UNIFORM INTEGRABILITY: AN INTRODUCTION (*)

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Acknowledgments. Through the years I've learned about the subject of uniform integrability through conversations with many mathematicians as well as studying the usual books and papers. I especially want to mention Professor Witold Bogdanowicz, Bob Huff, Olek Pelczynski and Jerry Uhl; in one way or another, they taught me most of what I know about the subject. For this I must thank them. This paper's preparation was supported in part by the Mathematics Department of the University of Pretoria. The author wishes to thank the Department for their hospitality, the members of their fine analysis group for their attention and particularly Anton Stroh and F. Y. Johan Swart for their helpful conversations.

Let (Ω, Σ, P) be a probability space.

A subset of K of $L_1(P)$ is called *uniformly integrable* if

$$\lim_{c \to \infty} \sup_{f \in K} \int_{[|f|>c]} |f| dP = 0 ,$$

that is, if given $\epsilon > 0$ there is a $c_\epsilon = c > 0$ so that for $t > c$, $\int_{[|f|>t]} |f| dP \leq \epsilon$ for all $f \in K$.

This definition, which is preferred by probabilists, has another version.

THEOREM 1. *A subset K of $L_1(P)$ is uniformly integrable if and only if K is bounded and given $\epsilon > 0$ there is a $\delta > 0$ so that any $E \in \Sigma$ with $P(E) \leq \delta$ has $\int_E |f| dP \leq \epsilon$ for all $f \in K$.*

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Proof. Suppose K is uniformly integrable. Take $E \in \Sigma$ and notice that

$$\begin{aligned}\int_E |f|dP &= \int_{E\cap[|f|\le c]} |f|dP + \int_{E\cap[|f|>c]} |f|dP \\ &\le \int_E c\,dP + \int_{[|f|>c]} |f|dP \\ &= cP(E) + \int_{[|f|>c]} |f|dP\,.\end{aligned}$$

This holds in particular for $E = \Omega$ and so for any $c > 0$

$$\|f\| \le c + \int_{[|f|>c]} |f|dP$$

which, if we choose $c_1 > 0$ so that $\int_{[|f|>c_1]} |f|dP \le 1$ for all $f \in K$, gives

$$\|f\| \le c_1 + 1\,.$$

K is bounded in $L_1(P)$. Again, for general $E \in \Sigma$

$$\int_E |f|dP \le cP(E) + \int_{[|f|>c]} |f|dP$$

tells us that if we choose c big enough then $\int_{[|f|>c]} |f|dP$ can be made small uniformly for $f \in K$; this having been done, careful control of $P(E)$ will ensure complete control of $\int_E |f|dP$ uniformly for $f \in K$. More precisely, if $\epsilon > 0$ is given then K's uniform integrability assures us of a $c > 0$ so that $\int_{[|f|>c]} |f|dP \le \epsilon/2$ for all $f \in K$; but now should $P(E) \le \frac{\epsilon}{2c}$ then $\int_E |f|dP \le \epsilon$ for all $f \in K$.

Conversely, suppose K is bounded and for each $\epsilon > 0$ there is a $\delta \ge 0$ such that whenever $P(E) \le \delta$ we have $\int_E |f|dP \le \epsilon$ for all $f \in K$. Then, regardless of $c > 0$, we have

$$c\chi_{[|f|>c]} \le |f|\chi_{[|f|>c]}$$

and so

$$cP[|f| > c] \le \int_{[|f|>c]} |f|dP \le \int |f|dP = \|f\|_1\,.$$

If we let $M = \sup\{||f||_1 : f \in K\}$ then we see that

$$P[\,|f| > c] \leq \frac{M}{c}$$

for all $f \in K$. It follows that if c is chosen so that $\frac{M}{c} < \delta$ (δ corresponding to the ever–present $\epsilon > 0$) then $P[\,|f| > c] < \delta$ so $\int_{[\,|f|>c]} |f| dP \leq \epsilon$ for all $f \in K$ and K is uniformly integrable.

EXAMPLES.

1. *Suppose $p > 1$. Then bounded subsets of $L_p(P)$ are uniformly integrable.* Indeed, sets that are bounded in $L_p(P)$ are bounded in $L_1(P)$ and Hölder's inequality ensures us that for $E \in \Sigma$ and $||f||_p \leq M$ we have
$$\int_E |f| dP \leq ||f||_p ||\chi_E||_q \leq M P(E)^{1/q}$$
where $\frac{1}{p} + \frac{1}{q} = 1$.
2. More generally, *if $\Phi : [0,\infty) \to [0,\infty)$ is an increasing function such that $\frac{\Phi(x)}{x} \to \infty$ as $x \to \infty$ and if $\int \Phi(|f(w)|)\, dP(\omega) \leq M < \infty$ for all $f \in K$, then K is uniformly integrable.* Indeed, if $\epsilon > 0$ is chosen then one can find T_o such that $\Phi(t)/t \geq M/\epsilon$ for all $t \geq T_o$. It follows that for all $f \in K$
$$\int_{[\,|f|>T_o]} |f| dP \leq \frac{\epsilon}{M} \int_{[\,|f|>T_o]} \Phi \circ |f| dP \leq \epsilon\,.$$

We hasten to add that conditions such as 2. arise frequently in both harmonic analysis and probability in the study of tail behaviour of special sums.

Actually, 2. is quite close to the heart of things with regards to uniform integrability. Here's an old gem of de la Vallée Poussin.

THEOREM 2. (de la Vallée Poussin). *For $K \subseteq L_1(P)$ to be uniformly integrable it is both necessary and sufficient that there exist a convex even function $\Phi : \mathbf{R} \to \mathbf{R}$ such that $\Phi(0) = 0$, $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$ and $\sup_{f\in K} \int \Phi(|f(w)|)\, dP(w) < \infty$.*

As one might expect, not all (bounded) subsets of L_1–spaces are uniformly integrable. To highlight a natural example we look to $L_1[0,1]$.

EXAMPLE. Suppose that $(f_n) \subset L_1[0,1]$ are disjointly supported functions, non–negative real–valued and having $\int_0^1 f_n(t)\,dt = 1$ for each n. DRAW the graphs of such! It ought to be plain that the supports of f_n necessarily shrink to a set of measure zero yet in no way can we force the indefinite integrals to behave. In fact, *and this* <u>*is*</u> *very instructive*, for such a sequence (f_n) it is so that regardless of scalars (a_n) we have

$$(\ell_1) \|\sum_n a_n f_n\|_1 = \sum_n |a_n| ;$$

after all, $|\Sigma_n a_n f_n| = \Sigma_n |a_n| f_n$ so integrating term–by–term soon reveals (ℓ_1).

Part and parcel of the study of uniform integrability is the remarkable fact that the above example is, in a very strong sense, the *only* obstruction to a bounded set's uniform integrability. Before we come to understand why this is so, it is important to relate uniform integrability with the "weak topology" of the Banach space $L_1(P)$.

Weak topologies on Banach spaces came to be because the norm topology of a Banach space is inadequate. There are two weak topologies of interest to us. They are delicate to the touch, each has its own character and we must be careful to be sympathetic to each.

First, let us talk about the *weak topology* of a Banach space X. Suppose we denote by X^* the linear topological dual of X, that is, X^* consists of the linear continuous functionals on X. X^* *is* itself a Banach space and there is a weakest locally convex linear topology on X which ensures each member x^* of X^* of its continuity; this is what's called the weak topology. It's a linear topology on X in which a net $(x_\alpha)_\alpha$ converges to $x \in X$ if $(x^*(x_\alpha))_\alpha$ converges to x^*x for each $x^* \in X^*$. Every weak neighborhood of zero contains a set of the form

$$W(x_1^*, \ldots, x_n^*, \epsilon) = \bigcap_{k \le n} [\,|x_k^*(x)| < \epsilon] ,$$

sets which contain *subspaces* of finite codimension. The weak topology *is* a locally convex linear topological Hausdorff topology on X such that the dual of X, when X is equipped with this topology, is X^*. *The weak topology is not complete nor is it metrizable if X is infinite dimensional.*

Nevertheless, the weak topology is a good friend in the study of the finer structure of X.

Starting with the Banach space X pass to X^* and on X^* we can define the "weak*" topology; a net $(x^*_\alpha)_\alpha$ in X^* converges in the weak* topology to an $x^* \in X^*$ if $(x^*_\alpha(x))_\alpha$ converges to $x^*(x)$ for each $x \in X$. Again, the weak* topology in X^* is a locally convex linear Hausdorff topology; the dual of (X^*, weak*), the topological dual is X! In fact, the weak* topology is defined so as to make such so. Every weak* neighborhood of zero contains a set of the form

$$W(x_1, \ldots, x_n, \epsilon) = \bigcap_{k \leq n} [\,|x^*(x_k)| < \epsilon] \,.$$

The weak topology is neither complete nor metrizable if X is infinite dimensional.*

The weak and weak* topologies are kin. They are not the same (in general) but each helps understand the other. Principal in this understanding is an appreciation of compactness in each topology.

The weak* topology appreciates bounded sets: if $B \subseteq X^*$ is a bounded set, then $\bar{B}^{wh^*}$ is weak* compact. This is a famous theorem of Alaoglu.

In the weak topology, compactness is more elusive but, like the fair maiden, it is worth pursuing. IN FACT, the famous theorem of Eberlein and Smulian tells us that a subset K of a Banach space is relatively weakly compact if and only if K is relatively weakly sequentially compact, a situation which occurs precisely when K is relatively countable compact; what's more, if K *is* a weakly compact subset of X and $A \subseteq K$ then every point of $\bar{A}^{weak}$ is the weak limit of a *sequence* of points from A. Weak compactness, once in hand, is an analyst's dream. All's well – sequences suffice!

How does one ascertain when a relatively weakly compact set is so? Here's the basic strategy – the only general strategy available. Take a set K in the Banach space X that's norm bounded (by the way the Banach–Steinhaus theorem should warn us off looking for weakly compact sets in all the wrong places – they are norm bounded). Look at K as a subset of X^{**} and take K's weak* closure $\bar{K}^{wk^*}$ up in X^{**}: if $\bar{K}^{wk^*}$ *never* passes outside of X then K is relatively weakly compact and $\bar{K}^{weak}$ is precisely, $\bar{K}^{wk^*}$. That this is so is a simple comparison–of–topologies argument made possible through the good graces of $\bar{K}^{wk^*}$'s weak* compactness.

Okay, so what? Here's what!

THEOREM 3. (Dunford–Pettis). *A subset K of $L_1(P)$ is relatively weakly compact if and only if K is uniformly integrable.*

This theorem is more than fifty years old. It is still stunning. What's more, its proof is still worthy of serious study.

On the one hand, we have the duality concepts so pertinent to the notion of weak compactness. How can they be handled, controlled? Here's how! The dual of $L_1(P)$ is $L_\infty(P)$ where the action of $g \in L_\infty(P)$ on an $f \in L_1(P)$ is given by

$$g(f) = \int_\Omega f(\omega)g(\omega)dP(\omega) ;$$

this is the Radon–Nikodym Theorem in action. Under this identification, $||g||_\infty = ||g||_{L_1(P)^*}$ and all is well in life.

