DETERMINISTIC FRACTALS AND FRACTAL MEASURES (*)

by CHRISTOPH BANDT (**)

This is a working paper, a basis for discussion. I am most grateful to Prof. Aljoša Volčič for inviting me to the Summer School and for urging me to write down these notes. Fractals are exciting and modern for me mainly since they give rise to questions from most different mathematical fields – so they can motivate mathematicians from various areas to cooperate. I admit that my knowledge is quite incomplete, I do not even know how to define "fractals". Incomplete knowledge is a starting point for research. So beside the rather simple exercises given in the text, you will find questions of moderate difficulty and problems which seem to be unsolved. After the school at Grado, these notes were revised, and a few problems cancelled, or modified, as a result of discussions with Prof. Siegfried Graf, Prof. Gerald Goodman, and others. Nevertheless, enough problems are left.

In recent years, a number of textbooks on fractals have been published. Schroeders popular exposition [50] resembles the pioneering work of Mandelbrot [41] and is more concise. The books by Falconer [26, 27] provide a readable and rigorous introduction to the theory of Hausdorff measure and dimension; the new book also surveys almost all classes of fractals in the mathematical literature. The beautifully illustrated book by Barnsley [11] and the more mathematical introduction by Edgar [23] deal mainly with iterated function schemes and self–similarity. In these lectures I shall concentrate on questions which have not been completely worked out while topics treated in the books will be discussed rather briefly.

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(**) Address of the Author: Fachbereich Mathematik, Ernst–Moritz–Arndt Universität, Jahnstr. 15a, 0-2200 Greifswald (Germany).

1. Cantor Sets.

Cantor sets are the basic examples for self–similar structures. We study their different aspects: trees, totally disconnected sets, shift spaces. The theorems in this section are standard material, and they are basic for our course.

fig. 1

1.1. TREES, WORDS AND SEQUENCES.

Trees are often used to describe hierarchical structures in real life. Think of family trees, the classification of plants and animals, or the structure of an administration. Mathematically, a tree is a connected graph without cycles. Thus any two vertices in a tree are connected by a unique path. We consider *rooted infinite trees*. The choice of one special vertex λ, the root, defines an order among the vertices. For each vertex v, the neighbouring vertex which lies on the path from v to λ is called the *predecessor* of v, all other neighbours are called *successors* of v. We assume that each vertex has *at least two* successors. If each vertex has exactly m successors, we have the m–ary tree.

EXERCISE 1.1. Describe the structure of the tree in fig. 2. Is there any connection with the graph in fig. 3? (Hint: consider directed paths)

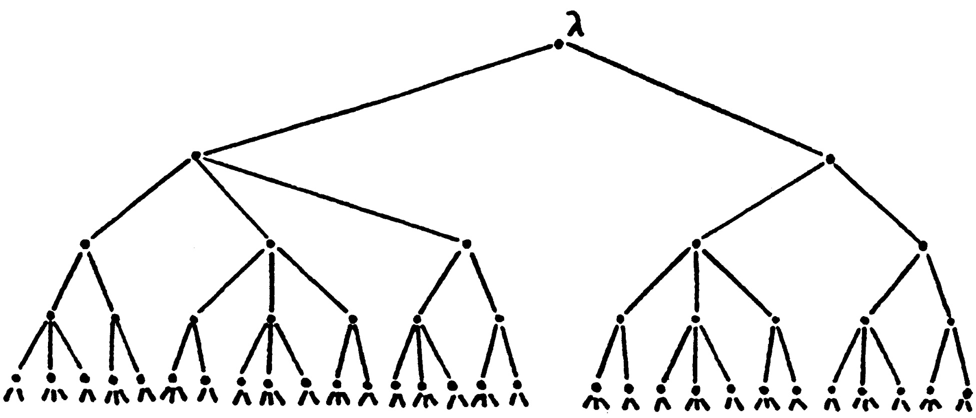


fig. 2

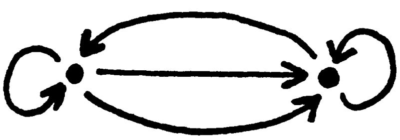


fig. 3

How can we describe the structure of a tree arithmetically? Beginning with λ, the edges from a vertex v to its successors are numbered $1, 2, \ldots$ and the endpoint of the edge i is called vi. In this way, *each vertex v is assigned a word $i_1 \ldots i_n$*. This word is obtained by writing the numbers of all edges in the path from λ to v. In the following, each vertex v will be identified with the corresponding word. Thus λ denotes the empty word, as usual. The vertices of the m–ary tree are all words from the alphabet $\{1, \ldots, m\}$. Note that any infinite path in a rooted tree corresponds to a *sequence* $i_1 i_2 \ldots$ of symbols from $N = \{1, 2, \ldots\}$

QUESTION 1.2. Suppose the m–ary tree is extended "backwards" beyond the root by defining predecessors of λ. How would you describe vertices and paths in such a "two–sided–infinite tree"?

When you think about this question, you will realize that two–sided–infinite sequences $\ldots j_2 j_1 j_0 i_1 i_2 \ldots$ do not describe one "two–sided tree" but a whole family of such trees. A more careful investigation of that family, and of the interplay of right–infinite and left–infinite one–sided sequences, should be helpful for the study of strange attractors and fractal tilings.

1.2. Geometric Realizations of Cantor Sets.

Throughout, we shall work in a complete metric space (X, d). For simplicity, you may assume we are in the Euclidean plane. By $|A| = \sup\{d(x, y) \mid x, y \in A\}$ we denote the diameter of a set A.

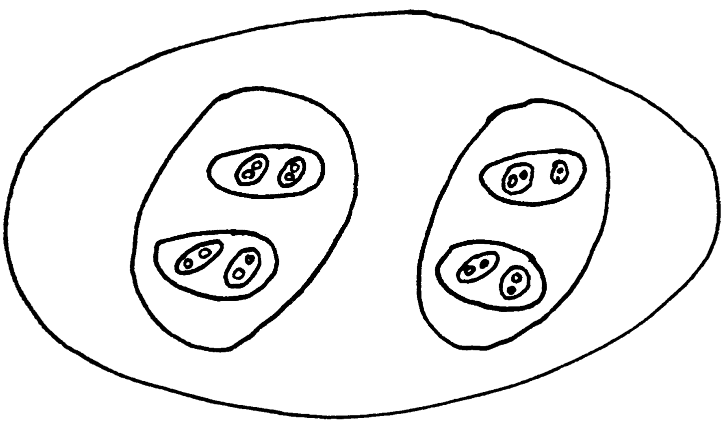


fig. 4

There are two methods using the structure of a rooted tree to generate figures in which the infinite paths correspond to points. The first method works with *downward filtering families of closed sets*. It is illustrated in fig. 4, the typical picture which most mathematicians will associate with the term "Cantor set".

One starts with a bounded set K_λ and chooses closed sets K_w for all words w in the tree, in such a way that $K_v \supset K_w$ whenever v is an initial word of w. The only condition we have to guarantee is that for each sequence $s = s_2, s_2 \ldots$, the diameter $|K_{s_1 \ldots s_n}|$ tends to zero with $n \to \infty$. Since the space is complete, $\bigcap_{n=1}^{\infty} K_{s_1 \ldots s_n}$ will then be a point which we denote by $p(s) = p(s_1 s_2 \ldots)$. The set of all these points is the desired figure.

To be more precise, the resulting figure is $A = \bigcap_{n=1}^{\infty} K^n$, where the sets

$$K^n = \bigcup \{K_w | w \text{ word of length } n\}$$

are called the n-th level approximations of A.

Fig. 1 shows Cantor sets obtained from a binary tree. In the classical example, the middle–third set, we start with $K_\lambda = [0,1]$, $K_0 = [0, \frac{1}{3}]$ and $K_1 = [\frac{2}{3}, 1]$. In each K_v, the left third is taken as K_{v0} and the right third as K_{v1}. It turns out that $K_{v_1 \ldots v_n} = [a, a + 3^{-n}]$ with $a = 2 \sum_{i=1}^{n} v_i 3^{-i}$.

EXERCISE 1.3. Modify this construction so that you obtain a Cantor set in [0,1] with positive measure. How to do this in plane?

If for different words, v, w of the same length, the sets K_v and K_w are disjoint, then p is a one–to–one function. (No two different sequences yield the same point $p(s)$.) In particular, since the set of sequences has cardinality of the continuum, the set of points has cardinality c too. This has the surprising consequence that in the middle–third set, practically all points are *not* endpoints of some interval K_w.

EXERCISE 1.4. Determine points in the middle third set which are not endpoints of some interval. (Hint: $\frac{1}{4}$)

REMARK 1.5. The middle–third construction can also be done "outwards" on the integers. Take a right–infinite sequence $\ldots s_2 s_1 s_0$. Let

$B_{s_0} = \{0\}$. If $s_0 = 0$, let $B_1 = \{2\}$. If $s_0 = 1$, let $B_0 = \{-2\}$. In both cases, put $B_{s_1,s_0} = B_0 \cup B_1$. Given $B = B_{s_n \dots s_0}$, let $B_{1 s_{n-1} \dots s_0} = B + 2 \cdot 3^n$ for $s_n = 0$ and $B_{0 s_{n-1} \dots s_0} = B - 2 \cdot 3^n$ in case that $s_n = 1$. In both cases, let $B_{s_{n+1} s_n \dots s_0} = B_{0 s_{n-1} \dots s_0} \cup B_{1 s_{n-1} \dots s_0}$. The limit set $B_s = \bigcup_{n=1}^{\infty} B_{s_n \dots s_0}$ is a pattern of integers. It is important to note that this pattern depends on the given sequence. In the case of "inside" constructions, each $s = s_1 s_2 \dots$ determines one point.

For outside constructions, each $s = \dots s_1 s_2$ determines a discrete pattern.

QUESTION 1.6. How do "outwards" constructions relate to the "two–sided tree"? For which $s = \dots s_2 s_1 s_0$ does B_s contain arbitrary large and arbitrary small integers?

The second and more general method for construction of figures from a tree structure uses *replacement* instead of downward filtering. The approximating sets K_w will usually be of one type: singletons, or intervals, or balls, for instance. There are certain rules according to which K_w is replaced by the union of the K_{wi} where wi is a successor of w in the given tree. We define the K^n as above, and $A = \lim_{n\to\infty} K^n$.

