next we select $X(\frac{1}{2})$ at random with respect to the normal distribution with mean $\frac{1}{2}(X(0) + X(1))$ and variance $\frac{1}{2}$. at the next step $X(\frac{1}{4})$ and $X(\frac{3}{4})$ are chosen, and so on. At the j-th stage the values $X(k2^{-j})$ for k odd are chosen independently with respect to a normal distribution with mean $\frac{1}{2}(X((k-1)2^{-j}) + X((k+1)2^{-j}))$ and variance 2^{-j}. This procedure determines $X(t)$ at all dyadic rationals t. One can show that, with probability one, this function on the dyadic rationals in uniformly continuous and, therefore, extends to a continuous function on [0,1].

5.2 Statistical Self-Similarity.

5.2.1. Scaling Behaviour of Sample Functions.

Let W be Wiener measure on $C_o(\mathbf{R}_+)$ and let $\gamma > 0$ be given. Define $T_\gamma : C_o(\mathbf{R}_+) \to C_o(\mathbf{R}_+)$ by $[T_\gamma(f)](t) = \gamma^{-1/2} f(\gamma t)$.

Then

$$W = W \circ T_\gamma^{-1} .$$

Proof. To prove the result we have to show that $P_\gamma = W \circ T_\gamma^{-1}$ satisfies

$$P_\gamma(X_{t+h} - X_t \le x) = (2\pi h)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2h)\,du\,,$$

where $(X_t)_{t\in\mathbf{R}_+}$ is defined as in Remark 5.1.4.

We have

$$\begin{aligned}
P_\gamma(X_{t+h} - X_t \le x) &= P_\gamma(\{f \in C_o(\mathbf{R}_+) | f(t+h) - f(t) \le x\}) \\
&= P\{f \in C_o(\mathbf{R}) | \gamma^{-1/2} f(\gamma(t+h)) - \gamma^{-1/2} f(\gamma t) \le x\}) \\
&= P(\{f \in C_o(\mathbf{R}) | f(\gamma(t+h)) - f(\gamma t) \le \gamma^{1/2} x\}) \\
&\underset{5.1.1}{=} (2\pi\gamma h)^{-1/2} \int_{-\infty}^{\gamma^{1/2}x} \exp(-u^2/2\gamma h)\,du \\
&\quad (v = \frac{u}{\sqrt{\gamma}}) \\
&= (2\pi h)^{-1/2} \int_{-\infty}^{x} \exp(-v^2/2h)\,dv \\
&= W(X_{t+h} - X_t \le x)\,.
\end{aligned}$$

It can be shown that P_γ satisfies (iii) in 5.1.1.

Thus the result is proved.

5.2.2 A CHARACTERIZATION OF WIENER MEASURE BY A SCALING PROPERTY.

Let $R : C_o(\mathbf{R}_+) \to C_o([0,1])$, $f \in f_{|[0,1]}$ and $W_o = W \circ R^{-1}$. Define $T : C_o([0,1]) \times C_o([0,1]) \to C_o([0,1])$ by

$$T(f,g)(t) = \begin{cases} \frac{1}{\sqrt{2}} f(2t), & t \le \frac{1}{2} \\ \frac{1}{\sqrt{2}}(f(1) + g(2t-1)), & t > \frac{1}{2} \end{cases}.$$

Then the probability W_o is the unique probability P on $C_o([0,1])$ with

$$(1) \qquad (P \otimes P) \circ T^{-1} = P$$

and

$$(2) \qquad \int |f(1)|^2 dP(f) = 1\,.$$

Proof.

(i) First we will show that W_o satisfying conditions (1) and (2). Condition (2) follows from the fact that the distribution of $X_1 : f \to f(1)$ is a normal distribution with mean 0 and variance 1.