What about $L_\infty(P)^*$? Here we hit a small, a very small, snag, $L_\infty(P)^*$ *can* be described but it requires us to pass into the nether–land of finitely additive measures. More precisely, if $\mu : \sigma \to \mathbf{R}$ is a bounded, finitely additive measure, then $\int g d\mu$ can be made sense of for any $g \in L_\infty(\mu)$. How? Well, if g were simple, then it'd be easy and it'd be easy to see that $|\int g d\mu| \leq ||g||_\infty |\mu|(\Omega)$ where $||g||_\infty$ is the (essential) supremum norm of $g \in L_\infty(P)$ and $|\mu|(\Omega)$ is μ's total variation. The density of simple functions in $L_\infty(P)$ tells us that $\int g d\mu$ is well–defined for each $g \in L_\infty(P)$. The careful student will note that we've told a small lie here – one ought to make sure that $\int_E g d\mu = \int_E h d\mu$ for each $E \in \Sigma$ ensures $g = h$ P–almost surely–so one must ask of μ that $|\mu|(E) = 0$ whenever $P(E) = 0$.

So be it.

Here's the punch line: $L_\infty(P)^*$ can be identified with the (Banach) space $ba_o(\Sigma)$ of all bounded additive measures $\mu : \Sigma \to \mathbf{R}$ that vanish on P–null sets. Here the identification of x^* with μ entails

$$x^*(g) = \int_\Omega g(\omega)d\mu(\omega) ,$$

where

$$||x^*|| = |\mu|(\Omega) .$$

All this is well–known and easy.

Take $K \subseteq L_1(P)$ and suppose K is uniformly integrable. Thanks to Theorem 1, we know that K is $L_1(P)$–bounded *and* that for each $\epsilon > 0$ there is a $\delta > 0$ so that $\int_E |f|dP \leq \epsilon$ for all $f \in K$ whenever $P(E) \leq \delta$. BUT K's boundedness ensures us that $\bar{K}^{weak^*} \subseteq L_\infty(P)^*$ is weak* compact. If we take $\mu \in \bar{K}^{weak^*}$ then $\mu(\chi_E)$ can be approximated by $\chi_E(f)$'s where $f \in K$, that is, we know that at least

$$\begin{aligned} |\mu(E)| = |\mu(\chi_E)| &\leq \sup_{f\in K} |\int_E f dP| \\ &\leq \sup_{f\in K} \int_E |f| dP \,. \end{aligned}$$

It follows that given $\epsilon > 0$ there is a $\delta > 0$ so that if $P(E) \leq \delta$ then $|\mu(E)| \leq \epsilon$: μ is countably additive and P–continuous, μ belongs to $L_1(P)$!

The converse is not so easy, nor should it be. There are bigger fish to fry. In fact, a critical argument in establishing the converse goes back to Lebesgue and Vitali (albeit their interest was in case of $\Omega = [0,1]$, Σ = {Lebesgue measurable sets} and P = Lebesgue measure). Here's what Lebesgue and Vitali had to say (about this).

THEOREM 4. (Lebesgue–Vitali). *If (f_n) is a bounded sequence in $L_1(P)$ such that for each $E \in \Sigma$ we have*

$$\lim_n \int_E f_n dP = 0 \,,$$

then $\{f_n : n \in \mathbf{N}\}$ is uniformly integrable.

If not(!) then there is an $\epsilon_o > 0$ such that regardless of $m \in \mathbf{N}$ and $\delta > 0$ there is an $E_\delta = E \in \Sigma$ with $P(E) \leq \delta$ and an $n \geq m$ such that

$$|\int_E f_n dP| \geq \epsilon_o \,.$$

THINK ABOUT IT!

OK?

Find $E_1 \in \Sigma$ and n_1 so that

$$|\int_{E_1} f_{n_1}\, dP| \geq \epsilon_o$$

Find $\delta_1 > 0$ so that if $P(E) \leq \delta_1$, then $\int_E |f_{n_1}| dP < \frac{\epsilon_0}{4}$.
Find $E_2 \in \Sigma$ and $n_2 > n_1$ so that $P(E_2) < \frac{\delta_1}{2}$ *yet*

$$|\int_{E_2} f_{n_2}\, dP| \geq \epsilon_o \, .$$

Find $\delta_2 > 0$ so that if $P(E) \leq \delta_2$, then $\int_E |f_{n_2}| dP < \frac{\epsilon_0}{4}$ and, if you must, ensure yourself that $\delta_2 < \delta_1/2$.

Continue in this way to find sequences (E_k) in Σ, positive integers $n_1 < n_2 < \ldots < n_k < \ldots$ and positive numbers $\delta_k > 0$ so that $\delta_{k+1} < \delta_k/2$ and

$$|\int_{E_k} f_{n_k}\, dP| \geq \epsilon_o$$

while

$$P(E_{k+1}) \leq \delta_k/2$$

and if $P(E) \leq \delta_k$, then $|\int_E f_{n_k}\, dP| \leq \frac{\epsilon_0}{4}$. OK?

Of course, it follows that

$$\begin{aligned} P(E_{k+1} \cup \ldots \cup E_{k+m} \cup \ldots) &\leq P(E_{k+1}) + \ldots + P(E_{k+m}) + \ldots \\ &\leq \frac{\delta_k}{2} + \frac{\delta_{k+1}}{2} + \ldots \\ &< \delta_k \end{aligned}$$

so that

$$\int_{E_{k+1} \cup \ldots \cup E_{k+m} \cup \ldots} |f_{n_k}| \leq \frac{\epsilon_0}{4} \, .$$

If we let

$$A_k = E_k \setminus (E_{k+1} \cup \ldots \cup E_{k+m} \cup \ldots) \, ,$$

then

$$\begin{aligned} |\int_{A_k} f_{n_k}\, dP| &= |\int_{E_k \setminus (E_{k+1} \cup \ldots)} f_{n_k}\, dP| \\ &\geq |\int_{E_k} f_{n_k}\, dP| - |\int_{E_k \cap (E_{k+1} \cup \ldots)} f_{n_k}\, dP| \\ &\geq 3\frac{\epsilon_0}{4} \, . \end{aligned}$$

But the A_k's are pairwise disjoint and $A_k \subseteq E_k$ so

$$P(A_{k+1} \cup \ldots \cup \ldots) \leq \delta_k \, .$$

Let $k_1 = 1$.

Let k_2 be any $m > k_1$ so that

$$\left| \int_{A_{k_1}} f_{n_m} \, dP \right| < \frac{\epsilon_0}{4} \, .$$

Let k_3 be any $m > k_2$ so that

$$\left| \int_{A_{k_1} \cup A_{k_2}} f_{n_m} \, dP \right| < \frac{\epsilon_0}{4} \, .$$

In general, k_j will be any $m > k_{j-1}$ such that

$$\left| \int_{A_{n_1} \cup \ldots \cup A_{n_{j-1}}} f_{n_{k_j}} \, dP \right| < \frac{\epsilon_0}{4} \, .$$

A consequence? Of course,

$$\left| \int_{A_{k_j}} f_{n_{k_j}} \, dP \right| \geq \frac{3}{4} \epsilon_0 \, .$$

Also,

$$\begin{aligned} P(A_{k_{j+1}} \cup A_{k_{j+2}} \cup \ldots \cup \ldots) \leq & \\ & \leq P(A_{k_j+1} \cup A_{k_j+2} \cup \ldots \cup \ldots) \\ & < \delta_{k_j} \end{aligned}$$

so that

$$\left| \int_{A_{k_{j+1}} \cup A_{k_{j+2}} \cup \ldots} f_{n_{k_j}} \right| \leq \frac{\epsilon_0}{4} \, .$$

Let $Q = A_{k_1} \cup A_{k_2} \cup \ldots$

Then regardless of j

$$\int_Q f_{n_{k_j}} \, dP = \left(\int_{A_{k_1} \cup \ldots \cup A_{k_{j-1}}} + \int_{A_{k_j}} + \int_{A_{k_{j+1}} \cup A_{k_{j+2}} \cup \ldots} \right) f_{n_{k_j}} \, dP$$

which in modulus must be

$$\geq \epsilon_0/4 \ .$$

OOPS!

How does the Lebesgue–Vitali Theorem relate to the Dunford–Pettis Theorem? Well, suppose $K \subset L_1(P)$ is relatively weakly compact. To show K is uniformly integrable consider the alternative: K is bounded so there must be an $\epsilon_0 > 0$ so that regardless of n we can find $f_n \in K$ and $E_n \in \Sigma$ so that even though $P(E_n) < \frac{1}{n}$, $|\int_{E_n} f_n dP| \geq \epsilon_0$. But (f_n) must have a weakly convergent *subsequence* (f_{n_k}) with weak limit f, say in $L_1(P)$; it follows that $(g_k = f_{n_k} - f)$ is a bounded sequence in $L_1(P)$ that goes to zero weakly. In particular, for each $E \in \Sigma$

$$\lim_k \int_E g_k dP = 0$$

$\{g_k : k \geq 1\}$ must be uniformly integrable thanks to the Lebesgue–Vitali Theorem. It follows that $\{g_k + f : k \geq 1\} = \{f_{n_k} : k \geq 1\}$ is uniformly integrable too. OOPS!

Crucial to the above argument is the fact that both relative weak compactness and uniform integrability are sequential in nature. A set $K \subseteq L_1(P)$ is relatively weakly compact (respectively uniformly integrable) if and only if every sequence (f_n) from K has a subsequence (f_{n_k}) such that $\{f_{n_k})$ such that $\{f_{n_k} : k \geq 1\}$ is relatively weakly compact (respectively, uniformly integrable).

ACTUALLY, the proof of the Lebesgue–Vitali Theorem holds promise for much more than what's delivered in the Dunford–Pettis Theorem. A careful inspection of the proof as presented above will soon uncover the following: if K is a bounded non uniformly integrable subset of $L_1(P)$ then one can find a sequence (f_k) in K, and $\bar{\epsilon} > 0$ and a sequence (E_k) of pairwise disjoint members of Σ such that for all k

$$|\int_{E_k} f_k dP| \geq \bar{\epsilon} \ .$$

Now for a real treat.

Rosenthal's Lemma. *Let (μ_n) be a bounded sequence of bounded real–valued finitely additive measures defined on Σ, $\epsilon > 0$ and (E_n) be a sequence of pairwise disjoint members of Σ.*

Then there is an increasing sequence (n_k) *of positive integers such that for each* k.

$$|\mu_{n_k}|(\cup_{j\neq k} E_{n_j}) < \epsilon .$$

Proof. Let $||\mu_n||_1 \leq M$ for all n. Split $\mathbf{N}$ into a countable union $\cup_p M_p$ of pairwise disjoint, infinite subsets M_p.

Optimistically speaking, MAYBE there's a p for which *no* $k \in M_p$ satisfies

$$|\mu_k|(\cup_{j\in M_p, j\neq k} E_j) \geq \epsilon .$$

If this happens, then I'm happy since it means that for each $k \in M_p$,

$$|\mu_k|(\cup_{j\in M_p, j\neq k} E_j) < \epsilon$$

and all that need be done is list the members of M_p in ascending order $M_p = \{m_1 < m_2 <\sim\}$.

Realistically, it may be that for each p there's a $k_p \in M_p$ such that

$$|\mu_{k_p}|(\cup_{j\in M_p, j\neq k_p} E_j) \geq \epsilon .$$

Notice that

$$\begin{aligned}&|\mu_{k_p}|(\cup_q E_{k_q}) + |\mu_{k_p}|(\cup_{j\in M_p, j\neq k_p} E_j)\\ &\leq |\mu_{k_p}|(\cup_q E_{k_q}) + |\mu_{k_p}|(\cup_n E_n \setminus \cup_q E_{k_q})\\ &\leq M .\end{aligned}$$

Hence

$$|\mu_{k_p}|(\cup E_{k_q}) \leq M - \epsilon .$$

Replacing (μ_k) by (μ_{k_p}) and (E_k) by (E_{k_p}) in our arguments above we can take an optimistic view, which will be quickly rewarded if applicable or we can take a realistic view. Realistically though we soon find ourselves with an inequality of the form

$$|\mu_{k_{p_r}}|(\cup E_{k_{p_s}}) \leq M - 2\epsilon .$$

M is only so big so realism only lasts so long. Sooner or later (but sometime) optimism wins the day. And when it does Rosenthal's lemma is proved.