Convergence of sets in (X, d) is commonly expressed by the Hausdorff distance

$$d_H(B, C) = \max\{\sup_{b\in B} d(b, C),\ \sup_{c\in C} d(c, B)\}\,.$$

This is a metric on the space $\mathcal{B}$ of all non–empty closed and bounded subsets of X, and the space $(\mathcal{B}, d_H)$ is known to be complete ([39], ch. 3). It is an exercise to check that $d_H(\cup_{i\in S} B_i, \cup_{i\in S} C_i) \leq \sup_{i\in S} d_H(B_i, C_i)$ for any index set S.

THEOREM 1.7 (A general construction for "fractals"). *Suppose that for each vertex w of a rooted infinite tree we are given a closed and bounded set K_w, such that*

$$d_H(K_w, \bigcup\{K_{wi} | i \text{ edge starting at } w\}) \leq C r^{-|w|}$$

where C *and* r *are positive constants,* $r < 1$*, and* $|w|$ *denotes the length of the word* w*. Then the sets* K^n *converge to a closed and bounded limit set* A*.*

Proof. Applying the inequalities for d_H to the unions $K^n = \cup K_w$ and $K^{n+1} = \cup K_{wi}$, we get $d_H(K^n, K^{n+1}) \leq Cr^{-n}$ for each n. By triangle inequality, $d_H(K^n, K^m) \leq C(r^{-n} + \ldots + r^{-m}) \leq Cr^{-n}/(1-r)$ for $m > n$, and a Cauchy sequence in a complete metric space converges. ◇

EXERCISE 1.8. Does the theorem remain true if the estimate $Cr^{-|w|}$ is replaced by $C/|w|^2$? Show that for each sequence s describing a path in the given tree, $p(s) = \lim_{n\to\infty} K_{s_1 \ldots s_n}$ exists and is a single point.

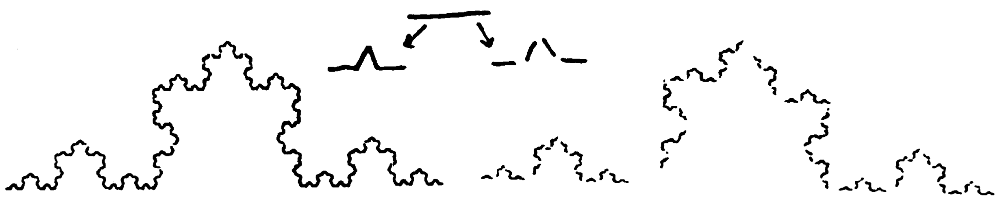


fig. 5

The classical construction is the Koch curve (fig. 5a). Mandelbrot [41] gives a lot of similar examples. For computer programming, the use of turtle graphics is convenient [23]. It should be pointed out that the Koch curve is not a Cantor set but a modification is (fig. 5b).

EXERCISE 1.9. Which pairs of sequences describe the same point in the Koch curve? Find a "downward" construction for the Koch curve. Can you modify the construction so that the resulting curve has positive measure (and no multiple points)? How should we define "outward constructions of patterns" corresponding to the Koch curve?

1.3. THE ABSTRACT CANTOR SET.

To give a rigorous definition of the Cantor set, let $S = \{1, \ldots, m\}$, $C = S^\infty$ the set of sequences $s = s_1 s_2 \ldots$ and $S^* = \cup_{k=0}^\infty S^k$ the set of words from S including the empty word λ. For $w \in S^*$, the length is denoted by $|w|$, and the concatenation with $s \in S^* \cup C$ is written ws. Further, $\bar{w}$ is the periodic sequence with period w, and the initial word of length k of s is denoted $s_{|k} = s_1 \ldots s_k$.

For $w \in S^k$, the set $C_w = \{s | s_{|k} = w\}$ is called the associated k–dimensional *cylinder*. Intuitively, cylinders are the sets drawn in fig. 4. Two cylinders can only intersect if one is contained in the other. So the set of all cylinders is a base for the open sets of a topology which is called the product topology on C. This topology is compact [34, 39]. It is important to note that all cylinders are open and closed, since the complement of a cylinder is a finite union of cylinders. For each r between 0 and 1, the following metric generates the product topology: $d_r(s,t) = r^{-k}$ if $s_1 = t_1, \ldots, s_k = t_k$ and $s_{k+1} \neq t_{k+1}$, $k = 0, 1, \ldots$, and $d_r(s,t) = 0$ for $s = t$.

From the topology or metric it is clear that the $C_{s_{|k}}$, $k = 1, 2, \ldots$ form a neighbourhood base for the point s. Thus a sequence $(t^j)_{j=1,2,\ldots}$ of points $t^j = t_1^j t_2^j \ldots$ converges to s iff for each coordinate i, there is $j(i)$ with $s_i = t_i^j$ for $j \geq j(i)$. We see that C has no isolated points.

EXERCISE 1.10. Show that for a downward filtering construction, the mapping $p : C \to A$ is continuous.

From the purely topological point of view, it turns out that the sets C are the same for all $m \geq 2$, and they coincide with many of their subsets.

THEOREM 1.11 (Uniqueness of Cantor set). *Any two compact spaces with a countable base of open–and–closed sets and without isolated points are homeomorphic.*

Proof. This classical theorem [34, 39] can be proved with downward filtering families. Suppose A has the given properties. Let $A = K_\lambda$ and construct disjoint open–and–closed K_w, $w \in \{0,1\}^n$, so that their union K^n is A, for $n = 1, 2$ and that $K_{wi} \subset K_w$. Using the compactness, one can choose these sets so that their diameter goes to 0 with $|w| \to \infty$. (If A is covered by m open–and–closed subsets of diameter $< \epsilon$, which we can assume to be disjoint, it is easy to obtain $|K_w| < \epsilon$ for $|w| = m$. Use induction for $\epsilon_k \to 0$). This way one constructs a continuous $p : \{0,1\}^\infty \to A$. Since p is one–to–one, it is a homeomorphism. ◇

The same method applies to arbitrary compact metric A when we require the K_w only to be closures of open sets and not necessarily disjoint. In this case, p need not be one–to–one:

THEOREM 1.12 (Universality of the Cantor set). *Each compact metric space A is a continous image of the Cantor set.* $\diamond$

EXERCISE 1.13. Construct a continuous map from C onto $[0,1]^2$.

Let us mention that for trees with an infinite number of branches, there are similar theorems on the space $B = N^\infty$, $N = \{1,2,\ldots\}$, with the above metric d_r, the Baire null space. "Compact" is then replaced by "completely metrizable".

Thus projections from the Cantor or Baire null space to other spaces are classical subjects of topology. "Fractals" will be those spaces for which the projection is defined in a particularly regular, recursive way.

1.4. FINITE MEASURES ON CANTOR SETS.

For our purposes, a measure μ on a metric space (X, d) is best defined as a *metric outer measure*: μ is defined for all sets, is non–negative, monotone, σ–subadditive, and $\mu(A \cup B) = \mu(A) + \mu(B)$ when A, B are closed sets with positive distance. It is known [26,27] that under these conditions, all Borel sets are μ–measurable.

Finite measures on C are particularly simple: they are determined by the values of cylinders, and additivity on cylinders is given by simple equations. In combinatorics, one would speak of a flow on the corresponding (directed) tree.

THEOREM 1.14 (Simplified measure theory on the Cantor set). *Let $\eta : S^* \to [0,\infty[$ fulfil $\eta(\omega) = \sum_{i\in S} \eta(wi)$ for each $w \in S^*$. Then there is a unique measure μ on $C = S^\infty$ with $\mu(C_w) = \eta(w)$.*

Proof. For the outer measure $\mu(D) = \inf\{\sum_{w\in W} \eta(w) \mid \bigcup_{w\in W} C_w \supset D\}$ we have $\mu(C_w) = \eta(w)$. (By compactness, we need only consider finite covers of a cylinder by cylinders, and these are essentially finite partitions.) Since any two disjoint closed sets in C have disjoint neighbourhoods which are finite unions of cylinders, μ is a metric outer measure by its definition. ◇

The simplest probability measures of this type are the *product measures*: choose $p_i \geq 0$ for $i \in S$ with $\sum_{i \in S} p_i = 1$, and let $\eta(w_1 \dots w_n) = p_{w_1} \cdot \dots \cdot p_{w_n}$. Intuitively, in the m–ary tree, the mass $\eta(w)$ coming into a vertex w divides into masses $p_i \eta(w)$, $i = 1, \dots, m$, which flow along the outgoing edges to the successors wi. You certainly know these measures from probability theory: they model a sequence of independent experiments with m outcomes.

The next examples come from Markov chains. Let p_i as above, and let $P = (p_{ij})_{i,j \in S}$ be a matrix with positive entries and $\Sigma_j p_{ij} = 1$ for every i. Then let $\eta(w_1, \dots, w_n) = p_{w_1} \cdot p_{w_1 w_2} \cdot \dots \cdot p_{w_{n-1} w_n}$ and μ the generated measure.

EXERCISE 1.15. If $p_i > 0$ for all i, the support of this measure is the so called topological Markov chain or *Markov subshift* corresponding to P:

$$S_P^\infty = \{s = s_1 s_2 \dots \in S^\infty | p_{s_i, s_{i+1}} \neq 0 \text{ for } i = 1, 2, \dots\} .$$

The name "shift space" for $C = S^\infty$ and "subshift" for S_P^∞ comes from the important "shift map" $\sigma : C \to C$ with $\sigma(s_1 s_2 s_3 \dots) = s_2 s_3 \dots$ This map is m–to–one and has m inverse mappings $\tau_i(s_1 s_2 \dots) = i s_1 s_2 \dots$ A subshift is a closed subset D of C with $\sigma(D) = D$. A measure μ is shift–invariant if $\mu(\sigma^{-1}(F)) = \Sigma_i \mu(\tau_i(F)) = \mu(F)$ for every $F \subset C$. Clearly, all product measures are shift–invariant.

EXERCISE 1.16. For which p_i and p_{ij} is a Markov measure shift–invariant?

In the next chapters, we shall often use *images of measures*. If $f : X \to Y$ is a continuous map between metric spaces and μ is a measure on X, the image $\nu = f(\mu)$ is defined by $\nu(F) = \mu(f^{-1}(F))$.

QUESTION 1.17. Give examples of measures in R and R^2 which are, or are not, images of product measures.

2. Iterated Function Schemes.

We consider fractals as continuous images of the Cantor set, where the projection is defined in a recursive way. It is very natural to use iteration of functions for such definitions. There are several classes of functions which generate fractals. The random algorithm will lead us to fractal measures.