To prove (1) it remains to check (ii) in definition 5.1.1 If $t \leq t+h \leq \frac{1}{2}$ then

$$(W_o \otimes W_o) \circ T^{-1}(\{f \in C_o([0,1]) | f(t+h) - f(t) \leq x\})$$
$$= W_o \otimes W_o(\{f,g) | \frac{1}{2}(f(2t+2h) - f(2t) \leq x\})$$
$$= (2\pi h)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2h)\, du\,.$$

If $t < \frac{1}{2} \leq t+h$ then

$$(W_o \otimes W_o) \circ T^{-1}(\{f \in C_0([0,1]) | f(t+h) - f(t) \leq x\})$$
$$= W_o \otimes W_o(\{(f,g) | \frac{1}{\sqrt{2}}(f(1) + g(2t+2h-1) - f(2t)) \leq x\})$$
$$= W_o \otimes W_o(\{(f,g) | (f(1) - f(2t)) \leq \sqrt{2}(x - g(2t+2h-1))\})$$
$$= \int \left[\frac{1}{\sqrt{2\pi(1-2t)}} \int_{-\infty}^{\sqrt{2}(x-g(2t+2h-1))} \exp(-u^2/(2(1-2t))\, du \right] dW_o(g)$$
$$= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi(1-2t)}} \int_{-\infty}^{\infty} \exp(-u^2/(2(1-2t))\, du \right] \exp(-y^2/(2(2t+2h-1))\, dy\,.$$

After some calculations this integral turns out to be equal to $\frac{1}{\sqrt{2\pi h}} \int_{-\infty}^{\infty} \exp(-v^2/(2h))\, dv$.

If $\frac{1}{2} \leq t < t+h$ then

$$(W_o \otimes W_o) \circ T^{-1}(\{f \in C_o([0,1]) | f(t+h) - f(t) \leq x\})$$
$$= W_o \otimes W_o(\{(f,g) | \frac{1}{\sqrt{2}}(f(1) + g(2t+2h-1) - (f(1) + g(2t-1))) \leq x\}) = \frac{1}{\sqrt{2\pi h}} \int_{-\infty}^{x} \exp(-u^2/2h)\, du\,.$$

Thus (ii) in 5.5.1 is satisfied.

(ii) Now let P be a probability on $C_o([0,1])$ satisfying conditions (1) and (2).

We will sketch the proof that $P = W_0$. First we will show that the distribution of X_1 w.r.t. P is a normal distribution with mean 0 and variance 1. Since $(P \otimes P) \circ T^{-1} = P$, X_1 has the same distribution as the sum of two independent copies of X_1 divided by $\sqrt{2}$. By Bauer [4], Korollar 53.4 this implies that X_1 has a normal distribution with mean 0. By assumption (2) the variance of X_1 equals 1. Let $Q_o = \{0,1\}$, $Q_n = \{\frac{j}{2^n} | j = 0, \ldots, 2^n\}$. By induction on n we can show that, for each $q_1, q_2 \in Q_n$ with $q_1 > q_2$ the random variable $X_{q_1} - X_{q_2}$ has a normal distribution with mean 0 and variance $q_1 - q_2$. Since $Q = \bigcup_{n \in \mathbf{N}} Q_n$ is dense in [0,1] and since P is a probability on $C_0([0.1])$ it follows that for all $t \in [0,1]$ and all $h > 0$ the random variable $X_{t+h} - X_t$ has a normal distribution with mean 0 and variance h. The independence condition (iii) in 5.1.1 can also be checked.

5.3 BROWNIAN BRIDGE AND LOCAL TIME.

Let $(X_t)_{t \in \mathbf{R}_+}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.

5.3.1 DEFINITION.

The stochastic process $Y_t := X_t - tX_1$ with $0 \le t \le 1$ is called a *Brownian bridge* on $(\Omega, \mathcal{F}, P)$.

5.3.2 THEOREM.

The random compact set $\Omega \to \mathcal{K}([0,1])$, $\omega \to \{t \in 0,1] | Y_t(\omega) = 0\}$ *has the same distribution as the* ν*–random Cantor set, where* ν *is the measure introduced in Example 2.1.7* c).

Idea of proof: Check that the distribution of the zero–set of Brownian bridge is ν–self–similar.

5.3.3 REMARK AND DEFINITION.

It can be shown that, for P–a.e. ω,

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{(-\epsilon,\epsilon)}(X_u(\omega))\,du$$

exists for all $t \in \mathbf{R}_+$.