We have in mind to apply Rosenthal's lemma to a sequence (f_n) in the norm bounded non–uniformly integrable set $K \subseteq L_1(P)$ which comes accompanied by an $\bar{\epsilon} > 0$ and a sequence (E_n) of pairwise disjoint memebers of Σ such that

$$\left|\int_{E_n} f_n dP\right| \geq \bar{\epsilon}\,.$$

The measures we wish to consider are $\mu_n(E) = \int_E f_n dP$ and $\epsilon = \bar{\epsilon}/2$. The result is a sequence $n_k \uparrow \infty$ such that

$$\int_{\bigcup_{j\neq k} E_{n_j}} |f_{n_k}| dP \leq \frac{\bar{\epsilon}}{2}\,.$$

So what? Well, here's what: if (a_n) is any scalar sequence then

$$\|\Sigma_k a_k f_{n_k}\|_1 \geq \int_{\bigcup_k E_{n_k}} |\Sigma_k a_k f_{n_k}(\omega)| dP(\omega)$$

$$\geq \int |\Sigma_k a_k f_{n_k}(\omega)\chi_{E_{n_k}}(\omega)| dP(\omega) - \int |\Sigma_k a_k f_{n_k}(\omega)\chi_{\bigcup_{j\neq k} E_{n_j}}(\omega)| dP(\omega)$$

$$\geq \Sigma_k \int_{E_{n_k}} |a_k f_{n_k}(\omega)| dP(\omega) - \Sigma_k |a_k| \int_{\bigcup_{j\neq k} E_{n_j}} |f_{n_K}(\omega)| dP(\omega)$$

$$\geq \bar{\epsilon}\Sigma_k |a_n| - \frac{\bar{\epsilon}}{2}\Sigma_k |a_k|\,.$$

Since (f_{n_k}) is bounded (say by M) we always have

$$\left\|\sum_k a_k f_{n_k}\right\|_1 \leq M \sum_k |a_k|$$

so

$$\frac{\bar{\epsilon}}{2}\sum_k |a_k| \leq \left\|\sum_k a_k f_{n_k}\right\|_1 \leq M \sum_k |a_k|\,.$$

We've proved the following.

THEOREM 6 (Kadec–Pelczynski). *If K is a bounded non–uniformly integrable subset of $L_1(P)$ then K contains a sequence which is equivalent to the unit coordinate vector basis of ℓ_1.*

In truth, more is so. The sequence extracted above is what Rosenthal called "relatively disjoint". Such sequences span *complemented* copies of

ℓ_1 in $L_1(P)$. Since there *are* uncomplemented copies of ℓ_1 in $L_1[0,1]$ (though they're not easily located), this indicates the special character of the above line of argumentation. Before broaching a new line of investigation involving uniform integrability and weak compactness in $L_1(P)$ it seems worth while to collect a number of equivalent conditions some of which we've seen and some of which we've not seen.

THEOREM 7. *Let K be a subset of $L_1(P)$. Then the following statements regarding K are equivalent.*

1. *K is uniformly integrable.*
2. *K is bounded and for each $\epsilon > 0$ there is a $\delta > 0$ such that if $P(E) \leq \delta$ then $\int_E |f| dP \leq \epsilon$ for all $f \in K$.*
3. *For each $\epsilon > 0$ there is a $c_o > 0$ so that for $c \geq c_o$*

$$\left|\int_{[|f|>c]} f dP\right| \leq \epsilon \quad \textit{forall } f \in K .$$

4. *K is bounded and for each $\epsilon > 0$ there is a $\delta > 0$ such that if $P(E) \leq \delta$ then $|\int_E f dP| \leq \epsilon$ for all $f \in K$.*
5. *K is relatively weakly compact.*
6. *No sequence in K is equivalent to the unit coordinate vector basis of ℓ_1.*
7. *K is bounded and given any sequence (E_n) of pairwise disjoint members of Σ*

$$\lim_n \left|\int_{E_n} f dP\right| = 0 \ , \ \textit{uniformly } f \in K .$$

There are a few surprises in this next theorem.

THEOREM 8. *Let X be a closed linear subspace of $L_1(P)$. The following statements about X are equivalent.*

1. *B_X is uniformly integrable.*
2. *B_X is weakly compact.*
3. *X is reflexive.*
4. *X contains no subspace isomorphic to ℓ_1.*
5. *X contains no subspace isomorphic to ℓ_1 that's complemented in $L_1(P)$.*

6. *X does not contain ℓ_1^n's uniformly. A Banach space contains ℓ_1^n's uniformly if there is a $K > 0$ so that for each n there is a $1-1$, linear operator $u_n : \ell_1^n \to X$ such that $||u_n||\,||u_n^{-1}|| \leq K$.*
7. *X does not contain ℓ_1^n's uniformly complemented.*
8. *There is $1 < p \leq 2$ and a probability μ on Σ such that X is isomorphic to a closed linear subspace of $L_p(\mu)$.*

One would be irresponsible if when talking about uniform integrability *no* notice was given to probability. To give a bit of the background we recall that if Σ_0 is a sub–σ–field of the σ–field Σ and $f \in L_1(P)$ then there is a Σ_o–measurable function, called the conditional expectation of f given Σ_o, denoted by $\mathbf{E}(f|\Sigma_o)$ defined by the relationship

$$\int_E f dP = \int_E \mathbf{E}(f|\Sigma_o)\, dP$$

whenever $E \in \Sigma_o$.

This follows, by the way, from the Radon–Nikodym Theorem applied to the measure $\int_E f dP(E \in \Sigma_o)$ which is absolutely continuous with respect to P's restriction to Σ_o.

We list here a number of properties enjoyed by the conditional expectation; their proofs may be found in many books on probability, a few in analysis and too few in measure theory.

THEOREM 10.

1. *$\mathbf{E}(|\Sigma_o)$ takes $L_1(\Sigma, P)$ into $L_1(\Sigma, P)$ in a linear, monotone non–increasing manner with $\mathbf{E}(f|\Sigma_o)$ actually in $L_1(\Sigma_o, P)$ for each $f \in L_1(\Sigma, P)$; further more, for $f \in L_1(\Sigma_o, P)$, $\mathbf{E}(f|\Sigma_o) = f$ $P|_{\Sigma_o}$–almost surely so that $\mathbf{E}(|\Sigma_o)$ is a linear projection of $L_1(\Sigma, P)$ onto $L_1(\Sigma_o, P)$.*
2. *If f_n, $f_o \in L_1(\Sigma, P)$ and $f_n \uparrow f_o$ almost surely, then $\mathbf{E}(f_n|\Sigma_o) \uparrow \mathbf{E}(f_o|\Sigma_o)\, P|_{\Sigma_o}$ almost surely.*
3. *If $f_n, f, g \in L_1(\Sigma, P)$ and $|f_n| \leq g$ almost surely while $f = \lim_n f_n$ almost surely, then $\mathbf{E}(f|\Sigma_o) = \lim_n \mathbf{E}(f_n|\Sigma_o)\, P|_{\Sigma_o}$ almost surely.*
4. *If Σ' is a sub σ–field of Σ_o and $f \in L_1(\Sigma, P)$, then $\mathbf{E}(f|\Sigma') = \mathbf{E}(\mathbf{E}(f|\Sigma_o)\,|\Sigma')\, P|_{\Sigma'}$ almost surely.*
5. *If f, g and $fg \in L_1(\Sigma, P)$ with $f \in L_1(\Sigma_o, P)$ then $\mathbf{E}(fg|\Sigma_o) = f\mathbf{E}(g|\Sigma_o)\, P|_{\Sigma_o}$ almost surely.*

6. *If* $1 \leq p \leq \infty$ *and* $f \in L_p(\Sigma_o, P)$ *then* $\mathbf{E}(f|\Sigma_o) \in L_p(\Sigma_o P)$ *and* $||\mathbf{E}(f|\Sigma_o)||_p \leq ||f||_p$.

A sequence (f_n) in $L_1(\Sigma, P)$ is called a *martingale* if there is a corresponding increasing sequence (Σ_n) of sub σ–fields of Σ such that f_n is Σ_n measureable and $\mathbf{E}(f_{n+1}|\Sigma_n) = f_n P|_{\Sigma_n}$ almost surely.

Though martingales have been under intense scrutiny for better than half a century, the grand dad of martingale theorems is still one of the most stunning.

THE MARTINGALE CONVERGENCE THEOREM (Doob) *Every L_1–bounded martingale sequence is almost surely convergent.*

It is natural to ask when a martingale is convergent in mean. The result, also due to Doob, is as follows:

MEAN CONVERGENCE OF MARTINGALES *An L_1–bounded martingale sequence converges in L_1–mean if and only if it is uniformly integrable.*

Of course the L_1–boundedness cited above is done so for emphasis and is not necessary since each of the pertinent conditions implies L_1–boundedness.

Doob's Theorem on mean convergence was proved by Doob directly using the very definition we started with of uniform integrability. However, it could have been culled from an old result of Vitali which in our terms goes as follows.

THEOREM 11 (Vitali). *A subset K of $L_1(P)$ is relatively norm compact if and only if it is relatively weakly compact and relatively $L_o(P)$–compact (i.e. relatively compact in the topology of convergence in probability).*

Proof. Suppose K is relatively weakly compact and relatively L_o–compact. Let $(f_n) \subseteq K$ and let $M = \sup\{||f|| : f \in K\}$. There is a subsequence (g_n) of (f_n) that converges in $L_o(P)$ to some f; the M–ball in $L_1(P)$ is L_o–closed, thanks to Fatou's Lemma and so $||f||_1 \leq M$. Of course $g_n - f \to 0$ in $L_o(P)$ and $\{g_n - f : n \geq 1\}$ is uniformly integrable. Therefore given $\epsilon > 0$ there's a $\delta > 0$ so that whenever $P(E) \leq \delta$ we have $\int_E |g_n - f| dP \leq \epsilon$, for all n. Moreover, there is an $N = N_\epsilon \in \mathbf{N}$ so that if $n \geq N_\epsilon$ then

$$P[\,|g_n - f| \geq \epsilon] \leq \delta\,;$$

after all $(g_n - f)$ is $L_o(P)$–null. Let $n \geq N$ and you realize that

$$||g_n - f||_1 = \int_{[\,|g_n - f| \leq \epsilon]} + \int_{[\,|g_n - f| > \epsilon]} |g_n - f| dP \leq \epsilon + \epsilon = 2\epsilon$$

and so $||g_n - f||_1 \to 0$.

Martingales are ever present in the study of weak convergence in $L_1(P)$. Here's why.

THEOREM 12 (Gaposhkin). *Suppose* (Ω, Σ, P) *is a non–atomic probability space and* (f_n) *is a weakly null sequence in* $L_1(P)$, *then there is a subsequence* (g_n) *of* (f_n) *and a sequence* $(d_n) \subseteq L_1(P)$ *such that*

$$\sum_n ||g_n - d_n||_1 < \infty$$

and yet for each n,

$$\mathbf{E}(d_n | d_1, \ldots, d_{n-1}) = 0 \quad \textit{almost surely}\ .$$

Such sequences are called *martingale difference sequences* because if we let $H_n = \Sigma_{k=1}^n d_k$ then (H_n) is a martingale sequence. Martingale difference sequences enjoy many of the properties of independent random variables and all the properties of orthogonal sequences.