2.1. Transformations on Metric Spaces.

A transformation $f : X \to X$ from a metric spaces (X, d) into itself is called a

$$\textit{Lipschitz map} \text{ if } d(f(x), f(y)) \leq L \cdot d(x, y) \text{ for some } L > 0$$

$$\text{and all } x, y \in X ,$$

$$\textit{non-expanding} \text{ if it is Lipschitz with } L \leq 1 ,$$

$$\textit{contraction} \text{ if it is Lipschitz with } L < 1 ,$$

$$\textit{similarity} \text{ with factor } L \text{ if } d(f(x), f(y)) = L \cdot d(x, y), \text{ and}$$

$$\textit{isometry} \text{ if } d(f(x), f(y)) = d(x, y) .$$

THEOREM 2.1 (The fixed point theorem for contractions). *A contraction f on a complete metric space has a unique fixed point.*

Proof. If two points x, y were fixed, then $d(f(x), f(y)) = d(x, y)$ and f would not be a contraction. Now take $x_0 \in X$ and let $x_{k+1} = f(x_k)$ for $k = 0, 1, 2, \ldots$ Writing f^k for k–fold composition of f with itself, we get

$$d(x_{k+1}, x_k) = d(f^k(x_1),\ f^k(x_0)) \leq L^n d(x_1, x_0) \text{ and}$$

$$d(x_{k+m+1}, x_k) \leq d(x_{k+m+1}, x_{k+m}) + \ldots + d(x_{k+1}, x_k) \leq$$

$$(L^{m+k} + \ldots + L^k) d(x_1, x_0) < \frac{L^k}{1-L} d(x_1, x_0)$$

which tends to zero for each m and $k \to \infty$. The Cauchy sequence (x_k) converges to a limit x^*, and clearly $f(x^*) = x^*$. ◇

EXERCISE 2.2. Isometries, similarities and bi–Lipschitz mappings (one–to–one Lipschitz maps f for which f^{-1} is also Lipschitz) from (X,d) onto itself form a group under composition. Each similarity which is not an isometry has a fixed point. Find two contractions f,g such that fg is not a contraction.

EXERCISE 2.3. Are the maps σ and τ_i on S^∞ Lipschitz or even similarity maps?

Now let us consider Euclidean R^n. The similarities are of the form $f(x) = rMx + b$, where M is an orthogonal $n \times n$–matrix, r the positive factor and b a translation vector. In R^2 we can also use complex numbers: $f(z) = re^{i\varphi}z + b$ describes a rotation by the angle φ together with a homothety by the factor r (with the same center, obtained from solving $f(c) = c$) or a translation (in case $\varphi = 0$), $f(z) = re^{i\varphi}\bar{z} + b$ is a reflection or glide reflection together with a homothety.

There are two possible generalizations: the affine mappings $f(x) = Mx+b$ and the conformal maps. Still larger classes are the quasiconformal and differentiable mappings, but we shall restrict ourselves to similarities, a few affine maps, and some quadratic maps $f(z) = z^2 + c$ in connection with Julia sets.

A non–empty, closed and bounded set A in a complete metric space (X,d) is called *self–similar set* with respect to given contractions $f_1,\ldots,f_m$ if the following equation holds.

$$A = f_1(A) \cup \ldots \cup f_m(A)$$

The important thing is that if we substitute $\cup f_j(A)$ for A on the right–hand side, we obtain $A = \cup f_i f_j(A)$. In this way, A divides into smaller and smaller copies of itself: $A = \cup\{A_w | |w| = n\}$ with $A_w = f_{w_1}\ldots f_{w_n}(A)$ for $w = w_1 \ldots w_n$.

EXERCISE 2.4. Which of the Cantor sets in fig. 1 are self–similar? By inspection, determine the mappings f_i for fig. 5 and 6. Note that for the interval, the f_i can be chosen in various ways.

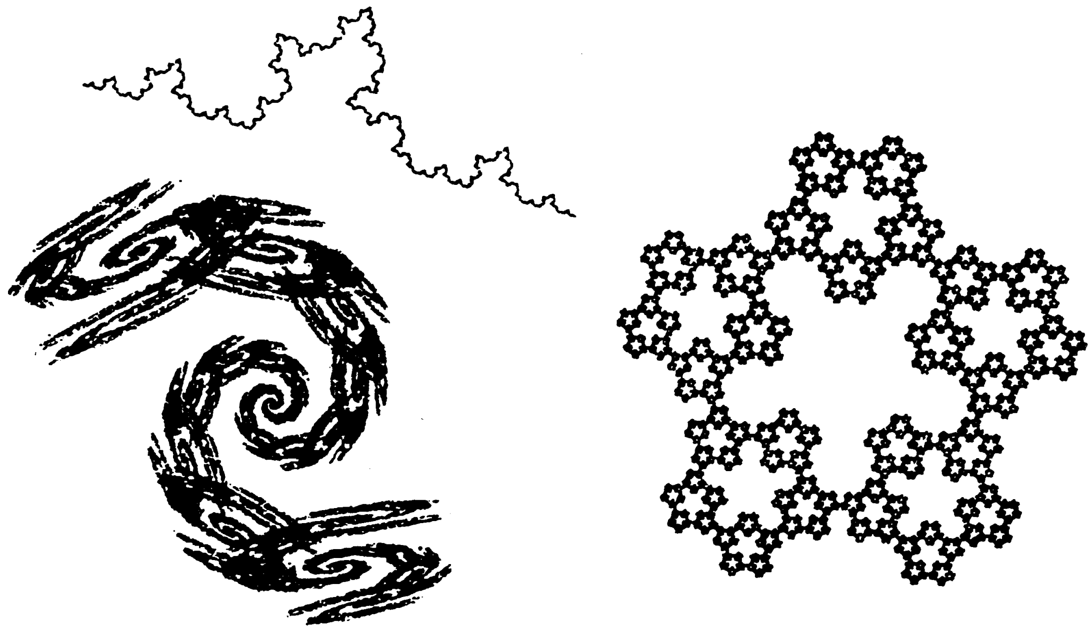


fig. 6

2.2. Self–Similar Sets.

Looking at the simple proof, one is surprised that the following theorem was proved only in 1980 by Hutchinson [35, 26, 27, 23, 11].

Theorem 2.5 (Existence and uniqueness of self–similar sets). *Given contractions $f_1, \ldots, f_m$ on a complete metric space (X, d) there exists a unique self–similar set A.*

Proof. The mapping $F : \mathcal{B} \to \mathcal{B}$ with $F(B) = \cup_{i \in S} f_1(B)$ is a contraction on the metric space $(\mathcal{B}, d_H)$ with factor $r = \max r_1$. To see this, we need only the inequality for Hausdorff distance of unions, and the estimate $d_H(f_i(B), f_i(C)) \leq r_i \cdot d_H(B_i, C_i)$, where $r_i < 1$ denotes the contraction factor of f_i. The contraction F must have a unique fixed point A. ◇

Remarks 2.6 a) The fixed point theorem implies that the self–similar set A can be approximated by successively applying F to an arbitrary closed and bounded initial set B_0. (Or K_λ. We can apply theorem 1.7 with $K_{wi} = f_i(K_w)$ since we just verified its assumption!) For computer constructions, one can choose B_0 as a large ball so that $f_i(B_0) \subset B_0$ for all i. In this case $B_1 = F(B_0), B_2 = F(B_1) \ldots$ is a downward filtering family as considered in sec. 1. (To convince yourself, draw B_2 for the middle third set!) If the f_i are similarities, only disks have to be drawn. However, the convergence is usually too slow to show fine details.

b) For this choice of B_0 the method shows that $A = p(C)$ with $p(i_1 i_2 \ldots) = \bigcap_{n=1}^{\infty} f_{i_1} \cdot \ldots \cdot f_{i_n}(B_0)$. Note that $f_i p = pr_i$. Incidentally, the mappings $g_n = f_{i_1} \cdot \ldots \cdot f_{i_n}$ converge uniformly on bounded sets to the constant map with value $p(i_1 i_2 \ldots)$. Thus one can consider the self–similar set as a limit set of a semigroup in the space of all similarities or contractions – this has not been studied so far, cf. [22].

c) An alternative construction starts with a singleton. The best is to choose it from A. For instance, let $B_0 = \{a\}$ be the fixed point of f_1. In this case $B_n = F^n(B_0)$ is a subset of A with m^n points which can be drawn quickly for small n. More precisely, we draw one point $f_w(a) = f_{w_1} \ldots f_{w_n}(a)$ from each $A_w = f_w(A) = p(C_w)$, $|w| = n$.

d) If the r_i are different, it is useful to estimate diameters; $|A_w| \le r_w \cdot |A|$ with $r_w = r_{w_1} \ldots r_{w_n}$, and stop drawing points when the diameter is small enough. Instead of drawing $f_w(a)$ with $|w| = n$, we better draw $f_w(a)$ with $r_w \ge \epsilon$.

The following statement was used by Barnsley and others [11] for finding equations of self–similar sets which approximate given real life pictures. The aim is data compression in image processing, and Barnsley's method was to try certain "collages" of a figure by similar copies of itself.

PROPOSITION 2.7 ("Collage theorem"). *If* $d_H(F(B), B) \le \epsilon$ *then for the self–similar set* $A = F(A)$ *holds* $d_H(A, B) \le \frac{1}{1-r}\epsilon$.

Proof. In the proof of the fixed point theorem, set $d(x_1, x_0) \le \epsilon$, $k = 0$ and $m \to \infty$. ◇

EXERCISE 2.8. As S. Graf told me in Grado, the last proof can be simplified, using only the triangle inequality.

EXERCISE 2.9. Discuss the existence and uniqueness of a solution of

$$A = f_1(A) \cup \ldots \cup f_m(A) \cup K\ ,$$

where the f_i are given contractions and K is a fixed compact set. Derive a formula for A using iteration of F with $B_0 = \emptyset$.

QUESTION 2.10. Discuss the existence and uniqueness of the solution of a system

$$C_i = \bigcup_{j=1}^{m} f_{ij}(C_j) \cup K_i,\ i = 1,\ldots,m$$

where the f_{ij} are given contractions or "empty maps", and K_i are (possibly empty) compact sets. (Hint: use the metric $d^*(B_1,\ldots,B_m), (C_1,\ldots,C_m)) = \max_{i=1}^{m} d_H(B_i,C_i)$ on the space $\mathcal{B}^m$.) Formulate and prove the collage theorem for this case. (Many authors independently studied such systems of equations, cf. [43, 14, 13, 2].)