Let $F_t : \Omega \to \mathbf{R}$ be defined by

$$F_t(\omega) = \overline{\lim}_{\epsilon\to 0} \frac{1}{2\epsilon} \int_0^t 1_{(-\epsilon,\epsilon)}(X_u(\omega))\,du\,.$$

The process $(F_t)_{t\in\mathbf{R}_+}$ is called the *local time at 0*. For P–a.e. $\omega \in \Omega$, the trajectory $t \to F_t(\omega)$ is continuous and increasing.

5.3.4 THEOREM (Taylor–Wendel [2]).

There exists a $c > 0$ such that for P–a.e. $\omega \in \Omega$ and all $t \in [0,1]$.

$$F_t(\omega) = c\mathcal{H}^h(\{s \in [0,t] : X_s(\omega) = 0\}$$

where

$$h(s) = s^{1/2}(\log|\log s|)^{1/2}\,.$$

5.4 HAUSDORFF MEASURE AND DIMENSION.

5.4.1 THEOREM.

Let $(X_t)_{t\in\mathbf{R}}$ be a Brownian motion. Then, for P–a.e. $\omega \in \Omega$,

$$H\text{-dim}(\{(t,X_t(\omega)) \in \mathbf{R}^2 | t \in \mathbf{R}_+\}) = \frac{3}{2}\,,$$

moreover

$$0 < \mathcal{H}^h(\{(t,X_t(\omega)) \in \mathbf{R}^2 | t \in [0,1]\}) < \infty\,,$$

where

$$h(t) = t^{3/2}(\log|\log t|)^{1/2}\,.$$

Proof. For the first part see, for instance, Falconer [11]. The second part is proved in Pruitt–Taylor [22].

5.4.2 DEFINITION.

Let $(\Omega,\mathcal{F},P)$ be a probability space. A stochastic process $X_t : \Omega \to \mathbf{R}^n$ $(0 \le t < \infty)$ is called a *Brownian motion in* $\mathbf{R}^n$ if $X_t = (X_t^{(1)},\ldots,X_t^{(n)})$ and the components $X_t^{(i)}$ are one–dimensional Brownian motions such that, for all $t_1, \ldots, t_n \geq 0$ the random variables $X_{t_1}^{(1)}, \ldots, X_{t_n}^{(n)}$ are independent.

5.4.3 THEOREM (Levy [18], Taylor [27], Ciesielski–Taylor [6], Ray [23]).
If $n \geq 2$ and $(X_t)_{t\in\mathbf{R}_+}$ is a Brownian motion in $\mathbf{R}^n$ then, for P–a.e. $\omega \in \Omega$,

$$H\text{-dim}(\{X_t(\omega) \,|\, t \in \mathbf{R}\}) = 2 \ ,$$

moreover,

$$0 < \mathcal{H}^h(\{X_t(\omega) \,|\, t \in [0,1]\}) < \infty$$

where

$$h(t) = \begin{cases} t^2 |\log t| \log \ \log |\log t|, & n = 2 \\ t^2 \log |\log t| & n \geq 3 \ . \end{cases}$$

Proof. For the first part see, for instance, Falconer [11]. The second part is proved in Levy [18], Taylor–Ciesielski [6], and Ray [23].

5.5 FRACTIONAL BROWNIAN MOTION.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X_t)_{t\in\mathbf{R}_+}$ a stochastic process, where $X_t : \Omega \to \mathbf{R}$.

5.5.1 DEFINITION.

$(X_t)_{t\in\mathbf{R}_+}$ is called a *fractional Brownian motion of index* α $(0 < \alpha < 1)$, if the following conditions hold:

(i) For P–a.e. $\omega \in \Omega$, $t \to X_t(\omega)$ is continuous and $X_o = 0$.

(ii) For all $t_1, \ldots, t_n \in \mathbf{R}_+$ and all $a_1, \ldots, a_n \in \mathbf{R}$ the random variable $\sum_{i=1}^n a_i X_i$ has a normal distribution.