Remarkable as Gaposhkin's Theorem is, its proof is more so.

Proof. Let $n_1 = 1$.

Choose $A_1^{(1)}$ and $A_2^{(1)}$ to be disjoint members of Σ with $A_1^{(1)} \cup A_2^{(1)} = \Omega$ and $P(A_1^{(1)}) = \frac{1}{2} = P(A_2^{(1)})$.

Define $d_1 = \chi_{A_1^{(1)}} - \chi_{A_2^{(1)}}$. Plainly, $\int d_1 dP = 0$ and $||f_{n_1} - d||_1 \leq \frac{\max(1, 2||f_1||+2)}{2}$. For future convenience, we let $B = \max(1, 2||f_1|| + 2)$.

Pick $n_2 > n_1$ so that

$$|\int_{A_i^{(1)}} f_{n_2} dP| \leq \frac{P(A_i^{(1)})}{3 \cdot 2^2} \quad i = 1, 2\ .$$

Approximate f_{n_2} on $A_1^{(1)}$ and $A_2^{(1)}$ by simple functions $f_1^{(2)}$ and $f_2^{(2)}$ so closely that

$$\int_{A_i^{(1)}} |f_{n_2} - f_i^{(2)}| dP \leq \frac{P(A_i^{(1)})}{3 \cdot 2^2} \quad i = 1,2 \ .$$

Set

$$\hat{f}_2 = f_1^{(2)} \chi_{A_1^{(1)}} + f_2^{(2)} \chi_{A_2^{(1)}} \ .$$

Put

$$d_2 = \hat{f}_2 - \mathbf{E}(\hat{f}_2 | d_1) \ .$$

Of course,

$$\mathbf{E}(d_2 | d_1) = 0 \quad \text{almost surely} \ .$$

It is plain and easy to see that on $A_i^{(1)}$,

$$\mathbf{E}(g|d_1) = \frac{1}{P(A_i^{(1)})} \int_{A_i^{(1)}} g dP \ ;$$

therefore, on $A_i^{(1)}$ we have almost surely that

$$\begin{aligned}
&|\mathbf{E}(\hat{f}_2 | d_1)| \\
&\quad \leq |\mathbf{E}(\hat{f}_2 - f_{n_2} | d_1)| + |\mathbf{E}(f_{n_2} | d_1)| \\
&\quad \leq \frac{1}{P(A_i^{(1)})} \int_{A_i^{(1)}} |\hat{f}_2 - f_{n_2}| dP + \frac{1}{P(A_i^{(1)})} \left| \int_{A_i^{(1)}} f_{n_2} \, dP \right| \\
&\quad \leq \frac{1}{P(A_i^{(1)})} \frac{P(A_i^{(1)})}{3 \cdot 2^2} + \frac{1}{P(A_i^{(1)})} \left| \int_{A_i^{(1)}} f_{n_2} \, dP \right| \\
&\quad \leq \frac{1}{3 \cdot 2^2} + \frac{1}{P(A_i^{(1)})} \frac{P(A_i^{(1)})}{3 \cdot 2^2} = \frac{2}{3 \cdot 2^2} \ .
\end{aligned}$$

Now we can compute $\|d_2 - f_{n_2}\|_1$: we do it piecemeal:

$$\begin{aligned}
\int_{A_i^{(1)}} |d_2 - f_{n_2}| dP &= \int_{A_i^{(1)}} |\hat{f}_2 - \mathbf{E}(\hat{f}_2 | d_1) - f_{n_2}| dP \\
&\leq \int_{A_i^{(1)}} |\hat{f}_2 - f_{n_2}| dP + \int_{A_i^{(1)}} \mathbf{E}(\hat{f}_2 | d_1) dP| \\
&\leq \frac{P(A_i^{(1)})}{3 \cdot 2^2} + \frac{2P(A_i^{(1)})}{3 \cdot 2^2} = \frac{P(A_i^{(1)})}{2^2}
\end{aligned}$$

Summing over the $A_i^{(1)}$'s gives

$$\|d_2 - f_{n_2}\|_1 \leq \Sigma_i \int_{A_i^{(1)}} |d_2 - f_{n_2}| dP$$
$$\leq \Sigma_i \frac{P(A_i^{(1)})}{2^2} = \frac{1}{2^2} \leq \frac{B}{2^2} .$$

The next stage starts by listing the sets of constancy $A_1^{(2)}, \ldots, A_{\nu_2}^{(2)}$ of d_2 and make no mistake about it d_2 is a simple function thanks to the utter simplicity of d_1. The sets $(A_i^{(2)})_{i \leq \nu_2}$ partition Ω into disjoint members of Σ and both d_1 and d_2 are constant on each $A_i^{(2)}$. It follows that the σ-field $\sigma(d_1, d_2)$ generated by d_1 and d_2 is just that generated by $\{A_i^{(2)} : i \leq \nu_2\}$ and so is finite. What's more, for any $g \in L_1(P)$, on $A_i^{(2)}$ we have

$$\mathbf{E}(g|d_1, d_2) = \frac{1}{P(A_i^{(2)})} \int_{A_i^{(2)}} g dP .$$

Now choose $n_3 > n_2$ so that

$$\left| \int_{A_i^{(2)}} f_{n_3} dP \right| \leq \frac{P(A_i^{(2)})}{3 \cdot 2^3} \quad i = 1, \ldots, \nu_2 .$$

Approximate f_{n_2} on $A_i^{(2)}$ by a simple $f_i^{(3)}$ so that

$$\int_{A_i^{(2)}} f_{n_3} - f_i^{(3)} | dP \leq \frac{P(A_i^{(2)})}{3 \cdot 2^3} .$$

Set

$$\hat{f}_3 = \sum_{i \leq \nu_2} f_i^{(3)} \chi_{A_i^{(2)}} .$$

Define

$$d_3 = \hat{f}_3 - \mathbf{E}(\hat{f}_3 | d_1, d_2) ;$$

note that d_3 is simple, $\mathbf{E}(d_3 | d_1, d_2) = 0$ almost surely.

Check that on each $A_i^{(2)}$

$$|\mathbf{E}(\hat{f}_3 | d_1, d_2)| \leq \frac{2P(A_i^{(2)})}{3 \cdot 2^3} .$$

Use this to conclude that

$$\int |\mathbf{E}(\hat{f}_3|d_1, d_2)|dP \leq \frac{2}{3 \cdot 2^3}$$

so that

$$||d_3 - f_{n_3}||_1 \leq \frac{B}{2^3} .$$

Tra la, tra la!

Let's pursue just one consequence of life as a martingale difference sequence. We have in mind the sequence provided by Gaposhkin.

THEOREM 13 (Freniche). *Any weakly null martingale difference sequence* (d_n) *in* $L_1(P)$ *has norm null arithmetic means.*

Proof. Keeping in mind the fact that a weakly null sequence in $L_1(P)$ is uniformly integrable, let $M > 0$ be (momentarily) fixed and supply an $\epsilon > 0$ to the proceedings.

Let $e_n = d_n \chi_{[|d_n| \leq M]}$ and let h_n be given as follows

$$h_1 = e_1, h_n = \mathbf{E}(e_n|d_1, \ldots, d_{n-1}) \quad n \geq 2 ;$$

by golly, $(e_n - h_n)$ is a martingale difference sequence that's uniformly bounded by $2M$ and adapted to the sequence $\sigma(d_1, \ldots, d_n)$. Moreover, $(e_n - h_n)$ is an orthogonal sequence!! THINK ABOUT IT!! If (Δ_n) is any martingale difference sequence adapted to the sequence (Σ_n) of sub σ fields with $\Delta_n \in L_2(P)$ then

$$\begin{aligned} \int \Delta_{m+n} \Delta_m dP &= \int \mathbf{E}(\Delta_{m+n} \Delta_m | \Sigma_m) dP \\ &= \int \Delta_m \mathbf{E}(\Delta_{m+n} | \Sigma_m) dP = 0 . \end{aligned}$$

$$\sum_{n \leq N} d_n = \sum_{n \leq N} (d_n - e_n) + \sum_{n \leq N} (e_n - h_n) + \sum_{n \leq N} h_n .$$

Uniform integrability lets us choose M so big that $||d_n - e_n||_1 \leq \epsilon$ for all n as well as assuring that $||h_n||_1 \leq \epsilon$ as well. It follows that

$$\begin{aligned}||\sum_{n\leq N} d_n||_1 &\leq ||\sum_{n\geq N}(d_n - e_n)||_1 + ||\sum_{n\leq N}(e_n - h_n)||_1 + ||\sum_{n\leq N} h_n||_1 \\ &\leq N\epsilon + ||\sum_{n\leq N}(e_n - h_n)||_2 + N\epsilon \\ &\leq N\epsilon + 2M\sqrt{N} + N\epsilon .\end{aligned}$$

So

$$\frac{1}{N}||\sum_{n\leq N} d_n||_1 \leq \epsilon + \frac{2M}{\sqrt{N}} + \epsilon .$$

and the band plays on.

Close on the heals of Gaposhkin and Freniche is the following beautiful result about $L_1(P)$.

COROLLARY (Szlenk). *Any weakly null sequence in $L_1(P)$ has a subsequence with norm null arithmetic means.*

In case you're wondering about atoms being present: don't worry, be happy. The atomic piece of $L_1(P)$ is isomorphic to a subspace of ℓ_1, a Banach space long known to enjoy the "Schur property" – weak and norm convergence of sequences in ℓ_1 coincide.

Armed with Gaposhkin's Lemma. Aldous and Fremlin took a close look at $L_1(P)$ and, with the help of some of the many beautiful inequalities available to martingale difference sequences, proved the following.

THE ALDOUS–FREMLIN DICHOTOMY *Let (f_n) be a bounded sequence in $L_1(P)$. Then either (f_n) has a norm convergent subsequence or (f_n) has a subsequence (g_n) which admits lower ℓ_2–estimates, that is, for some subsequence (g_n) of (f_n) and some $C > 0$, regardless of the scalar sequence a_n we have*

$$C(\sum_n |a_n|^2)^{1/2} \leq ||\sum_n a_n g_n||_1 .$$

It has always been so that uniform integrability's role in functional analysis was tied closely to the study of operators, particularly between classical Banach spaces and general Banach spaces. The Dunford–Pettis Theorem itself was used immediately by Dunford and Pettis to study "representable" operators. Here's what they proved.

THEOREM 14 (Dunford–Pettis). *Let $u : L_1(P) \to X$ be a representable operator, that is, suppose there is a strongly measurable $g : \Omega \to X$ that's P–essentially bounded such that*

$$uf = \text{Bochner} \int f g dP \quad f \in L_1(P) .$$

Then u takes weakly compact sets in $L_1(P)$ to compact sets in X.

Proof. If u were representable by a simple g then u would be a finite rank operator hence compact and so in such a case u would take bounded sets into compact sets.

Generally, representable operators are close enough to the above set–up to ensure they take uniformly integrable sets into compact sets.

If $\epsilon > 0$ is given and K is uniformly integrable, then using the definition of strong measurability and Egoroff's Theorem we can find a simple function $h : \Omega \to X$ such that $P[\,\|h - g\| > \epsilon]$ is small–small enough that all the integrals $\int_{[\,\|h-g\|>\epsilon]} |f| dP$ are very small regardless of $f \in K$. The result will be

$$\begin{aligned} uf &= \int f g dP = \int f(g-h)\,dP + \int f h dP \\ &= \int_{[\,\|f-g\|\le\epsilon]} f(g-h)\,dP + \int_{[\,\|f-g\|>\epsilon} f(g-h)\,dP + \int f h dP . \end{aligned}$$

For $f \in K$ we get that

$$uf = \underset{\text{norm}}{\underset{\text{small}}{\text{something of}}} + \underset{\text{norm}}{\underset{\text{small}}{\text{something of}}} + \int f h dP .$$

Conclusion: $\{uf : f \in K\}$ is totally bounded in X.