2.3. THE RANDOM ALGORITHM.

Instead of drawing the points $f_w(a), w \in S^*$ systematically according to the order of words, one can also use a random algorithm: let $x_0 = a \in A$, and let $x_{k+1} = f_i(x_k)$, where i is randomly chosen from $S = \{1,\ldots,m\}$. The simplest choice is to select each i with the same probability $p_i = \frac{1}{m}$. This algorithm is absolutely easy to program on a computer, using a random number generator. It was studied and propagated by Barnsley and his co–workers [12, 11 13, 14]. They call the family $(f_1,\ldots,f_m,p_1,\ldots,p_m)$ an *iterated function system*. We prefer Falconer's modification "iterated function scheme" with the same abbreviation, IFS. Romanian authors had studied such processes without computer in the sixties, with applications to models of learning behaviour. A one–dimensional IFS was first investigated in this context by Karlin in 1953 (see the references by Goodman [29] and Elton and Piccioni [25]).

Such random programs construct a self–similar set rather fast, drawing points of all pieces at the same time. The contours of the set are seen from the very beginning, and details come with time. In contrast, deterministic algorithms work locally, drawing full detail at a specific place.

At our school in Grado, G. Goodman pointed out the probabilistic shortcomings of the random algorithm for the case of the Sierpinski gasket (cf. sec. 5). For this particular case, he suggested a deterministic method which performs much better [29]. For cases with different r_i, or p_i, this seems very difficult, however. I think that IFS are important, not only as fast algorithms, but also as a starting point for studies and experiments concerning fractal measures.

Let us analyse whether the random method eventually draws points from all pieces A_w (cf [29]). For fixed $w = w_1 \dots w_n$ and k, the probability that $x_{k+1} = f_{w_n}(x_k)$, $x_{k+2} = f_{w_{n-1}}(x_{k+1}), \dots, x_{k+n} = f_{w_1}(x_{k+n-1})$ is $q_w := p_{w_1} \cdot \ldots \cdot p_{w_n}$. Thus the probability that all x_{k+m} for $k = 1, \dots, k_0$ are *not* in A_w is $(1 - q_w)^{k_0}$. In probabilistic terms: the first arrival time of our random point at A_w follows a geometric distribution. (For a more careful analysis, note that $x_k \dots x_{k+n}$ and $x_l \dots x_{l+n}$ with $k < l$ are independent only if $k + n < l$.)

EXERCISE 2.11. To construct a self-similar set with three pieces on level 8 by the deterministic method, $3^8 = 6561$ points have to be drawn. How many points are needed with the random method, $p_i = \frac{1}{3}$, when we want to have 99 per cent probability for each A_w with $|w| = 8$ to be represented in the picture?

As usual with random methods, the practical performance is much better than the "worst-case analysis". There is one particular advantage if the r_i are different, so that the A_w have unequal size for words of the same length. One can adapt the p_i so that pieces with larger size will be hit by more points. To show this effect, take a right-angled triangle T with angles of $30°$ and $60°$ and with area 1. The altitude divides T into two triangles T_1 and T_2 of area $\frac{1}{4}$ and $\frac{3}{4}$ which are similar to T.

EXERCISE 2.12. Find the similarities which make T a self-similar set.

Fig. 7 shows the unexpected result of the random algorithm with $p_1 = p_2 = \frac{1}{2}$. Half of the points fall into the small triangle T_1, the other half into the large T_2. This unequal distribution extends to the subpieces. An easy calculation shows that T_{1111} accounts for 0.4 per cent of the area and is hit by $\frac{1}{16}$ of the points. $T_{22222222222222222222}$ has the same area and will only get two points from a million! That is why some pieces of the triangle remain white.

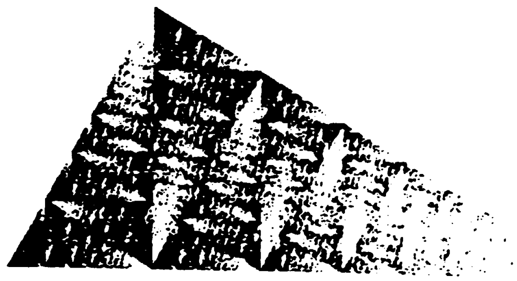


fig. 7

In the present example, one has only to choose $p_1 = \frac{1}{4}$ in order to get a completely uniform distribution of points in the triangle. If the self–similar set has positive area, then it is natural to choose p_i according to the area of A_i. In general, one determines the unique number α for which

$$r_1^\alpha + \ldots + r_m^\alpha = 1$$

which is called the *similarity dimension.* Then one puts $p_i = r_i^\alpha$. In sec. 3, we justify this method for the case that the A_i "do not overlap too much".

EXERCISE 2.13. Show that $\alpha = 2$ if A is positive area and the sets $A_i \cap A_j$ with $i \neq j$ have area zero.

PROBLEM 2.14. Prove that a self–similar set in the plane with positive area must have non–empty interior (cf. [3, 5, 49]). It would also be nice to classify the self–similar sets A with open interior and with a few pieces [3]. If A has interior points, one can show that copies of A will tile the plane. Is it true that such tilings must be periodic?

2.4. SELF–SIMILAR MEASURES.

The random algorithm shows how mass can be distributed unevenly over an area. Such experiments as we described give concrete material for measure theory: one does not only theoretically know about the existence of singular measures, but one can see them and work with them.

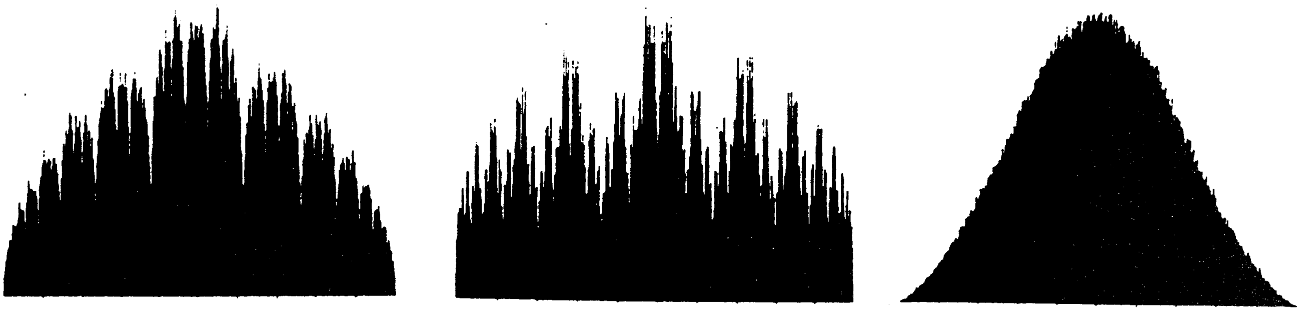
Let μ, λ be measures on X. The support of μ is the set of all points which have no neighbourhood with measure zero. We call μ absolutely continuous with respect to λ if for each ϵ there is δ such that $\lambda(B) < \delta$ implies $\mu(B) < \epsilon$. Moreover, μ and λ are orthogonal (or singular) if there are disjoint sets B, C with $\mu(X\backslash B) = \lambda(X\backslash C) = 0$.

EXERCISE 2.15. Show that the measure of fig. 7 has support T and is orthogonal to Lebesgue measure.

In order to avoid convergence discussions, let us just state here that the iterated function system $(f_1, \ldots, f_m, p_1, \ldots, p_m)$ generates the measure $\nu = p(\mu)$ on A, where μ is the product measure on C given by the p_i, and $p : C \to A$ is the projection onto the self–similar set. (The exact statement reads as follows: for each x_0 and with probability 1, the discrete measures $\frac{1}{n+1}\sum_{k=0}^{n}\delta_x$ converge weakly to μ [24, 11]. Goodman [29] stresses the fact that weak convergence, shown by means of the ergodic theorem, does not properly describe what happens on a computer screen. More definite estimates of convergence, perhaps in terms of the metric ρ defined below, are highly desirable.) It is known that different choices of the p_i yield orthogonal product measures on C. So if the A_i do not overlap too much (that is $p^{-1}(A_i \cap A_j)$, $i \neq j$ has measure zero with respect to all product measures) then the measures ν are also mutually orthogonal.

A simple example was studied by the young Erdös in 1939. $f_1(x) = rx$ and $f_2(x) = rx + (1 - r)$ generate a self–similar Cantor set in R for $0 < r < \frac{1}{2}$, and the interval $A = [0, 1]$ for $\frac{1}{2} \leq r < 1$. Let $p_1 = p_2 = \frac{1}{2}$. The measure ν is clearly singular (with respect to Lebesgue measure λ) for $r < \frac{1}{2}$, and $\nu = \lambda$ for $r = \frac{1}{2}$.

PROBLEM 2.16. For which $r > \frac{1}{2}$ is ν singular?



$r = (\sqrt{5}-1)/2$ $r = 9/16$ $r = 0.7549$

fig. 8

This problem is still unsolved [28]. Erdös proved that ν is absolutely continuous, with k times differentiable density, for almost all points r in a neighbourhood of 1 (this was already known for $r = 1/\sqrt[n]{2}$, $n \geq 2$), and ν is singular for numbers like $(\sqrt{5}-1)/2$, the golden mean. Fig. 8 shows "empirical densities" for ν obtained by the random algorithm.

It is possible to transmit the basic statements on self–similar sets and their proofs to measures. This was already done by Hutchinson [35, 26]. A finite measure ν is called self–similar with respect to (f_i, p_i), $\Sigma p_i = 1$, if

$$\nu = p_1 f_1(\nu) + \ldots + p_m f_m(\nu) \ .$$

To show existence and uniqueness, one has to introduce a suitable metric on the space of probability measures on R^n [35,11].

$$\rho(\mu,\nu) = \sup\{|\mu(h) - \nu(h)| \, |h : R^n \to R \text{ is non-expanding}\}$$

where $\mu(h) = \int_{R^n} h d\mu$. Then one has to show that this metric generates the weak topology, at least when we restrict ourselves to probability measures with support in a compact set (this was mentioned but not proved in [35, 11], for a proof compare [46], sec. 1). This proves that we have a complete space and can apply the fixed point theorem. It is not hard to see that $F(\mu) = \Sigma p_i f_i(\mu)$ is a contraction on $(\mathcal{M}, \rho)$ [35, 11]. Finally, one has to show that one can choose a large compact set D such that a self–similar measure must have its support within D.

EXERCISE 2.17. Work out some steps of this proof.

QUESTION 2.18. What about solutions of the inhomogeneous equations

$$\nu = p_1 f_1(\nu) + \ldots + p_m f_m(\nu) + \kappa \, .$$

where $\Sigma p_i \leq 1$ and κ is a given measure?