(iii) For all $t \geq 0$ and $h > 0$ the increment $X_{t+h} - X_t$ has a normal distribution with mean 0 and variance $h^{2\alpha}$, i.e.

$$P(X_{t+h} - X_t \leq x) = (2\pi)^{-1/2} h^{-\alpha} \int_{-\infty}^{x} \exp(-u^2/2h^{2\alpha})\, du \ .$$

5.5.2 REMARKS.

a) One can show that, for each $\alpha \in (0,1)$, a fractional Brownian motion of index α exists. One can choose $\Omega = C_o(\mathbf{R}_+)$, $\mathcal{F}$ to be the Borel field of $C_o(\mathbf{R}_+)$, and $X_t(f) = f(t)$. Denote the corresponding probability by W_α.

b) There are no good methods known to construct sample paths of fractional Brownian motion. This results from the fact that for $\alpha \neq 1/2$ and $0 \leq t_1 < \ldots < t_n$ the increments $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are not independent.

5.5.3 THEOREM.

Let $\alpha \in (0,1)$ and $\gamma > 0$. Let $T_\gamma : C_o(\mathbf{R}_+) \to C_o(\mathbf{R}_+)$ be defined by

$$T_\gamma(f)(t) = \gamma^{-\alpha} f(\gamma t) .$$

Then

$$W_\alpha = W_\alpha \circ T_\gamma^{-1} .$$

Proof. Direct calculation as in 5.2.1.

5.5.4 PROBLEM.

Is there a characterization of W_α $(0 < \alpha < 1)$ by a scaling property similar to that for W given in 5.2.2.

5.5.5 THEOREM.

Let $(X_t)_{t \in \mathbf{R}}$ be a fractional Brownian motion of index α. Then, for P–a.e. ω,

$$H\text{-dim}(\{(t, X_t(\omega)) \in \mathbf{R}^2 | t \in \mathbf{R}_+\}) = 2 - \alpha .$$

Proof. See, for instance, Falconer [11]).

5.6 BROWNIAN SURFACES.

5.6.1 DEFINITION.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\alpha \in (0,1)$. A family $(X_t)_{t \in \mathbf{R}^2}$ of real–valued random variables on Ω is called a Brownian function of index α if the following conditions hold:

(i) For P–a.e. $\omega \in \Omega$, $t \to X_t(\omega)$ is continuous and $X_{(0,0)} = 0$.

(ii) $X_{(t_1+h, t_2+k)} - X_{(t_1,t_2)}$ has a normal distribution with mean 0 and variance $(h^2 + k^2)^\alpha$.

5.6.2 THEOREM.

Let $(X_t)_{t\in\mathbf{R}^2}$ be a Brownian function of index α. Then, for P–a.e. $\omega \in \Omega$,

$$H\text{-dim}(\{(t, X_t(\omega)) \in \mathbf{R}^2 \times \mathbf{R} \mid t \in \mathbf{R}^2\}) = 3 - \alpha .$$

Proof. See, for instance, Falconer [11].

5.6.3 DEFINITION.

Let $(X_t)_{t\in\mathbf{R}^2}$ be a Brownian function of index α. For $0 \leq u \leq 1$ define $Y_u : \Omega \to \mathbf{R}$ by

$$Y_u(\omega) = \sup_{0 \leq v \leq 1} X_{(u,v)}(\omega) .$$

$(Y_u)_{u\in[0,1]}$ is called the *horizon* of the Brownian function $(X_t)_{t\in[0,1]^2}$.

5.6.4 THEOREM (Falconer [11]).

For P-a.e. ω, $\max(1, \frac{3}{2} - \alpha) \leq H\text{-dim}\{(u, Y_n(\omega)) : u \in [0,1]\} \leq 2 - \alpha$.

5.6.5 PROBLEM (see Falconer [11]).

Is the Hausdorff–dimension of the horizon almost everywhere equal to $2 - \alpha$?

6. Other types of random fractals.

In this section we will just hint at a few other types of random fractals. We do not strive for completeness.

6.1 RANDOM WEIERSTRASS FUNCTIONS.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(C_k)_{k\in\mathbf{N}}$ be a sequence of independent real–valued random variables, each having a normal distribution with mean 0 and variance 1. Let $(A_k)_{k\in\mathbf{N}}$ be sequence of independent real–valued random variables which are also independent of the sequence

In this section we will just hint at a few other types of random fractals. We do not strive for completeness.