Dunford and Pettis went on to prove that *if $u : L_1(P) \to X$ is a weakly compact linear operator with a separable range, then u is representable.* Soon thereafter, Phillips noted the following

THEOREM 15. (Phillips). *Every weakly compact linear operator* $u: L_1(P) \to X$ *has a separable range.*

Proof. Again uniform integrability plays a role.

Let $u: L_1(P) \to X$ be a weakly compact linear operator. To prove $u(L_1(P))$ is separable it's enough to prove $\{u(\chi_E) : E \in \Sigma\}$ is separable, and this will follow if we can show $\{u(\chi_E) : E \in \Sigma\}$ is relatively compact. So let (E_n) be a sequence from Σ and look at $\Sigma_o = \sigma(\Sigma_n)$ the σ–field generated by the E_n's. If we look at $L_1(\Sigma_o, P|_{\Sigma_o})$ we get a separable closed linear subspace of $L_1(P)$ and so $u_o = u|_{L_1(\Sigma_o, P-\Sigma_o)}$ is weakly compact and has separable range. By the Dunford–Pettis result cited above, u_0 is representable hence $\{u_o\chi_E) : E \in \Sigma_o\}$ is the image of uniformly integrable family–namely, $\{\chi_E : E \in \Sigma_o\}$. Theorem 15 follows now from Theorem 14.

So weakly compact linear operators from $L_1(P)$ to any X are representable and hence map weakly compact sets to compact sets. This enunciates what is usually called the Dunford–Pettis theorem for operators on $L_1(P)$.

A particularly striking consequence of the above circle of ideas is a frequently useful theorem of Grothendieck (which he ascribes to Phillips!): *if K is a weakly compact subset of a Banach space and μ is a regular Borel (probability) measure defined on (K, weak) then μ's support is separable.* The idea behind the proof is so elegant it deserves a few words. We may as well suppose μ is a probability. Next, the Krein–Smulian Theorem says K's closed convex hull and its absolutely (=balanced) closed convex hull are both weakly compact so we might just as well assume K is a absolutely convex and weakly compact. Now μ has a barycenter in K: there is a unique $x \in K$ such that for each $x^* \in X^*$, $x^*(x) = \int_K x^*(k)\,d\mu(k)$; it follows that if $f \in B_{L_1)\mu)}$ then there is a unique $u(f) \in K$ such that for each $x^* \in X^*$, $x^*(u(f)) = \int_K x^*(k) f(k)\,d\mu(k)$. The operation $u: B_{L_1(\mu)} \to K$ extends to a weakly compact linear operator $u: L_1(\mu) \to K$ by homogeneity. u has a separable range, i.e., there is a separable closed linear subspace X_o of X such that $u(L_1(\mu) \subseteq X_o$. Plainly, $K \cap X_o$ contains μ's support.

Broaching the subject of representable operators leads us to the question of uniform integrability in the spaces of vector–valued Bochner integrable functions. Here the picture is understandably murkier and it has only been in the past decade or so that any real understanding has emerged.

To put things in context we start with the probability space (Ω, Σ, P) as before, let X be a (real) Banach space and look for $1 \leq p \leq \infty$ at the space of all (equivalence classes of) functions $f : \Omega \to X$ that're strongly P–measurable and such that $\|f(\cdot)\| \in L_p(P)$ with $\|f\| = \| \|f(\cdot)\| \|_p$.

The study of the Lebesgue–Bochner spaces is replete with examples of results sure to elicit a shrug and an "Oh yes, that's nice..." from the uninitiated. Too bad. Much is missed in this way.

For instance, a beautiful result of Kwapien says "X contains a copy of c_o if and only if for any $1 \leq p < \infty$ and any (Ω, Σ, P), $L_p(P, X)$ contains a copy of c_o"; while this might be met with a blasé attitude it in fact covers a very pretty piece of mathematics. Indeed Kwapien was interested in bigger fish and he caught one: he showed that in order for the almost sure boundedness of the sum $\Sigma_n f_n$ of independent integrable random variables with values in a Banach space to imply the almost sure convergence of the sum it is both necessary and sufficient that the space be void of subspaces isomorphic to c_o.

In this case $L_p(P, X)$ possesses another mysterious quality. It is plain that if X contains c_o, then $L_p([0,1]X)$ does too; ACTUALLY, and here we see the mixture of $L_p[0,1]$ and X really work, *if X contains an isomorphic copy of c_o, then $L_p([0,1]), X)$ contains a complemented copy of c_o*. This result is due to Giovanni Emmanuelle, inspired no doubt by an earlier gem of like *ilk* due to Pilar Cembranos. Let's see just why c_o's presence in X forces its complemented presence in $L_p([0,1]), X)$.

Basic to our considerations is the following: suppose Z is a Banach space and $u : Z \to c_o$ is a bounded linear operator such that for some series $\Sigma_n z_n$ in Z, $\Sigma_n u z_n$ is *not* unconditionally convergent in c_o even though $\Sigma |z^* z_n| < \infty$ for each $z^* \in Z^*$. The series $\Sigma_n u z_n$ is a wuc (weakly unconditionally Cauchy) but not an uc (unconditionally convergent). Now Bessaga and Pelczynski showed that in such a case there's subsequence (v_n) of (z_n) such that (v_n) is equivalent to c_o's unit coordinate vector basis and $u|_{[v_n]}$ is an isomorphism onto $Y = [u(v_n)]$, where [ ] denotes the closed linear space of its enclosure. But c_o is complemented in any separable super space (this is a famous theorem of Sobczyk) so $[u(v_n)]$ is the range of a bounded linear projection S. If $P : Z \to Z$ is the operator

$(u|_{[v_n]})^{-1}Su$, then P is a bounded linear projection of Z onto $[v_n]$, an isomorph of c_o.

To build on this we recall that a subset K of the Banach space Z is *limited* if whenever (z_n^*) is a weak* null sequence in Z^* we have

$$\lim_n \sup_{k\in K} |z_n^*(k)| = 0 ;$$

$K \subset Z$ is limited if and only if given any bounded linear operator $u : Z \to c_o$, uK is relatively compact. *If Z contains an unlimited sequence (z_n) that's equivalent to the unit coordinate vector basis of c_o, then Z contains a complemented subspace that's isomorphic to c_o.* Indeed, if (z_n) is such a sequence it must be because for some weak* null sequence (z_m^*) in Z^* and some subsequence (z_{n_m}) of (z_n) and some $\epsilon_o > 0$ we have $z_m^*(z_{n_m}) > \epsilon_o$ for all m. Define $u : Z \to c_o$ by $uz = (z_m^*(z))_m$. $\Sigma_m z_{n_m}$ is a wuc but $\Sigma_m uz_{n_m}$ is not an uc simply because $||uz_{n_m}|| \geq |z_m^*(z_{n_m}| > \epsilon_o$ for all m. Plug in the general procedure described in the previous paragraph, turn it on with the work of this paragraph and find your complemented copy of c_o in Z.

How does the above procedure apply to $L_p([0,1],X)$? Well, in $L_1[0,1]$ look to the Rademacher sequence (r_n); (r_n) is biorthogonal to itself, i.e. (r_n) viewed in $L_1[0,1]^*$ satisfies $r_m(r_n) = \int_0^1 r_m(t)r_n(t)\,dt = \delta_{mn}$. Let $(x_n) \subseteq X$ be equivalent to c_o's unit coordiante vector basis and let $(x_n^*) \subseteq X^*$ be a bounded sequence that's biorthogonal to (x_n). Since there's k, $K > 0$ such that

$$k \max |a_n| \leq ||\sum a_n x_n|| \leq K \max |a_n|$$

for any $(a_n) \in c_o$ we have

$$k \max |a_n| \leq ||\sum_n a_n r_n(t) x_n|| \leq K \max |a_n|$$

for any $(a_n) \in c_o$ and any $t \in [0,1]$. On integrating we see that

$$\begin{aligned} k \max |a_n| &\leq \int_0^1 ||\Sigma_n a_n r_n(t) x_n|| dt \\ &= ||\Sigma_n a_n r_r \otimes x_n||_{L_1[0,1],X)} \leq K \max(a_n) . \end{aligned}$$

So $(r_n \otimes x_n)$ is equivalent to c_o's unit coordinate vector basis. But $(r_n \otimes x_n^*) \subseteq L_1([0,1],X)^*$ is weak* null yet $(r_n \otimes x_n,\ r_m \otimes x_m^*) = \int_o^1 r_n(t) r_m(t) x_m^*(x_n)\,dt$ does not go to zero uniformly in n as $m \to \infty$ – after all, for $m = n$ the above integal is =1. So $(r_n \otimes x_n)$ is unlimited. Fini. The same holds for $1 < p < \infty$, too.

So surprises exist in the study of the Lebesgue – Bochner spaces. So, too, do difficulties in the study of weak compactness therein. Uniform integrability *still* plays a central role.

Uniform integrability in its pristine form remains the same: $K \subset L_1(P,X)$ (as we call the space of Lebesgue–Bochner integrable functions) is uniformly integrable if

$$\lim_{c\to\infty} \sup_{f\in K} \int_{[\|f\|>c]} \|f(\omega)\| dP(\omega) = 0 \ ;$$

as before, with nary a skip in heart beat, K is uniformly integrable if and only if K is L_1–bounded and for each $\epsilon > 0$ there is a $\delta > 0$ such that if $P(E) \leq \delta$ then $\int_E \|f(\omega)\| dP(\omega) \leq \epsilon$ for all $f \in K$. Once more if we try to mimic the Lebesgue–Vitali Theorem's proof we discover that for bounded $K \subseteq L_1(P,X)$, K is uniformly integrable if and only if given any sequence (E_n) of pairwise disjoint events

$$\lim_n \sup_{f\in K} \int_{E_n} \|f(\omega)\| dP(\omega) = 0$$

Vitali's Theorem holds, too, and we even have *by precisely the same reasoning as before* the following

THEOREM 16. (Uhl's vectorial version of the Kadec–Pelczynski Theorem). *If K is a bounded non–uniformly integrable subset of $L_1(P,X)$, then there is a sequence (f_n) in K such that (f_n) is equivalent to the unit coordinate vector basis of ℓ_1.*

Anyone with a nodding acquaintance with the unit coordinate vector basis of ℓ_1 can soon realize the following

COROLLARY. *IF K is a conditionally weakly compact subset of $L_1(P,X)$, that is, if every sequence in K has a subsequence that's weakly Cauchy, then K is uniformly integrable.*

Generally speaking Uhl has given the best consequence regarding uniformly integrable collections without resorting to hypotheses on X. A natural hope would be to recapture something akin to the Dunford–Pettis Theorem. Sorry folks but it's not in the cards. What can be said of the Dunford–Pettis Theorem in its pristine form was known by Dunford and Pettis.

THEOREM 17. (Dunford–Pettis in souped–up style). *suppose X is a* reflexive *Banach space. Then $K \subset L_1(P,X)$ is relatively weakly compact if and only if K is uniformly integrable.*

A word or two of warning are in order.

The proof of this vector–valued version of the Dunford–Pettis Theorem is not *a* trivial *modification of its scalar counterpart.* New ideas come to the fore to save the day as well as give warning of obstructions to further generalization. Let's talk about some of these ideas in more detail.