QUESTION 2.19. What about systems of equations of measures? There are some papers [14, 11, 43, 2] but something is left to be written down. Which random algorithms generate which solutions of such systems?

Finally, let us mention that the "outward" constructions mentioned in sec. 1 have been neglected by mathematicians, while they are preferred by the physicists. They can be described in terms of the f_i^{-1} in a similar way as the self–similar sets A are described by the f_i. A special case are quasiperiodic (Penrose) tilings. I think it is really worthwhile to study outward constructions systematically. Recent papers by D. Sullivan, his student Y. Jiang and Bedford and Fisher [15] indicate that they have theoretical importance, too.

3. Invariant Measures and Hausdorff Measures.

Invariant measures play such an important role that it is impossible to survey them on a few pages. We concentrate on metric invariance concepts and Hausdorff measures. In contrast to these "uniformly distributed" measures, the notion of multifractal is mentioned.

3.1. INVARIANCE PROPERTIES FOR MEASURES.

A measure μ on X is invariant with respect to some mapping $f : X \to X$ if $f(\mu) = \mu$, i.e. $\mu(f^{-1}(F)) = \mu(F)$ for all sets F. A central question of ergodic theory is the existence and uniqueness of locally finite measures which are invariant under a group of transformations. The best–known case is Haar measure on a locally compact group, which is unique up to a constant. Existence is also known for compact spaces X where we can apply the Schauder fixed theorem to the set of probability measures. Uniqueness does not hold, as the example of the shift on C shows (sec.

1). Thus one tries to get uniqueness by additional conditions, as maximal entropy.

Here we are more interested in geometric aspects. We search for a "*natural volume function*" on a metric space. A natural condition is that "*congruent sets have equal measure*".

Mycielski [44] proved the existence of measures on locally compact spaces which assume equal values on open isometric sets. They are not unique. Let us say μ is *metrically invariant* if any two isometric sets have the same measure. Haar measure on a locally compact group with translation–invariant metric has this property [4] and is unique. However, the space $Y = [0,1] \cup [0,\frac{1}{2}]^2 \cup \ldots \cup [0,\frac{1}{n}]^n \cup \ldots$ with Euclidean metric does not possess a metrically invariant measure (why?) and can be made compact by addition of one point. Moreover, the half–parabola $\{(t,t^2)\,|t \geq 0\}$ does not possess any two isometric sets with more than four points. Any non–atomic measure will be invariant in this case. Thus a certain degree of homogeneity must be required from the space if it should admit a metrically invariant measure, and quite strong homogeneity is needed for uniqueness.

Kolmogorov [38] introduced the notion of *strong invariance*: $\mu(F) \leq \mu(D)$ whenever there exists a non–expanding map $f : D \to F$. This characterizes the volume function at least for the parabola, and certainly also for manifolds.

PROBLEM 3.1. Is it true that on a differentiable manifold (embedded in R^n or Riemannian) the volume function is the only strongly invariant measure? How is the situation for self–similar sets?

Another approach is to look for measures which assign the same measure to balls of the same size [4], or at least in the limit $\epsilon \to 0$. Such requirements yield uniqueness (cf. Mattila [42]) but they are not fulfilled for fractals (see Falconer [26], Preiss [46]).

A third way to define natural volume functions are the *equidistributions* studied by Dembski and Graf [30]. A finite set $F \subset X$ is ϵ–separated in (X,d) if $d(x,y) > \epsilon$ for all $x \neq y$ in F. The maximum cardinality of an ϵ–separated set in X is the *packing number* $N(X,\epsilon)$. A probability measure μ is called an equidistribution on X if there exists a sequence (ϵ_k) which tends to zero and a corresponding sequence of maximal ϵ_k–

separated sets F_k, card $F_k = N(X, \epsilon_k)$ such that the discrete measures $\mu_k = \frac{1}{N(X,\epsilon_k)} \sum_{x \in F_k} \delta_x$ converge weakly to the measure μ. Graf has shown that on self–similar sets "without overlap" there is a unique equidistribution [30].

We consider now the classical construction of metrically invariant measures.

3.2. HAUSDORFF MEASURES.

Beside the packing number, there is also the *covering number* $M(B, \epsilon)$, the minimum cardinality of sets of diameter $\leq \epsilon$ needed to cover the set B.

EXERCISE 3.2. Prove that $N(B, \epsilon) \leq M(B, \epsilon) \leq N(B, \frac{\epsilon}{2})$ for any metric space.

Packing and covering numbers give some impression of the size of X, but for a more accurate measurement one has to use coverings by sets of varying size. In the following, let $h : [0, \infty[\to [0, \infty[$ be strictly monotonous, continuous and $h(0) = 0$. Later, we shall mainly use the functions $h(t) = t^\alpha$. We can roughly estimate the size of a set B by $h(|B|)$. Next, we can use this estimate for a better estimate:

$$M(B, h, t) = \inf \{ \sum_{k=1}^{\infty} h(|F_k|) | B \subseteq \bigcup_{k=1}^{\infty} F_k \ , \ |F_k| \leq t \} .$$

Finally, one can the limit for finer and finer coverings which exists (but may be infinite) since $M(B, h, t)$ increases when t goes down. This limit is called the *h–dimensional Hausdorff measure of B.*

$$\mu^h(B) = \lim_{t \to 0} M(B, h, t)$$

For $h(t) = t^\alpha$ we say α–dimensional Hausdorff measure and write μ^α. Note that μ^1 is the length measure in R or on a curve in R^n, μ^2 is (up to a factor) the area measure in R^2 or on a surface in R^n etc. Thus Hausdorff abstractly defined the measures which are used in analysis.

At first sight, the definition looks somewhat complicated but it is really the simplest and most elegant way to measure exactly the size of sets in metric spaces. It is very easy to show that μ^h is a metric outer measure [48, 26, 23]. There are sets for which the definition of μ^h is very simple.

EXERCISE 3.3. Prove that if α is the similarity dimension of a self–similar set A, then $\mu^\alpha(B) = \inf\{\Sigma|F_k|^\alpha | B \subset \cup F_k\}$ for $B \subseteq A$ [5]. Hint: "$\geq$" is obvious. Consider $B = A$, then work with the complement.

PROBLEM 3.4. Are there other examples for which the Hausdorff measure can be simplified this way? Perhaps Julia sets of conformal maps?

REMARK 3.5. Any Hausdorff measure is obviously metrically invariant and even strongly invariant. The covering sets F_k for B can be taken as subsets of B, so they will be taken to the image under an isometry or a non–expanding map. If f is a Lipschitz map with factor L, then $h(|f(F)|) \leq h(L \cdot |F|)$. Thus for $h(t) = t^\alpha$ we get $\mu^\alpha(h(B)) \leq L^\alpha \mu^\alpha(B)$ In particular, this implies the following invariance property of α–dimensional Hausdorff measures: if f is a similarity with factor r, then $\mu^\alpha(h(B)) = r^\alpha \mu^\alpha(B)$ for every B.

EXERCISE 3.6. Prove that these statements are also true for $h(t) = t^\alpha g(t)$, where $\lim_{t\to 0} \frac{g(2t)}{g(t)} = 1$.

Thus Hausdorff measures fit the metric structure very well. The difficulty is that for a specific set B almost all of them are either zero or infinite. One has to care to select the proper h so that $0 < \mu^h(B) < \infty$. This is the problem of dimension.

For self–similar sets with disjoint pieces, it is not hard to find the right α, using the above property. Since

$$\mu^\alpha(A) = \sum_i \mu^\alpha(f_i(A)) = \sum_i r_i^\alpha \mu^\alpha(A) ,$$

$\mu^\alpha(A)$ is positive and finite iff α is the similarity dimension.

This also holds for "just–touching" [11] self–similar sets defined by the *open set condition*: there exist an open set U with $f_i(U) \subseteq U$ and $f_i(U) \cap f_j(U) = \emptyset$ for all $i, j \in S$, $i \neq j$. The proof goes back to Moran 1946, see [35, 26, 27, 23].

An algebraic condition for a self–similar set A to have positive finite μ^α–measure was recently given by Bandt and Graf [5]. Schief [49] has shown that this condition is in fact equivalent to the open set condition. Roughly, the idea of [5, 49] is that the open set is the fundamental domain of a certain group determined by the f_i.

PROBLEM 3.7. What can be said about the "natural measure" of a self–similar set when the open set condition is not fulfilled?

3.3. PACKING MEASURES AND SRB–MEASURES.

Although Hausdorff measures work very well, there is a "dual" construction which is sometimes more appropriate (cf. [23, 27]). Instead of measuring B by covers with balls or sets of certain diameter, one can try to fill B from inside with disjoint balls of different radii r_k. This will only work if B has open interior, still for other sets one can work with the centers x_k of the balls. Two balls $B(x_1, r_1)$, $B(x_2, r_2)$ are disjoint iff $d(x_1, x_2) \geq r_1 + r_2$. Thus let

$$N(B,h,t) = \sup\{\sum_{k=1}^{\infty} h(2r_k) | x_k \in B,\ r_k \leq t,\ d(x_k, x_j) \geq r_k + r_j$$

$$\text{for } j, k \in N, j \neq k\}$$

This time $N(B,h,t)$ decreases with t, but the limit $\nu_0^h(B) = \lim_{t\to 0} N(B,h,t)$ is not yet a measure since it assumes positive values on countable dense sets. We have to apply outer measure once again.

$$\nu^h(B) = \inf\{\sum_k \nu_0^h(F_k) | B \subseteq \bigcap_k F_k\}$$

is called the *h–dimensional packing measure* of B. It is a metric outer measure, and it has the same invariance properties as the Hausdorff measure, as one can easily check.

These invariance properties imply that on a self–similar set with open set condition, packing measure and Hausdorff measure of the similarity dimension α coincide up to a constant factor. In other cases, there is considerable difference. For limit sets of certain Kleinian groups Sullivan [51] defined a "natural volume function" which is a Hausdorff measure in some cases and a packing measure in others. The same phenomenon for certain Julia sets was discovered by Denker and Urbanski [20]. It would be nice to know a principle which unifies both constructions and/or clearly points out their differences.