6.1 Random Weierstrass Functions.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(C_k)_{k\in\mathbf{N}}$ be a sequence of independent real–valued random variables, each having a normal distribution with mean 0 and variance 1. Let $(A_k)_{k\in\mathbf{N}}$ be sequence of independent real–valued random variables which are also independent of the sequence $(C_k)_{k\in\mathbf{N}}$ and have the uniform distribution on $[0, 2\pi]$ as their distribution. For $t \geq 0$ define $X_t : \Omega \to \mathbf{R}$ by

$$X_t(\omega) = \sum_{k=1}^{\infty} C_k(\omega)\lambda^{-\alpha k}\sin(\lambda^k t + A_k(\omega)) ,$$

where $\lambda > 1$. The stochastic process $(X_t)_{t\geq 0}$ is often used as a replacement for fractional Brownian motion of index α in fractal modelling. A value $\alpha = 0.8$ can be used to draw a realistic looking random skyline (see Falconer [11], p. 248).

There are generalizations to higher dimensional parameter spaces (see Falconer [11], p. 252).

6.2 Random Sets Arising in Random Covering Problems.

Let $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ be equipped with normalized Haar measure μ (which is equal to normalized 1–dimensional Hausdorff measure on S^1). Let $(a_n)_{n\in\mathbf{N}}$ be a sequence of real numbers with $a_n \downarrow 0$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X_n : \Omega \to S^1$ be a sequence of independent random variables, whose distributions equal μ. For $\omega \in \Omega$ define

$$K(\omega) = S^1 \setminus \bigcap_{n\in\mathbf{N}} \bigcup_{k\geq n} [X_n(\omega), X_n(\omega) + a_n]$$

where $[X_n(\omega), X_n(\omega) + a_n]$ denotes the positively oriented arc on S^1 which starts at $X_n(\omega)$ and has length a_n. If $a_n = \frac{a}{n}$, $0 < a < 1$ then

$$H\text{-}\dim K(\omega) = 1 - a$$

for P–a.e. ω.

The covering problem described above has been considered by Shepp [24]. It has been generalized in many directions.

Zähle [30] investigated random compact sets generated by random cutouts.

Kahane, in a series of papers (see [17]) developed his notion of multiplicative chaos which allows him to deal with the covering problem as a special case.

6.3 ZÄHLE'S SELF-SIMILAR QUASI-DISTRIBUTIONS.

6.3.1 DEFINITIONS AND REMARKS.

Let $\mathcal{M}$ be the set of all locally finite Borel measures on $\mathbf{R}^m$. Let $\mathcal{F}$ be the σ-field on $\mathcal{M}$ generated by the functions $\nu \to \nu(B)$, where B runs through all Borel sets in $\mathbf{R}^m$.

For $D > 0$ and $r > 0$ define $S_{D,r} : \mathcal{M} \to \mathcal{M}$ by $S_{D,r}\nu(B) = r^D\nu(r^{-1}B)$. For $x \in \mathbf{R}^m$ define $T_x : \mathcal{M} \to \mathcal{M}$ by $T_x\nu(B) = \nu(B - x)$. A σ-finite measure Q on $\mathcal{F}$ is called a *quasi-distribution*, if $Q(\{0\}) = 0$. A quasi-distribution Q is called *stationary* if $Q \circ T_x^{-1} = Q$ for all $x \in \mathbf{R}^m$. For a quasi-distribution Q define the *intensity measure*

$$\mu_Q : \mathcal{B}(\mathbf{R}^m) \to \mathbf{R}_+$$

by $\mu_Q(B) = \int \nu(B)dQ(\nu)$.

A non-finite quasi-distribution Q is called *D-self-similar*, if the following conditions are satisfied:

(i) Q is stationary

(ii) The intensity measure μ_Q is a finite positive multiple of λ^m.

(iii) $\forall\, r > 0 : Q \circ S_{D,r}^{-1} = r^{D-m}Q$.

6.3.2 THEOREM (Zähle [31]).

Let Q be a non-finite D-self-similar quasi-distribution. Suppose that there are $s, t \in \mathbf{R}_+$ with $0 < s < t < \infty$ and

$$\int \nu(\{x \in \mathbf{R}^m | s < ||x|| < t\}dQ(\nu) < \infty$$

(where $||\ ||$ is the euclidean norm).

Then the carrying dimension of Q–a.e. ν equals D.