Naturally, if we want to discuss duality results (and we are talking about relative and conditional weak compactness) then some care must be taken to describe $L_1(P,X)^*$. Is it $L_\infty(P,X^*)$? Sometimes it is and sometimes it's not; when it is then for $f \in L_1(P,X)$ and $g \in L_\infty(P,X^*)$ it's easy to show that $g(\cdot)(f(\cdot)) \in L_1(P)$ and

$$g(f) = \int_\Omega g(\omega)(f(\omega)dP(\omega)$$

makes sense and handles duality perfectly well with $||g||_\infty = ||g||_{L_1(P,X)^*}$. HOWEVER, this duality is available if and only every operator $u : L_1(P) \to X^*$ is representable! So, if X is reflexive (ensuring X^* is) then we know a bit about the problem: at least duality is behaving!. In a word, reflexivity is *just* what's called for to conclude to the equivalence of uniform integrability and relative weak compactness – indeed *if X is any Banach space for which these notions agreee for all subsets of $L_1(P,X)$, then X must be reflexive.*

Since the case of X reflexive is so old and so well–documented we turn to some of the more delicate relationships involving uniform integrability in the vector–valued setting. To make clear what we're about we have to discuss $L_1(P,X)^*$ when not every operator $u : L_1(P) \to X^*$ is representable. In this case, $L_1(P,X^*)$ is not $L_\infty(P,X^*)$ but it's still describable and the description is better than a kick in the knee.

Using the lifting theorem, one can identify $L_1(P,X)^*$ with the space $L^\sigma_\infty(P,X^*)$ of all $g : \Omega \to X^*$ such that $g(\cdot)(x) \in L_\infty(P)$ for each $x \in X$ and such that $\|g(\cdot)\| \in L_\infty(P)$, too. In this case, perfectly good sense can be made of

$$g(f) = \int g(\omega)(f(\omega))dP(\omega)$$

since f is strongly measurable. It's this pairing that completely describes the duality between $L_1(P,X)$ and $L_1(P,X)^* = L^\sigma_\infty(P,X^*)$. (In fact, such a modification also allows a complete description of $L_p(P,X)^* = L^\sigma_q(P,X^*)$ in case ($1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$).

Using this duality, Pisier (with a bit of help from Maurey) was able to show that *conditional* weak compactness follows from uniform integrability in a very special class of spaces.

THEOREM 18 (Pisier & Maurey). *In order that uniformly integrable subsets of $L_1(P,X)$ be conditionally weakly compact regardless of (Ω,Σ,P) it is necessary and sufficient that X not contain any subspace isomorphic to ℓ_1.*

The proof of the above Theorem relies essentially on Rosenthal's ℓ_1 theorem and some ideas the roots of which are found in the theory of operator ideals. In fact, Pisier relates the phenomenon of uniform integrability implying conditional weak compactness with the validity of certain vector–valued versions of the Riemann–Lebesgue lemma.

In 1982, Talagrand characterized weakly Cauchy sequences in general $L_1(P,X)$'s. His results involved sequences that were not necessarily subsequences of a given sequence but rather were convex combinations of tails of such a sequence. It was to wait until 1990 before anyone saw the convenience of Talagrand's formulations (by the way, the results of Talagrand in his 1984 paper are quite pretty, non–trivial in the extreme, of great interest to anyone who's reading *this* stuff but not necessary to repeat herein!). In 1990, A. Ülger formulated a criterion for weak compactness in $L_1(P,X)$; his criteria were the first of their kind but lacked the crispest formulation possible. In any case, Ülger's work evolved with the end result:

THEOREM 19 (Ülger–Schachermayer–Ruess–Diestel). *The following statements regarding a subset K of $L_1(P,X)$ are equivalent.*

1. *K is relatively weakly compact.*
2. *K is uniformly integrable and given any sequence (f_n) in K there is a sequence (g_n) so that $g_n \in co\{f_n, f_{n+1} \ldots\}$ such that $(g_n(\omega))$ is norm convergent for almost all $\omega \in \Omega$.*
3. *K is uniformly integrable and given any sequence (f_n) in K there is a sequence (h_n) so that $h_n \in co\{f_n, f_{n+1} \ldots\}$ such that $(h_n(\omega))$ is weakly convergent for almost all $\omega \in \Omega$.*

Proof. We'll rely on a folklore Lemma unearthed by Ülger, which he derived from James's theorem on attainment of suprema by linear functionals on potentially weakly compact sets. Ülger showed the following fact:

For a subset A of a Banach space X the following are equivalent:

(a) *A is relatively weakly compact.*

(b) *Given any sequence (a_n) in A we can find a norm convergent sequence (b_n) so that $b_n \in co\{a_n, a_{n+1} \ldots\}$.*

(c) *Given any sequence (a_n) in A we can locate a weakly convergent sequence (c_n) so that $c_n \in co\{a_n, a_{n=1}, \ldots,\}$.*

This Lemma in hand suppose (1) of Theorem 19 holds. By UHL's version of Kadec–Pelczynski, K is uniformly integrable. Take a sequence (f_n) from K. By (1), (f_n) has a subsequence (f_n') which is weakly convergent to some $f \in L_1(P,X)$. By Mazur's theorem, there is a sequence (f_n'') of convex combinations of (f_n')'s tails such that $f_n'' \to f$ in $L_1(P,X)-$norm. We plainly have that $f_n'' \in co\{f_n' f_{n+1}' \ldots\} \subset co\{f_n, f_{n+1}, \ldots\}$. Since (f_n'') converges in mean, (f_n'') converges in probability and so (f_n'') has a subsequence (g_n) that is almost everywhere convergent; it is clear that $g_n \in co\{f_n, f_{n+1}, \ldots\}$. How about that?

(2) implies (3) as a matter of nature.

Let's suppose (3) holds. Suppose $(f_n) \subseteq K$ and pass to $g_n \in co\{f_n, f_{n+1} \ldots\}$ with the idea being that (g_n) is almost everywhere weakly convergent. Let $g(\omega) :=$ weak $\lim_n g_n(\omega)$ (if each exists) and let $g(\omega) = 0$ if not. How about g? g is scalarly measurable and $P-$essentially separable valued. Pettis's measurability theorem tells us that g is strongly measurable. Of course, $||g(\omega|| \leq \underline{\lim}||g_n(\omega)||$ for almost all $\omega \in \Omega$ so $g \in L_1(P,X)$ thanks to Fatou's lemma. ACTUALLY, g is the weak limit of (g_n) in $L_1(P,X)$. In fact, we might as well assume X is separable and view a typical $h \in L_\infty^\sigma(P,X^*) = L_1(P,X^*)$: $h(\omega)(g(\omega)) = \lim_n h(\omega)(g_n(\omega))$ holds for almost all $\omega \in \Omega$. K (and its convex hull) is uniformly integrable so $\{h(\cdot)(g_n(\cdot)): n \geq 1\}$ is uniformly integrable. Vitali's Theorem comes on the scene:

$$\begin{aligned} h(g) &= \int h(\omega)(g(\omega))\,dP(\omega) \\ &= \lim_n \int h(\omega)(g_n(\omega))\,dP(\omega) \\ &= \lim_n h(g_n) \,. \end{aligned}$$

Ülger's Lemma is ready for the <u>*KILL*</u>: K is relatively weakly compact. And that's all she wrote!

Uniformly integrable sets also play a central role in the study of weakly compact sets in Banach lattices. Without going into all the details let's see how this comes about. Incidentally, the study of Banach lattices has proved a fertile outlet for measure theory and nowhere is that more so than in the representation theory of Banach lattices.

Let E be an order complete order continuous separable Banach lattice. It is a well–known byproduct of the representation theory of Banach lattices that there exists a probability space (Ω, Σ, P) such that E is an order ideal in $L_1(P)$ with the following properties: $L_\infty(P) \subseteq E \subseteq L_1(P)$ with $L_\infty(P)$ dense in E and the inclusions are continuous. What's more, E^* may be identified with those measurable functions h on Ω such that $\|h\|_{E^*} = \sup\{\int hf dP : \|f\|_E \leq 1\} < \infty$ with $\int hf dP = h(f)$ defining the duality. It is easy to describe the relatively weakly compact subsets of E.

THEOREM 20. (Dieudonné). *Suppose E is as above. Then a bounded set A in E is relatively weakly compact if and only if for each $h \in E^*$, $hA = \{hf : f \in A\}$ is uniformly integrable.*

Proof. Since the map $g \to hg$ is a continuous linear operator of E into $L_1(P)$, hA is relatively compact if A is. Apply the Dunford–Pettis Theorem.

On the other hand, if we assume hA is a uniformly integrable for each $h \in E^*$ and let (g_n) be a sequence of members of A, then (g_n) has a subsequence (f_n) which converges weakly in $L_1(P)$ to some $f \in L_1(P)$ – after all $A = 1 \cdot A$ must itself be uniformly integrable hence a relatively weakly compact subset of $L_1(P)$, thanks again to the Dunford–Pettis theorem. Now if $\epsilon > 0$ is given, and $h \in E^*$ is fixed (though arbitrary), then we look at $\int h(f_n - f)\,dP$: for any $c > 0$, we have

$$\int h(f_n - f)\,dP = \Big(\int_{[|h|\leq c]} + \int_{[|h|>c]}\Big) h(f_n - f)\,dP .$$

Now we can choose c big enough that $P[\,|h| > c]$ is quite small and hence $\int_{[|h|>c]} h(f_n - f)\,dP \leq \epsilon/2$ for all n, this because $\{hA - hf\}$ is uniformly integrable. This being done, we can make $\int_{[|h|\leq c]} h(f_n - f)\,dP$ small simply by noting $\int_{[|h|\leq c]} h(f_n - f)\,dP$ is just $\int h\chi_{[|h|\leq c]}(f_n - f)\,dP$ and so can be made $\leq \epsilon/2$ by picking n large enough. That's all we need. Theorems 19 and 20 can be parlayed into a characterization of relatively weakly compact subsets of $L_p(P, X)$ for $1 < p < \infty$. So used, here's the result.

THEOREM 21 (Diestel–Ruess–Schachermayer). *Suppose $1 < p < \infty$. Then the following are equivalent statements regarding a bounded subset K of $L_p(P, X)$.*

1. *K is relatively weakly compact.*
2. *given any sequence (f_n) in K there is a sequence (g_n) so that $g_n \in co\{f_n, f_{n+1}, \ldots\}$ such that $(g_n(\omega))$ is norm convergent for almost all $\omega \in \Omega$.*
3. *given any sequence (f_n) in K there is a sequence (g_n) so that $g_n \in co\{f_n, f_{n+1} \ldots\}$ such that $(g_n(\omega))$ is weakly convergent for almost all $\omega \in \Omega$.*

We close these discussions with a few words about spaces of measures and relatives of "uniformly integrable sets" that live in such spaces. Our main objects of study will be $ca(\Sigma)$, the space of all real–valued countably additive measures defined on a σ–field Σ of subsets of a given set Ω and $rca(Bo_\Omega)$, the space of all regular members of $ca(Bo_\Omega)$ where Ω is a compact Hausdforff space and Bo_Ω denotes the Borel σ–field of Ω, i.e. Bo_Ω is the σ–field generated by the topology of Ω. Each is a Banach space when equipped with the variation norm. Moreover, each is very L_1–like; indeed, each is an $L_1(\mu)$–space for some unruly μ. The study of relatively weakly compact subsets in each of these spaces is fascinating and touches on some of the prettiest limit theorems in general analysis.