Let us at least mention one method which plays an important role in constructing "natural" invariant measures on strange attractors and Julia sets. Coming from statistical mechanics, Sinai, Ruelle and Bowen about 1972 constructed measures on Cantor product sets which they called Gibbs measures but which today are called SRB–measures. One formulation of the main theorem for $C = S^\infty$ reads as follows [16]:

THEOREM 3.8 (Existence and uniqueness of SRB–measures). *Let $g : C \to R$ be a function which is Lipschitz with respect to some metric d_r (cf. sec. 1). Then there is a unique shift–invariant measure μ on C and a unique constant P and constants $c_1, c_2 > 0$ such that*

$$c_1 \le \frac{\mu(C_w)}{\exp(-Pn + S_n g(x))} \le c_2 ,$$

for every $w \in S^n$ and each $x \in C_w$. Here $S_n g(x) = \sum_{k=0}^{n-1} g(\sigma^k(x))$.

This theorem will be applied to conformal maps φ which are conjugate to the shift σ by some projection $p : C \to A$. (Most authors consider two–sided Markov shifts, but this is not essential.) The map g will be $-\beta \cdot \log|\varphi'|$, and β can be chosen so that $P = 0$. In that case, $\mu(C_w)$ coincides with $|A_w|^\beta$ up to a constant, and the β–dimensional Hausdorff measure is positive and finite. The proofs are quite involved, and it is a challenging task to simplify them.

3.4. DENSITIES AND TANGENT MEASURES.

If the h–dimensional Hausdorff or packing measure is positive and finite on B, we can study densities. The limsup and liminf of $q(t) = \mu B(x,t)/h(2t)$ for $t \to 0$ are called upper and lower density of μ which denotes μ^h or ν^h restricted to B. For smooth manifolds, they coincide for almost every x, but on fractals a limit will exist almost nowhere. For self–similar sets it can at least be shown that the upper density and lower density are constant almost everywhere. Bedford and Fisher have studied $q(t)$ on self–similar sets and found that $q(\ln t)$ is rather periodic. Thus applying a Cesaro mean should give a limit:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mu B(x, e^{-t})}{h(2e^{-t})}\, dt$$

which is called second–order or Cesaro density. These densities are currently studied.

Another topic which I consider still more important for self–similarity are the *tangent measures* of a measure μ in a point x which were introduced by Preiss [46]. These are locally finite measures which are limits in the vague topology of the form $\lim_{k\to\infty} c_k \mu_{x,r_k}$, where $c_k > 0$, $r_k \to 0$ and $\mu_{x,r_k}(B) = \mu(x + rB)$ is the image of μ when the space is blown up with factor $1/r$ and center x. Preiss considers measures μ which have Lebesgue measures on linear subspaces as tangent measures and turn out to be rectifiable. However, it seems very interesting to study the tangent measures of self–similar measures. Even for simple examples, this has not been done. In the context of sets, Bedford and Fisher [15] have studied the limit models obtained by blowing up a self–similar set at a point.

4. Fractal Dimensions.

So much has been written on fractal dimension that I shall try to keep this chapter short. We explain some general principles and hint to the literature on multifractals.

4.1. Power Laws.

If x and y denote positive real variables and y is a function of x, we say that a *power law with exponent* β is fulfilled if there are constants $c_1, c_2 > 0$ with

$$c_1 \cdot x^\beta \le y(x) \le c_2 \cdot x^\beta \text{ for all } x\,.$$

When dealing with fractals, one always assumes power laws (cf. Schroeder [50]), and the exponents are often called dimensions even if the do not correspond to an intuitive notion of dimension. Usually one has to do with small quantities, so that the exponent is obtained from

$$\beta = \lim_{x \to 0} \frac{\ln y(x)}{\ln x}$$

but in some cases also limits for $x \to \infty$ are used.

In physical or numerical experiments, where one has only a finite number of values (x, y) one plots the points in a logarithmic coordinate system since $y = cx^{\beta}$ is the same as $\ln y = \ln c + \beta \cdot \ln x$. The assumption of a power law means that all values gather in a small strip around a line. If the limsup and liminf of $\ln y / \ln x$ are different, the values are rather in a cone (fig. 9). In this case we have only an upper and lower dimension. Do not be too optimistic about the existence of an approximating line since the logarithmic scale usually optically decreases the error. The line and its slope β can be determined by linear regression.

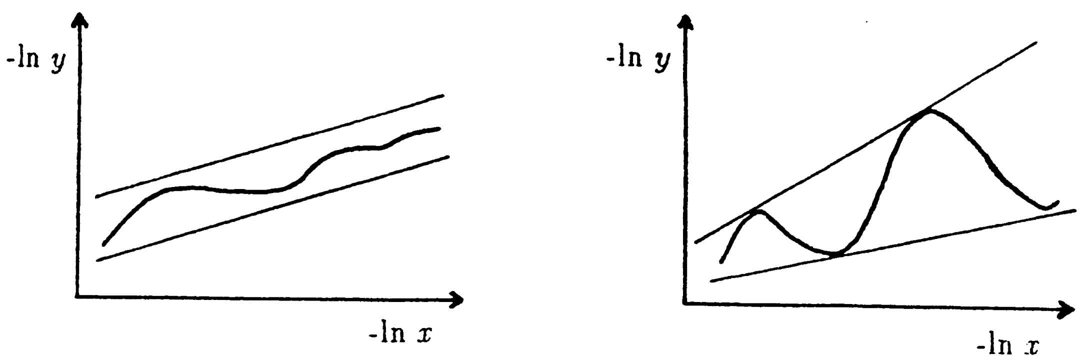


fig. 9

If two quantities $y(x)$, $z(x)$ differ at most by a constant factor, then the corresponding dimensions (or upper and lower dimensions) coincide.

When we have some kind of self-similarity, we will try to derive a law of the type

$$y(rx) = s \cdot y(x) \text{ with } r, s < 1, \text{ which implies } \beta = \frac{\ln r}{\ln s}.$$

EXERCISE 4.1. Verify these statements.

Thus fractal exponents are usually quotients of logarithms of "magnifying factors" of two quantities.

4.2. Simple Concepts of Dimension.

The *metric dimension* or capacity of a set B in a metric space (X, d) studies the change of the covering number $M(B,t)$ in dependence on t:

$$\dim_C B = \lim_{t\to 0} \frac{\ln M(B,t)}{\ln t} .$$

There are a number of variations. If B is in R^n, we may prefer to cover by cubes from a given grid (say with dyadic coordinates) rather than with balls or arbitrary sets. This leads to the box dimension which plays an important role in the numerical study of attractors. It coincides with $\dim_C$.

EXERCISE 4.2. What happens if we take packing numbers instead of covering numbers?

The *Minkowski dimension* of a set $B \subset R^n$ is obtained from the scaling of the Lebesgue measure of the t–neighbourhood (or parallel set) $U(B,t) = \{v|d(v,B) < t\}$ of B.

$$\dim_M B = n - \beta \text{ with } \beta = \lim_{t\to 0} \frac{\ln \lambda(U(B,t))}{\ln t}$$

When we imagine that a line B has cylinders with radius t, and a plane area has plates of thickness t as neighbourhoods, we see why $n-\beta$ is considered as dimension rather than β. In fact it is easy to prove that Minkowski and metric dimension coincide.

Other kind of exponents studied mainly by physicists concern random walk (see sec. 5), resistance [32], or the dimension of shortest paths [9, 32]. Note that for all these quantities the power law (and the existence of a limit) is only an assumption which has to be proved when one wants to be rigorous. Otherwise there are upper and lower exponents.

When one has the possibility to generate points randomly with respect to some probability distribution μ, as in the numerical study of attractors, and also in the random algorithm for self–similar measures, one can consider the *correlation dimension*. To this end, we consider the distance function $d(x, y)$ as a random variable d on the probability space $(X \times X,\ \mu \times \mu)$ and define its distribution function $F(t) = P[d(x, y) < t]$. Practically, $F(t)$ is the relative frequency of pairs of distance $< t$ among all pairs (x, y) of generated points. Then

$$\dim_2 \mu = \lim_{t \to 0} \frac{\ln F(t)}{\ln t}\ .$$

This is already a parameter which characterizes a measure rather than a set.

There is a simpler parameter of measures, the *pointwise dimension* $d_x(\mu)$. It is the mass exponent which expresses the scaling of measure in a ball around x.

$$d_x(\mu) = \lim_{t \to 0} \frac{\ln \mu(B(x, t))}{\ln t}\ .$$

The problem with this simple parameter is that it depends on x. There are cases where $d_x(\mu)$ is constant for almost all x.

EXERCISE 4.3. Prove that for all points x in a self–similar set with finite positive α–dimensional Hausdorff measure, $d_x(\mu) = \alpha$.

4.3. HAUSDORFF AND PACKING DIMENSION.

Both Hausdorff and packing dimension are refined ways to measure the size (dimension) of sets, and both can be refined further by taking Hausdorff functions h instead of numbers α or by taking the respective measures. An advantage is that these dimensions always exist (no upper or lower Hausdorff dimension).

The definition uses the fact that $\mu^\alpha(B) < \infty$ implies $\mu^\beta(B) = 0$ for all $\beta > \alpha$. (To remember: area and volume of a curve of finite length is zero.) Conversely, $\mu^\beta(B) > 0$ implies $\mu^\alpha(B) = \infty$ for all $\alpha < \beta$. Thus there is a unique number $\alpha = \dim_H(B)$ such that $\mu^\beta(B)$ is zero for $\beta > \alpha$ and infinite for $\beta < \alpha$. This is the Hausdorff dimension, and the packing dimension $\dim_P(B)$ is defined in the same way using the packing measures ν^β.

It is not easy to determine these dimensions for given sets. There is a complicated theory [26, 27] which will not be treated here. It is easy to show that

$$\dim_H(B) \leq \dim_P(B) \leq \dim_C(B)$$

(if the latter does not exist, take the upper dimension). There are examples where strict inequality holds, but for self–similar sets with open set condition all dimensions coincide. The equality of Hausdorff and packing dimension was used by Tricot as a definition of "fractals" [23].

As concerns Hausdorff dimension of finite measures, $\dim_H(\mu)$ is often defined as the minimum Hausdorff dimension of a set of full measure. Usually, this is considerably smaller than the dimension of the support, since sets of full measure need not be closed. If all sets of positive measure do already have this dimension, the term "carrying dimension" of a measure has also been used. In general, however, a finite measure μ can be so non–uniform that one can consider the pointwise dimension as a random variable (with respect to μ itself) and study its distribution function. This was done by Cutler. We recommend her survey [19] which also treats statistical aspects of estimation of dimensions.