Proof. see Zähle [31]

6.3.3 REMARKS.

There are close connections between Zähle's self–similar quasi–distributions and the statistically self–similar sets considered in Section 4 as well as Brownian motion. For details see the papers of Zähle [32], [33], [34].

6.4 ARBEITER'S RANDOM MEASURES.

The idea of Arbeiter's construction is to consider a sequence of random measures on $\mathbf{R}^d$ generated by a (time discrete) branching process: In each generation each mother particle splits up into a fixed number N of daughter particles with random location and random mass.

Next we will give a more detailed description of the construction: Let $x \in \mathbf{R}^d$ and let a sequence $(\kappa_n)_{n\in\mathbf{N}}$ of stochastic kernels from $\mathbf{R}^d \times [0,1]$ to $\mathbf{R}^{dN} \times [0,1]$ be given (i.e. $\kappa_n : (\mathbf{R}^d \times [0,1] \times \mathcal{B}(\mathbf{R}^{dN} \times [0,1]^N) \to \mathbf{R}_+$ s.t. for each $(y,r) \in \mathbf{R}^d \times [0,1]$ $\kappa_n((y,r),\cdot)$ is a probability on $\mathcal{B}(\mathbf{R}^{dN} \times [0,1]^N)$ and for each $B \in \mathcal{B}(\mathbf{R}^{dN} \times [0,1]^N)$ the map $\kappa_n(\cdot, B)$ is measurable)

First step:

Choose a random variable $(y_1,\ldots,y_N,p_1,\ldots,p_N)$ with distribution $\kappa_1(x,1,\cdot)$ and define a finite random measure ψ_1 on $\mathbf{R}^d$ by

$$\psi_1 = \sum_{i=1}^{N} p_i \epsilon_{x+y_i}$$

Let $\mathcal{F}_1$ be the σ–field generated by $(y_1,\ldots,y_N,p_1,\ldots,p_N)$.

n-th step: For each $\sigma \in \{1,\ldots,N\}^{n-1}$ choose a random variable $(y_{\sigma*1},\ldots,y_{\sigma*N},p_{\sigma*1},\ldots,p_{\sigma*N})$ such that the conditional distribution

$$P((y_{\sigma*1},\ldots,y_{\sigma*N},p_{\sigma*1},\ldots,p_{\sigma*N}) \in B|\mathcal{F}_{n-1})$$

$$= \kappa_n(x + y_{\sigma|1} + \ldots + y_\sigma,\ p_{\sigma|1} \cdots \cdot p_\sigma, B)$$

for all B Borel in $\mathbf{R}^{dN} \times [0,1]^N$.

Define a random measure by

$$\psi_n = \sum_{\sigma \in \{1,\dots,N\}^n} p_{\sigma|1} p_{\sigma|2} \cdots \cdot p_\sigma \epsilon_{x+y_{\sigma|1}+\dots+y_\sigma}$$

Under certain conditions ψ_n converges weakly a.e. to a random measure ψ which is the random object Arbeiter considers.

Details of the construction and the connection with statistically self–similar probabilities are described in Arbeiter [2].

REFERENCES

[1] ARBEITER M., *Random recursive construction of self–similar fractal measures.* Prob. Theory Rel. Fields, to appear.

[2] ARBEITER M., *Construction of random fractal measures by branching processes.* Preprint 1991.

[3] ADLER R.J., *The geometry of random fields*, John Wiley, Chicester etc. 1981.

[4] BAUER H., *Wahrscheinlichkeitstheorie und Grundzüge der Masstheorie*, de Gruyter, Berlin 1968.

[5] BEDFORD T. and FISCHER A.M., *Analogues of the Lebesgue density theorem for fractal sets of reals and integers*, Report 89-88, Delft University of Technology, Delft 1989.

[6] CIESIELSKI Z. and TAYLOR S.J., *First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path*, Trans. Amer. Math. Soc. *103* (1962), 434–450.

[7] CHAYES J.T., CHAYES L. and DURRETT R., *Connectivity of Mandelbrot's percolation process*, Prob. Theory Rel. Fields *77* (1988), 307–329.

[8] DEKKING M. and MEESTER R., *On the structure of Mandelbrot's percolation process and other random Cantor sets.* J. Statistical Physics *58* (1990), 1109–1126.