Plainly related to uniform integrability is the notion of uniform (absolute) continuity. If $\mu \in ca(\Sigma)$ is non–negative (denoted by $\mu \in ca^+(\Sigma)$), then $K \subseteq ca(\Sigma)$ is said to be uniformly μ–continuous if given $\epsilon > 0$ there is a $\delta > 0$ such that $|\eta(E)| \leq \epsilon$ whenever $\mu(E) \leq \delta$ for all $\eta \in K$. Now it is easy to show that for any $\eta \in ca(\Sigma)$ and any $E \in \Sigma$

$$\begin{aligned} &\sup\{|\eta(F)| : F \in \Sigma, F \subseteq E\} \\ &\leq |\eta|(E) = \text{variation of } \eta \text{ over } E \\ &\leq 2\sup\{|\eta(F)| : F \in \Sigma,\ F \subseteq E\}\,; \end{aligned}$$

from this one quickly deduces that K is *uniformly μ–continuous if and only if* $|K| = \{|\eta| : \eta \in K\}$ *is.*

Here's a classic.

VITALI–HAHN–SAKS THEOREM. *Let* $(\mu_n) \subseteq ca(\Sigma)$ *and suppose each* μ_n *is absolutely continuous with respect to* $\mu \in ca^+(\Sigma)$. *Assume* $\lim_n \mu_n(E)$ *exists for each* $E \in \Sigma$.

Then $\{\mu_n : n \geq 1\}$ *is uniformly μ–continuous and* $\mu_0(E) \equiv \lim_n \mu_n(E)$ *defines a member of* $ca(\Sigma)$ *which is absolutely continuous with respect to* μ.

Just as we found in our proof of the Lebesgue–Vitali Theorem, behaviour on disjoint sequences of events is worthy of note. $K \subseteq ca(\Sigma)$ is *uniformly additive* if given $\epsilon > 0$ and a sequence (E_n) of pairwise disjoint members of Σ we have an $n_\epsilon \in \mathbb{N}$ such that if $n \geq n_\epsilon$ then

$$\sum_{n \geq n_\epsilon} |\mu(E_n)| \leq \epsilon$$

for all $\mu \in K$. Again K *is uniformly additive if and only if* $|K|$ *is.*

Another classic.

NIKODYM'S CONVERGENCE THEOREM. *Let* $(\mu_n) \subseteq ca(\Sigma)$ *be such that* $\lim_n \mu_n(E)$ *exists for each* $E \in \Sigma$.

Then $\{\mu_n : n \in \mathbb{N}\}$ *is uniformly additive and* $\mu(E) = \lim_n \mu_n(E)$ $(E \in \Sigma)$ *defines a member of* $ca(\Sigma)$.

These *are* theorems about weak compactness and a key to why this is so is a beautiful result of Bartle, Dunford and Schwartz.

THE BARTLE–DUNFORD–SCHWARTZ THEOREM. *Suppose $K \subseteq ca(\Sigma)$ is relatively weakly compact. Then there exists a probability measure μ on Σ such that K is uniformly μ–continuous.*

It is worth mentioning that μ can be chosen in the closed linear space of K.

Finally, we mention another of Nikodym's contributions that is often overlooked in topological vector space texts even though it provides a stunning non–trivial example in barrelled spaces.

NIKODYM'S BOUNDEDNESS THEOREM. *Suppose $K \subseteq ca(\Sigma)$ satisfies $\sup\{|\mu(E)| : \mu \in K\} < \infty$ for each $E \in \Sigma$. Then $\sup\{|\mu(E)| : \mu \in K,\ E \in \Sigma\} < \infty$. In particular, K is bounded in $ca(\Sigma)$.*

Weak compactness? Let's just state the facts.

THEOREM 22. *Let $K \subseteq ca(\Sigma)$. Then the following statements regarding K are equivalent.*

1. *K is relatively weakly compact.*
2. *Given a sequence (μ_n) in K there is a subsequence (η_n) of (μ_n) such that for each $E \in \Sigma$, $\lim_n \eta_n(E)$ exists.*
3. *K is bounded and uniformly additive.*
4. *K is bounded and there exists a probability measure μ on Σ such that K is uniformly μ–continuous.*
5. *$|K|$ is relatively weakly compact.*

In truth the proof of the above theorem makes frequent call on the results involving uniformly integrable sets that we discussed earlier.

For instance, suppose K is bounded and uniformly μ–continuous for a given probability measure μ. Then each $\eta \in K$ has a Radon–Nikodym derivative $f_\eta \in L_1(\mu)$; moreover, for any $f \in L_1(\mu)$ the measure

$$\eta(E) = \int_E f\,d\mu$$

satisfies

$$|\eta|(E) = \int_E |f|\,d\mu\,.$$

It follows that the map $f \to \int_{(\cdot)} f d\mu$ defines an isometric isomorphism of $L_1(\mu)$ into$ca(\Sigma)$. A subset of $L_1(\mu)$ is relatively weakly compact in $L_1(\mu)$ if and only if its image under this isometric isomorphism is relatively weakly compact in $ca(\Sigma)$. But K's uniform μ–continuity, and hence that of $|K|$, quickly translates to the following about the family $\{f_\eta : \eta \in K\}$: given $\epsilon > 0$ there is a $\delta > 0$ such that if $\mu(E) \leq \delta$, then $\int_E |f_\eta| d\mu \leq \epsilon$ for all $\eta \in K$. The Dunford–Pettis Theorem now tells all.

Though we make no use of them in our discussion, we'd be remiss in our duty if we didn't mention the situation of weak compactness in $rca(Bo_\Omega)$. In a nutshell, here's what's so.

THEOREM 23 (Dieudonné–Grothendieck). *Let K be a bounded subset of $rca(Bo_\Omega)$, then the following are equivalent statements about K.*

1. *K is relatively weakly compact.*
2. *K is uniformly regular, that is, given $\epsilon > 0$ and a Borel set B there is a compact set $F \subseteq B$ and an open set $U \subseteq B$ such that $|\mu|(U \backslash F) \leq \epsilon$ for all $\mu \in K$.*
3. *K is uniformly additive an disjoint open subsets of K, that is, given $\epsilon > 0$ and a sequence (U_n) of pairwise disjoint open subsets of K, there is $n_\epsilon \in \mathbf{N}$ such that for $n \geq n_\epsilon$, $\Sigma_{n \geq n_\epsilon} |\mu|(U_n) \leq \epsilon$ for all $\mu \in K$.*

To see just one example of how one can use these results about spaces of measures we present a famous result about operators on spaces of continuous functions.

THEOREM 24. (Grothendieck). *Let $u : C(\Omega) \to X$ be a weakly compact linear operator, where Ω is a compact Hausdorff space and $C(\Omega)$ is the Banach space of continuous real–valued functions defined on Ω.*

Then u takes weakly convergent sequences in $C(\Omega)$ to norm convergent sequences in X.

To begin, we recall that $C(\Omega)^*$ is $rca(Bo)_\Omega$–thanks to Messrs F. Riesz, S. Saks, A. Markov and S. Kakutani. It follows from this and the Lebesgue Bounded Convergence Theorem that a sequence (f_n) in $C(\Omega)$ is weakly null precisely when (f_n) is uniformly bounded and $\lim_n f_n(\omega) = 0$ for each $\omega \in \Omega$.

A moment of reflection will soon convince you that Grothendieck's theorem plainly holds if the operator in question is the inclusion $C(\Omega) \hookrightarrow L_1(\mu)$ for some non-negative $\mu \in rca(Bo_\Omega)$. The inclusion is weakly compact – after all on its way from $C(\Omega)$ to $L_1(\mu)$ it passes through the reflexive space $L_2(\mu)$. Further, by our remarks about weak convergence in $C(\Omega)$ and Lebesgue's theorem (again) weakly convergent sequences in $C(\Omega)$ are norm convergent in $L_1(\mu)$.

The proof we are about to employ will try to show that within epsilonics every weakly compact linear operator is more-or-less dominated by a multiple of the natural inclusion $C(\Omega) \hookrightarrow L_1(\mu)$ for some $\mu \in rca^+(Bo_\Omega)$.

Let $u : C(\Omega) \hookrightarrow X$ be a given weakly compact linear operator. Then $u^* : X^* \to C(\Omega)^*$ is weakly compact, too. Hence $u^*(B_{X^*}$ = closed unit ball of X^* is weakly compact in $C(\Omega)^* = rca(Bo_\Omega) \subseteq ca(Bo_\Omega)$. Hence by the Bartle–Dunford–Schwartz Theorem, there is a regular Borel probability μ on Bo_Ω such that $\{u^*x^* : ||x^*|| \leq 1\}$ is uniformly μ–continuous.

CLAIM: Given $\epsilon > 0$ there is $K_\epsilon > 0$ such that for any $f \in C(\Omega)$,

$$||uf|| \leq K_\epsilon ||f||_{L_1(\mu)} + \epsilon ||f||_\infty .$$

In other words, u's behaviour is almost dominated by the behaviour of a multiple of the inclusion $C(\Omega) \hookrightarrow L_1(\mu)$.

Suppose the CLAIM is false, It'd be because for some $\epsilon_0 > 0$ no matter what $n \in \mathbf{N}$ we choose there'd be an $f_n \in C(\Omega)$, $||f_n||_\infty = 1$, say, such that

$$||uf_n|| \geq n||f_n||_{L_1(\mu)} + \epsilon_0 . \qquad (*)$$

On dividing everything in $(*)$ by n, we see

$$||u(\frac{f_n}{n})|| \geq ||f_n||_{L_1(\mu)} + \frac{\epsilon_0}{n} .$$

If we let $n \to \infty$, then $||f_n||_{L_1(\mu)} \to 0$ soon follows.

By passing to an appropriate subsequence we can assume that (f_n) goes to zero μ–almost surely. For each n there is an $x_n^* \in X^*$ with $||x_n^*|| = 1$ such that $x_n^*(uf_n) = ||uf_n||$. Let $\mu_n = u^*x_n^*$; $\{\mu_n : n \in \mathbf{N}\}$ is uniformly μ–continuous. Here's the catch: (f_n) is uniformly bounded and μ–almost surely null; Egoroff's theorem tells us that f_n is μ–almost uniformly null. Simple epsilonics tell us that $\lim_n \int f_n d_{\mu_n} = 0$ too. BUT $(*)$ says

$$\begin{aligned}\int f_n d_{\mu_n} &= u^* x_n^*(f_n)\\ &= x_n^* u f_n\\ &= \|u f_n\| > \epsilon_0\ .\end{aligned}$$

OOPS!

Once the claim is in hand it is easy to see that if (f_n) is weakly null in $C(\Omega)$ then (uf_n) is norm null in X. After all, if $\epsilon > 0$ is given we can find $K_\epsilon > 0$ so that

$$\|uf\| \le K_\epsilon \|f\|_{L_1(\mu)} + \epsilon \|f\|_\infty\ .$$

Since (f_n) is weakly null in $C(\Omega)$, (f_n) is $L_1(\mu)$–norm null. Hence, there is an n_ϵ such that for $n \ge n_\epsilon$, $\|f_n\| \le \epsilon / K_\epsilon$. It follows that if $n \ge n_\epsilon$ we have

$$\begin{aligned}\|uf_n\| &\le K_\epsilon \|f_n\|_{L_1(\mu)} + \epsilon \|f_n\|_\infty\\ &\le \epsilon + \epsilon \sup_n \|f_n\|_\infty\end{aligned}$$

which is good enough for even the roughest of knowledgable critics.

Notes and Remarks

Since most of what we're discussing herein is old and well–documented we will be brief. Anyone interested in uniform integrability really ought to read Dunford–Schwartz [DS] and Diestel–Uhl [DU] on the subject.

Theorem 1 is as old as the hills and is usually reserved for exercises in advanced courses in measure theory.