4.4. Multifractals.

In recent years, first physicists and then mathematicians have become aware of the necessity to study the "non–uniformities" of singular ("fractal") measures more thoroughly. Numerical computations of the local dimension show that this quantity may depend heavily of the point x. If this is so, one should distinguish those x where the measure is very thin and those where it is concentrated. Unfortunately, in typical cases both kinds of points lie dense in the support of the measure – so they cannot be separated from each other. It is the mixture of thin and fat places which makes the measure non–uniform.

It was then suggested by a number of different authors to take the dimension of all points with pointwise dimension α, and to study the dependence of this function from α. The function

$$f(\alpha) = \dim_H(D_\alpha), \text{ where } D_\alpha = \{x | d_x(\mu) = \alpha\}$$

is called the multifractal spectrum of μ. Typically, it has a parabolic shape, with a unique maximum which gives the Hausdorff dimension of the support of μ.

Physicists have determined multifractal spectra for strange attractors, turbulence, random resistor networks and diffusion–limited aggregation. For the mathematician, however, there are some problems left: can one neglect those points for which the pointwise dimension does not exist? Is it really true that $f(\alpha)$ must be a smooth function – is it not possible that some D_α are just empty? Is it true that the Legendre transform associates $f(\alpha)$ with the family of so–called Renyi dimensions?

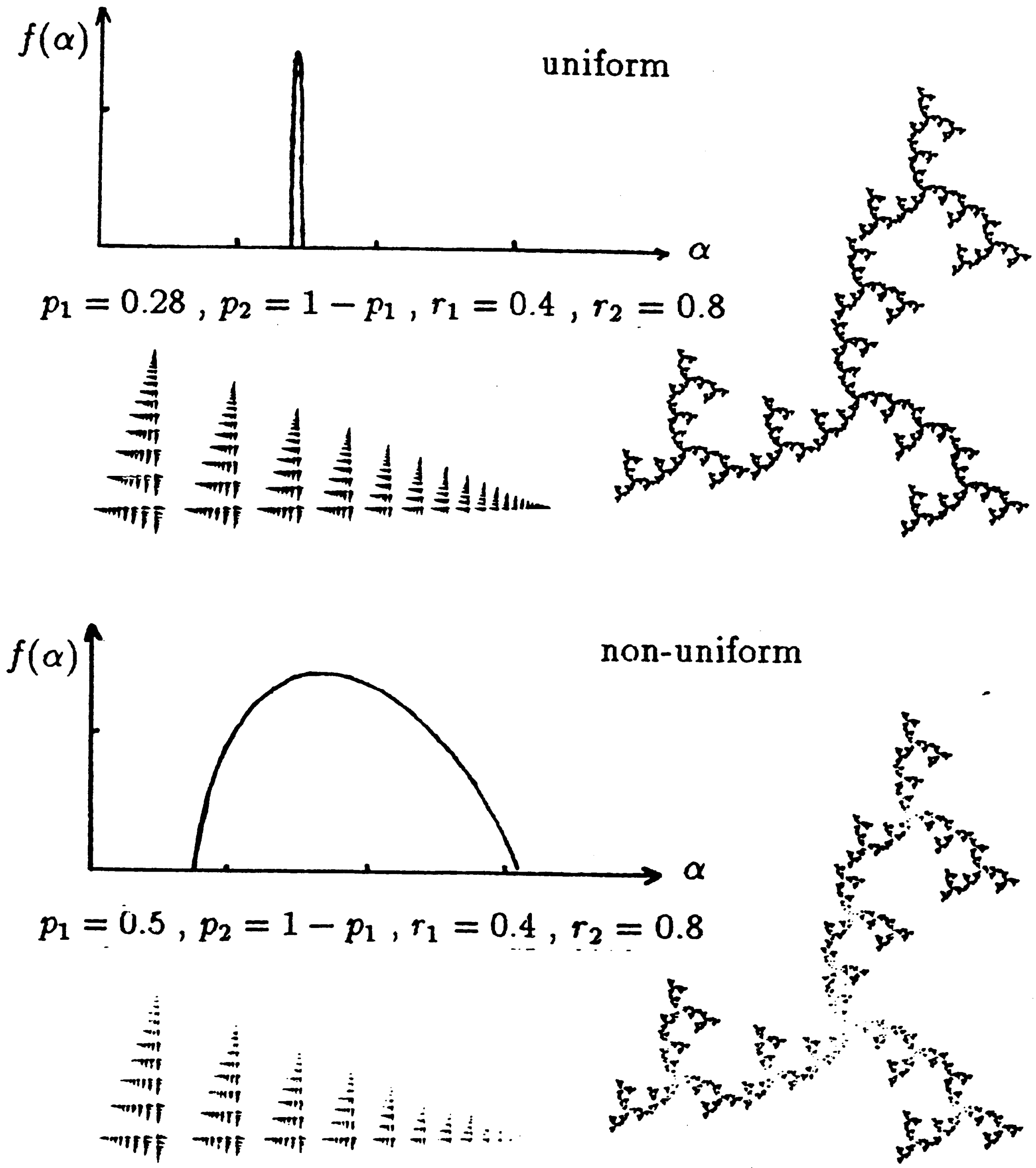


fig. 10

For particular cases, there are now rigorous proofs which confirm the physicists conjectures. For hyperbolic Cantor set attractors on the real line see the papers of Collet et al. [18] and Rand [47]. For the case of self–similar measures, which we study in these lectures, there is a rigorous investigation by Cawley and Mauldin [17]. A simple discussion can also be found in [27].

THEOREM 4.4 (Self–similar measures as multifractals) *Let $f_1, \dots, f_m$ be similarities fulfilling the open set condition, and let $p_1, \dots, p_m \geq 0$ with $\sum p_i = 1$. For each $q \in R$, let $\beta(q)$ be the unique number for which $\sum_i p_i^q r_i^\beta = 1$. Let $\alpha(q) = -\beta'(q)$. Then α runs through all values of an interval $[\alpha_1, \alpha_2]$ and for each of these α holds* $\dim_H(D_\alpha) = q\alpha(q) + \beta(q)$ *which can theoretically be written as a function $f(\alpha)$.*

Instead of inserting technical details here, we give a few pictures of self–similar measures and their $f(\alpha)$–function (fig. 10). For arbitrary measures the situation is complicated. Aversa and Bandt [1] considered a discrete measure which has a linear multifractal spectrum, but only with the lower pointwise dimension. For the upper pointwise dimension $f(\alpha)$ would be non–zero only at one point.

I think a search for other non–uniformity parameters of a fractal measure is useful, also for practical and psychological reasons, since the parabolic shape of an $f(\alpha)$–function does not yield much information.

5. Analysis on the Sierpinski Gasket.

The Sierpinski gasket is probably the best–known fractal. Its remarkable symmetry makes it a universal model for physicists. We shall argue that it is an immediate generalization of the unit interval, and that even now one can discover new properties of this space.

5.1 BASIC PROPERTIES OF THE SIERPINSKI SPACES.

The Sierpinski gasket, shown in the fig. 11, was introduced by Sierpinski 1915 as an example of a one–dimensional continuum in which each point x (except for the three vertices) is a branching point, that is, x can be approached by three paths which are disjoint up to the point x.

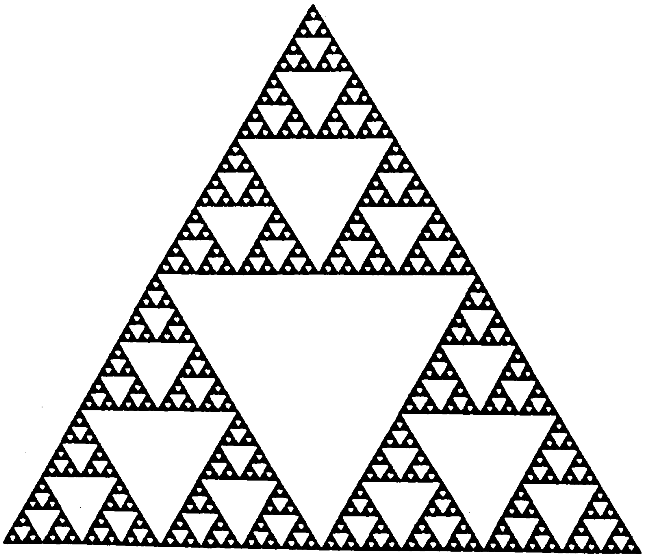


fig. 11

EXERCISE 5.1. Prove this property. Show that the vertices are not branching points.

We shall now define the Sierpinski n–space in R^n as the self–similar set $A = f_1(A) \cup \ldots \cup f_{n+1}(A)$ with respect to the mappings $f_i(x) = \frac{1}{2}(x + e_i)$, $i = 1 \ldots n+1$ where the e_i are vertices of a regular n–simplex [36]. Note that the Sierpinski 1–space is an interval, the 2–space is the gasket. For convenience we work with barycentric coordinates, i.e. we take R^n as hyperplane $\{x = (x_1, \ldots, x_{n+1}) \,|\, \sum x_i = 1\}$ and the e_i as coordinate unit vectors in R^{n+1}, so that $f_i(x) = \frac{1}{2}(x_1, \ldots, x_{i-1}, 1 + x_i, x_{n+1})$. When we use binary representations $x_j = 0.s_{1j}s_{2j}\ldots$ of the coordinates, $s_{ij} \in \{0, 1\}$, we see that f_i shifts all digits of the x_j one place to the right, and it puts 1 as the first digit in x_i and 0 as the first digit of all other x_j. Now remember from sec. 2 that a self–similar set contains the points which may be approached by repeated application of the f_i.

This leads to the following alternative definition of the Sierpinski n–space which says that in the binary number system, $\sum x_i = 1$ for $x \in A$ holds in a special way:

$$A = \{x \in R^{n+1} \,|\, \text{for each } m > 0,\ \text{there is } i_m \text{ with } s_{mi_m} = 1$$

$$\text{and } s_{mj} = 0 \text{ for } j \neq i_m\}$$

So each point of A is given by a sequence $i_1 i_2 \ldots$ There is a slight ambiguity in using binary numbers since $0.0111\ldots = 0.1$ Consequently, two sequences $ijjj\ldots$ and $jiii\ldots$ describe the same point. These points are called critical points.

The topology of a self–similar fractal A for which all $A_i \cap A_j$ with $i \neq j$ are finite can be best defined by listing those sequences in C which are mapped to the $A_i \cap A_j$. We called this description "generating rules" in [6]. For the Sierpinski n–space, they have the simplest form: $ijjj\ldots \sim jiii\ldots$, $1 \leq i < j \leq n+1$.