[9] DUBINS L.E. and FREEDMAN D.A., *Random distribution functions*, in Proceedings Fifth Berkeley Symp. on Math. Statistics and Probability" (L.M. LeCam, J. Neyman Eds.), pp. 183–214, Univ. of California Press, Berkeley – Los Angeles 1967.

[10] FALCONER K.J., *Random fractals*, Math. Proc. Cambridge Phil. Soc. *100* (1986), 559–582.

[11] FALCONER K.J., *Fractal geometry*, John Wiley, Chicester 1990.

[12] FALCONER K.J. *The horizon problem for random surfaces*, Math. Proc. Cambridge Phil. Soc. *109*, (1991), 211–219.

[13] GRAF S., *Statistically self–similar fractals*, Prob. Theory Rel. Fields *74* (1987), 357–392.

[14] GRAF S., MAULDIN R.D. and WILLIAMS S.C., *Random homeomorphisms*, Adv. Math. *60* (1986), 239–359.

[15] GRAF S., MAULDIN R.D. and WILLIAMS S.C., *The exact Hausdorff dimension in random recursive constructions*, Mem. Amer. Math. Soc. *381* (1988), 1–121.

[16] HUTCHINSON J.E., *Fractals and self–similarity*, Indiana Univ. Math. J. *30* (1981), 713–747.

[17] KAHANE J.-P., *Produits de poids aléatoires indépendenants et applications*, in: Fractal geometry and analysis (J. Bélair – S. Dubuc, Eds.). Kluwer Academic Publ. 1990.

[18] LÉVY P., *La mésure de Hausdorff de la courbe du mouvement brownien*, Giorn. Ist. Ital. Attuari *16* (1953), 1–37.

[19] MANDELBROT B.B., *Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier*, J. Fluid Mechanics *62* (1974), 331–358.

[20] MAULDIN R.D. and WILLIAMS S.C., *Random recursive constructions: asymptotic geometric and topological properties*, Trans. Amer. Math. Soc. *295* (1986), 325–346.

[21] MEESTER R., *Two models of dependent percolation*, Dissertation Delft 1990.

[22] PRUITT W.E. and TAYLOR S.J., *Sample path properties of processes with stable components*, Z. Wahrscheinlichkeitstheorie *12* (1969), 267–289.

[23] RAY D., *Sojourn times and the exact Hausdorff measure of the sample path for planar Brownian motion*, Trans. Amer. Math. Soc. *106* (1963), 436–444.

[24] SHEPP L.A., *Covering the circle with random arcs*, Israel J. Math. *11* (1972), 328–345.

[25] SHIRYAYEV A.N., *Probability*, Springer, New York, 1984.

[26] TAYLOR S.J., *The α–dimensional measure of the graph and set of zeros of a Brownian path*, Proc. Cambridge Phil. Soc. *51* (1955), 265–274.

[27] TAYLOR S.J., *The exact Hausdorff measure of the sample path for planar Brownian motion*, Proc. Cambridge Phil. Soc. *60* (1964), 253–258.

[28] TAYLOR S.J., *The measure theory of random fractals*, Math. Proc. Cambridge Phil. Soc. *100* (1986), 383–404.

[29] TAYLOR S.J. and WENDEL J.G., *The exact Hausdorff measure of the zero set of a stable process*, Z. Wahrscheinlickeitstheorie *6* (1966), 170–180.

[30] ZÄHLE U., *Random fractals generated by random cutouts*, Math. Nachr. *116* (1984), 27–52.

[31] ZÄHLE U., *Self-similar random measures I. Notion, carrying Hausdorff dimension and hyperbolic distribution*, Prob. Theory Rel. Fields *80* (1988), 79–100.

[32] ZÄHLE U., *Self-similar random measures II. Generalization to self-affine measures*, Math. Nachr. *146* (1990), 85–98.

[33] ZÄHLE U., *Self-similar random measures III. Self-similar random processes*, Math. Nachr. *151* (1991), 121–148.

[34] PATZSCHKE N. and ZÄHLE U., *Self-similar random measures IV. The recursive construction model of Falconer, Graf and Mauldin/Williams*, Math. Nachr. *149* (1990), 285–302.