Theorem 2 is proved, for example, in Meyer's "Probabilities and Potentials" [MPP] wherein uniform integrability is viewed from a different vantage point. Recently, J. Alexopoulos [A] has sharpened the theorem of de la Vallée Poussin. Here are a couple of his more striking results.

THEOREM (Alexopoulos). *Let $K \subseteq (P)$ be uniformly integrable. Then there exists a convex, even function $\Phi : \mathbf{R} \to \mathbf{R}$ such that $\Phi(0) = 0$, $\lim_{x\to\infty} \Phi(x)/x = \infty$, $\Phi(xx') \leq M\Phi(x)\Phi(x')$ for x, x' large enough and K is relatively weakly compact in $L_\Phi(P)$.*

If K is relative compact in $L_1(P)$, then Φ can be chosen as above with K being relatively compact in $L_\Phi(P)$.

Our proof that uniformly integrable sets are relatively weakly compact was shown us by D.J.H. "Ben" Garling. It seems to be the most natural proof from general principles of abstract analysis and affords us the opportunity of talking about bounded finitely additive measures.

The proof we follow of the Lebesgue–Vitali theorem is much like that of the original.

The Lebesgue–Vitali theorem, the Vitali–Hahn–Saks theorem and both theorems of Nikodym can be proved by either following a sliding–hump approach or calling on the Baire Category Theorem. Each approach has its benefits.

Elegance is apparent when the Baire Category theorem is used, as it is, for instance in Dunford–Schwartz. On the other hand, the sliding hump arguments are "elementary" but not easy and, oftentimes, let one see just what the obstructions to a given phenomenon are.

Rosenthal's lemma provides one with the ultimate tool to go looking for a "sliding hump". My understanding of this remarkable lemma comes from numerous conversations with one J. Jerry Uhl Jr. as indeed does my understanding of many of the topics discussed herein. The interested reader really ought to acquire a copy of Uhl's unpublished lecture notes on Rosenthal's lemma which were the basis for the treatment of the basic limit theorems of Vitali–Hahn–Saks and Nikodym in [DU] but contain so much more.

The last two conditions cited in Theorem 8 are due to H.P. Rosenthal [HPR] and are a motivation for the development of the theory of type and cotype in Banach spaces.

Of course, the Doob Martingale Convergence theorem is found in virtually every advanced graduate text in probability and its uses in analysis continue to grow.

In connection with Vitali's theorem, we hasten to note a stunning recent result of Maria Girardi used by her to characterize the completely continuous operators on $L_1(P)$.

Let $f \in L_1(P)$ and $E \in \Sigma$ with $P(E) > 0$.

The *Bocce oscillation of f on E* is given by

$$\frac{1}{P(E)} \int_E |f - \frac{1}{P(E)} \int_E f dP| dP .$$

A subset K of $L_1(P)$ satisfies the *Bocce criterion* if given $\epsilon > 0$ and $E \in \Sigma$ with $P(E) > 0$ there are finitely many subsets of E each having positive probability such that the Bocce oscillation of any $f \in K$ over any of these sets never exceeds ϵ.

Here's a companion to Vitali's theorem.

THEOREM (Maria Girardi). *If $K \subseteq L_1(P)$ is uniformly integrable and satisfies the Bocce criterion, then K is relatively norm compact.*

Gaposhkin's theorem is found as a special case of Lemma 1 in [G]. It was used by Aldous and Fremlin [AF] to derive their dichotomy. They needed as well some martingale inequalities of the Burkholder–Gundy type; Leonard Dor also derived inequalities of this sort that can be used in this connection. My lectures [DM] at Complutense University in Madrid discuss Dor's work as well as a derivation of the Aldous-Fremlin result therefrom. Freniche's paper [F] derives the Szlenk result in a beautiful way from Gaposhkin's work.

We did not mention in the text (but did in the lectures) a result of Komlos which makes frequent use of the idea of uniform integrability. The result

KOMLOS' THEOREM. *Let (f_n) be a bounded sequence in $L_1(P)$. Then there is an $h \in L_1(P)$ and a subsequence (g_n) of (f_n) such that for each subsequence (h_n) of (g_n) we have $h = \lim_N \frac{1}{N} \sum_{n \leq N} h_n P$–almost surely.*

The theorems of Dunford, Pettis and Phillips mentioned in connection with representable operators are given detailed treatment in both [DS] and [DU].

Uhl's version of the Kadec–Pelczynski theorem appears in [DU] as does the souped–up version of the Dunford–Pettis theorem.

The theorem of Pisier & Maurey was independently derived by J. Bourgain whose proof in [B] gives important insights into just how to recognize ℓ_1 inside vector–valued function spaces.

Emmanuelle's theorem also has a short note of Bourgain [BE] as a precursor; Bourgain shows that if ℓ_∞ embeds in X, then $L_p(P, X)$ is not isomorphic to a dual for $1 \leq p < \infty$. This is, by far, the easiest paper Bourgain has written; naturally, it answers a question of mine!

Again, Talagrand's paper on "Weakly Cauchy Sequences in $L^1(E)$" is worth reading and studying in detail. One really ought to have mastered Haskell Rosenthal's ℓ_1–theorem and its proof first and there is no better place to learn of it than from [HPR ℓ_1], after close study of [HPRℓ_1], [DSS] makes sense.

Ülger's paper [U] was really a pleasant surprise. Though Ülger only obtained the cited characterization of weak compactness in $L_1(P, X)$ for uniformly bounded families, he hit on the correct *formulation* of the result.

The result of Dieudonné [DK] is old but still pretty. It has been used by Creekmore [Cr] to characterize weakly compact sets in the Lorentz spaces $L_{w,p}$ and by Ando to characterize weak compactness in Orlicz spaces.

The results on weak compactness in $ca(\Sigma)$ are discussed in [DS] and [DU] along with generalizations. Our proof of the theorem of Grothendieck about weakly compact operators on $C(K)$ is different from his [G] or that of Bartle, Dunford and Schwartz [BDS]. Rather, this proof is due to A. Pelczynski who showed it to me because it modifies to the case of weakly compact operators on the disk algebra; corollary to the proof of Theorem 24 is a result of Chuck Seifert and myself [DSei] to the effect that a weakly compact operator $u : C(\Omega) \to X$ takes bounded sequences to sequences admitting of subsequences with norm convergent arithmetic means. The same hold for weakly compact operators on the disk algebra, any C^*–algebra [J] or into the predual of a von Neumann algebra [BD].

REFERENCES

[AF] ALDOUS D.J. and FREMLIN D.H., *Co-lacunary sequences in L–spaces*, Studia Math. *71* (1981), 297–304.

[A] ALEXOPOULOS J., *Weakly compact sets in Banach spaces*, Ph.D. dissertation, Kent

State University, Kent, Ohio, 1992.

[BD] BELANGER A. and DIESTEL J., *A remark on weak convergence in the dual of a C^*-algebra,* Proc. AMS *98* (1986), 185–186.

[B] BOURGAIN J., *An averaging result for ℓ^1-sequences and applications to weakly conditionally compact sets in L^1_X*, Israel J. Math. *32* (1979), 289–298.

[BE] BOURGAIN J., *A note on the Lebesgue spaces of vector-valued functions,* Bul. Soc. Mat. Belg. Sér *31* (1979) 45–47.

[C] CEMBRANOS P., *$C(K, E)$ contains a complemented copy of c_0*, Proc. AMS *91* (1984), 556–558.

[Cr] CREEKMORE J., *Geometric and linear topological properties of Lorentz function spaces,* Ph.D. dissertation, Kent State University, Kent, Ohio, 1980.

[DSS] DIESTEL J., *Sequences and Series in Banach Spaces,* Graduate Texts in mathematics, volume 92, Springer–Verlag, New York–Berlin–Heidelberg–Tokyo, 1984.

[DSSL$_1$] DIESTEL J., *Sequences and Series in L_1*, Publications of Department of Mathematics Complutense Univ. Madrid, 1986.

[DRS] DIESTEL J., RUESS W. and SCHACHERMAYER W., *Weak compactness in the Lebesgue–Bochner spaces,* Proc. AMS, to appear.

[DSe] DIESTEL J. and SEIFERT C.J., *The Banach–Saks Ideal. I. Operators acting on $C(\Omega)$*, Special issue dedicated to Wladyslaw Orlicz on the occasion of his 75th birthday, Comment Mat. Special Issue (1978), 109–118.

[DU] DIESTEL J. and UHL JR. J.J., *Vector measures,* AMS Surveys volume 15, Providence, RI, 1977.

[DK] DIEUDONNÉ J., *Sur les espaces de Köthe,* J. Analyse Math. *1* (1951), 81–115.

[Do] DOR L.E., *Some inequalities for martingales and applications to the study of L_1*, Math. Proc. Cambridge Phlos. Soc. *89* (1982), 135–148.

[Dor] DOR L.E., *On co-lacunary sequences in L_1*, Bull. London Math. Soc. *14* (1982), 410–414.

[DS] DUNFORD N. and SCHWARTZ J.T., *Linear Operators I,* Interscience Publishers, Inc., New York, 1966.

[E] EMMANUELE G., *On complemented copies of c_0 in L^p_X, $1 \leq p < \infty$*,, Proc. AMS *104* (1988).

[F] FRENICHE F.J., *Césaro convergence of martingale difference sequences and the Banach–Saks and Szlenk theorems,* Proc. AMS *103* (1988), 234–236.

[GR] GAPOSHKIN V.F., *Convergence and limit theorems for sequences of random variables,* Theor. Prop. Applic. *17* (1972), 379–399.

[MG] GIRARDI M., *Compactness in L_1, Dunford–Pettis operators, Geometry of Banach Spaces,* Proc. AMS *111* (1991), 767–777.

[G] GROTHENDIECK A., *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. *5* (1953), 129–173.

[J] JARCHOW H., *On weakly compact operators on C^*-algebras,* Math. Ann. *273* (1986), 341–343.

[KP] KADEC M.I. and PELCZYNSKI A., *Bases lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. *21* (1962), 161–176.

[K] KOMLOS J., *A generalization of a problem of Steinhauss,* Acta Math. Acad. Sci. Hanger *181* (1967), 217–229.

[MPP] MEYER P.A., *Probabilities and Potentials,* Blaisdell, Waltham, Mass. 1966.

[P] PIESTCH A., *Operator Ideals*, North Holland, Amsterdam, 1978.

[PM] PISIER G., *Une propriété de stabilité de la classe des espaces ne contenant pas* ℓ^1, C.R. Acad. Sci. Paris Ser A-B *286* (1978), A747–749.

[HRP] ROSENTHAL H.P., *On subspaces of* L_p, Ann of Math *97* (1973), 344–373.

[HPRℓ^1] ROSENTHAL H.P., *A characterization of Banach spaces containing* ℓ^1, Proc. Nat. Acad. Sci. USA *71* (1974), 2411–2413.

[SS] SAAB E. and SAAB P., *On complemented copies of* c_0 *in injective tensor products*, Contemp. Math. AMS *52* (1986).

[S] SCHLUMPRECHT T., *Limited Sets in Banach Spaces*, Ph.D. dissertation, Ludwigs–Maximillians–Universität, München, 1988.

[Sz] SZLENK W., *Sur les suites faiblement convergents dans l'espace* L, Studia Math. *25* (1965), 337–341.

[T] TALAGRAND M., *Weak Cauchy sequences in* $L^1(E)$, Amer. J. Math. *106* (1984), 703–724.

[Ü] ÜLGER A., *Weak compactness in* $L^1(\mu, X)$, Proc. AMS *113* (1991), 143–150.