A curious property of these spaces is that the topology of A completely determines the fractal structure, for $n \geq 2$ [8]. In fact, the critical points can be characterized by certain separation properties. Moreover, each one–to–one continuous map from A into A is a similarity map with respect to the Euclidean metric! In other words, each homeomorphism from A onto a subspace of A is the composition of an isometry of the simplex and a mapping f_w. It turns out that also in many other cases, "fractal structure" is a topological phenomenon, not a metric one [8].

Now a few words on "shortest paths" in the Sierpinski n–space. A metric d on a space X is called *interior metric* if for each x, y there is $z \neq x, y$ with $d(x,y) = d(x,z) + d(z,y)$. Since we deal with compact spaces, this implies existence of a path of length $d(x,y)$ between x and y. The Euclidean metric d_e on A induces an interior metric $d(x,y) = \inf\{d_e$–lengths of paths within A between x and $y\}$. (The paths connecting two points in A are polygons with infinitely many sides, so their length is well–defined.) We shall normalize d so that the diameter of A (the side length of the simplex) is 1.

EXERCISE 5.2. Using the above remarks, prove that the metric d_i is the unique (up to a constant factor) interior metric on A which transforms each homeomorphism into a similarity. (This is very interesting. I do not know any other spaces in topology which have a canonical metrization.)

Since the open set condition is fulfilled (with the interior of the basic simplex), the Sierpinski n–space has Hausdorff dimension $\alpha = \frac{\ln(n+1)}{\ln 2}$. The α–dimensional Hausdorff measure on A is positive and finite. The corresponding probability measure will be called μ in the sequel.

5.2. Two Simple Renormalization Schemes.

Self–similarity is the basis for a method which physicists call *renormalization* and which we already know from dimension calculations. We assume a power law, and we establish a connection between two consecutive levels. Here is one example. What is the average interior distance $\bar{d} = Ed(x, e_1)$ of a point $x \in A$ from the vertex e_1? This is a question in geometric probability. You can first solve it for $n = 1$, the interval. Dividing A into the A_i, gives a partition of the probability space (A, μ), the overlap does not matter. The conditional expectation of $d(x, a)$ on A_1 is $\frac{\bar{d}}{2}$. On all other A_i, we have $d(x, e_1) = d(x, a) + d(a, e_1) = d(x, a) + \frac{1}{2}$ with $\{a\} = A_1 \cap A_i$ which leads to the conditional expectation $\frac{1}{2} + \frac{\bar{d}}{2}$. Since each A_i has probability $\mu(A_i) = \frac{1}{n+1}$, we obtain

$$\bar{d} = \frac{\bar{d}}{2}\frac{1}{n+1} + (\frac{1}{2} + \frac{\bar{d}}{2}) \cdot \frac{n}{n+1}$$

which yields $\bar{d} = \frac{n}{n+1}$.

For the average distance of two points of the Sierpinski gasket a slightly more complicated calculation gives $\frac{466}{885}$ [33]. This calculation can also be done for the n–space, and it gives a value of about $\frac{n}{n+1} - \frac{1}{2n^2}$ [7]. This is very near to 1. Thus we can say that in a higher–dimensional Sierpinski space, the distance between most pairs of points is almost equal to the diameter of A. For a higher–dimensional Euclidean ball or sphere this is not true.

Exercise 5.3. Determine the average distance of two points in [0, 1], using renormalization.

Our second example of renormalization was done by physicists who prefer the discrete setting of the outward construction [32]. Imagine a random walk starting in the edge e_1. Assuming that the average time t needed to reach another vertex a of A_1 is finite, we determine the average time t' needed to reach one of the vertices $e_2, \ldots, e_{n+1}$. Let t_a be the average time required to reach some e_j, $j \neq 1$, from a vertex of type a, and t_b the time when starting in a vertex b of A_j, $j \neq 1$, which is not a vertex of A or A_1. (Draw a figure!) Now it is clear that $t' = t + t_a$ since each path from

e_1 to some other e_j passes through some point of type a. Moreover, if we assume that if we start the random walk from a vertex of $A_i \cap A_j$, then all other vertices of A_i and A_j have the same probability $\frac{1}{2n}$ to be met next, we have two other equations

$$t_a = t + \frac{1}{2n}(t' + (n-1)t_a + (n-1)t_b + 0), \quad t_b = t + \frac{1}{n}(t_a + (n-2)t_b + 0)$$

which give the solution $t_b = (n+1)t,\ t_a = (n+2)t,\ t' = (n+3)t$.

Thus to come twice as far, a random walker on the Sierpinski n–space must walk $n+3$ times longer. This contrasts with a random walk on the Euclidean lattice Z^n. Starting in 0 there, after k steps $u_1, \ldots, u_k$ we reach the point $u = u_1 + \ldots + u_k$ with average norm square $E||u||^2 = E\langle u, u\rangle = E(||u_1||^2 + \ldots + ||u_k||^2) = k$. (Since the u_j are independent and assume the value $+e_i$ and $-e_i$ with the same probability, $E(u_j u_{j'}) = 0$.) Thus to move twice as far in Euclidean space of *any* dimension, one has to walk four times longer. As was to be expected, walking on fractals is more difficult.

PROBLEM 5.4. Analyse non–symmetric random walk on Sierpinski spaces. (Some results were obtained recently by Kumagai in Osaka.)

In the spirit of sec. 4, if one assumes the power law $t = s^\gamma$, where $s = Es(t)$ is the average distance a random walker has moved from its starting place after time t, then γ is called the *random walk dimension.* We have $\gamma = 2$ in R^n and $\gamma = \frac{\ln\,(n+3)}{\ln\,2}$ for the Sierpinski n–space.

PROBLEM 5.5. Find the random walk dimension for other self–similar sets. (Very few of them are known.)

5.3. HARMONIC FUNCTIONS.

Harmonic functions play a central role in analysis. You probably remember that their characteristic property says that $h(x_0)$ is the average of the values $h(x)$ on any sphere with midpoint x_0. They were also defined on graphs: a function from the vertex set to R is harmonic if the value at any vertex is the average of the values on the neighbouring vertices. This was the starting point for Kigami [36] and Metz to define harmonic functions on the Sierpinski spaces. Our discussion will focus on the gasket.

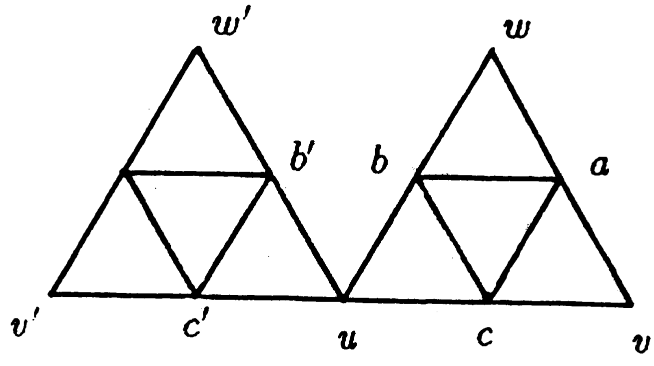


fig. 12

Suppose the values u, v, w of a real harmonic function h on A at the vertices are given and we want to find the values a, b, c at the remaining vertices of the A_i (see fig. 12). The mean value property gives the equations

$$4a - b - c = v + w, \quad -a + 4b - c = u + w, \quad -a - b + 4c = u + v$$

which are solved by $a = \frac{1}{5}(u + 2v + 2w)$, $b = \frac{1}{5}(2u + v + 2w)$ and $c = \frac{1}{5}(2u + 2v + w)$. Now if we go to the next stage and determine the values of the function at the vertices of the A_{ij}, there is a remarkable property of the gasket: the larger system of equations will reproduce those values of the first stage which we have already determined.

To see this, suppose the vertex with u belongs to another triangle with values v', w', a', b', c' which fulfil analogous equations. Assuming that $4u = v+w+v'+w'$ we shall show that $4u = b+c+b'+c'$. Adding the second and third of the above equations to the first, we get $5(b+c) = 4u+3v+3w$ and similarly, $5(b' + c') = 4u + 3v' + 3w'$. Adding these two equations and using the assumption we get what we want.

Thus one can construct harmonic functions on all vertices of the pieces of the k–th level successively, always using the values already constructed at the level before. This gives the values of h on a dense set, and if h is uniformly continuous there then we can extend h to all of A. But from the above equations we get $b - c = \frac{1}{5}(w - v)$, $b - u = \frac{1}{5}(2(w - u) + v - u)$ and $c - u = \frac{1}{5}(2(v - u) + w - u)$. Thus the difference of the values of two vertices of A_1 is $\frac{3}{5}$ times smaller than the difference on A. This implies $|h(A_w)| \leq (\frac{3}{5})^k \cdot |h(A)|$ for all $w \in S^k$. It is now not hard to show that

h is even Hölder continuous with exponent $\beta = \gamma - \alpha = \frac{\ln 5/3}{\ln 2}$. Another consequence of the above equations is the "maximum principle".

THEOREM 5.6 (The Dirichlet problem on the Sierpinski gasket). *Given values u, v, w for the vertices of the Sierpinski gasket, there is exactly one continuous function h on A which assumes the given values and is harmonic in the graph–theoretical sense when restricted to all vertices of the A_w, $w \in S^k$. The maximum of h is assumed at a vertex of A.*

PROBLEM 5.7. Define harmonic functions on other self–similar sets, and on self–similar measures on the Sierpinski gasket.

5.4. FURTHER RESULTS AND QUESTIONS.

Above, we have dealt with random walk on the Sierpinski "lattice" and on Z^n, but a mathematician better likes the calculus of Brownian motion. In R^n, Brownian motion was constructed as a limit of random walks on refining lattices, and the result $\gamma = 2$ is a basic property of Brownian motion. There were several attempts to construct Brownian motion on the Sierpinski gasket in a similar way, the most successful and thorough one by Barlow and Perkins [10]. They found transition densities with respect to μ, defined a Laplacian as infinitesimal generator of the Brownian motion semigroup and gave a lot of precise estimates, among others for the eigenvalues of the Laplacian.

So far, few results go beyond Sierpinski spaces. Lindstrøm [40] and Kigami [37] defined Brownian motion and harmonic calculus, respectively, on certain self–similar sets with pieces which intersect in finitely many points. We proved that interior metric on such spaces can be described by a suitable Hausdorff measure for the connecting paths. Barlow and Bass tried to define Brownian motion on the Sierpinski carpet. There are forthcoming papers by Barlow, Fukushima (who applies the theory of Dirichlet forms), and Kumagai. In this context, it would be helpful to develop a mathematical concept for electrical resistance (cf. [21]).

These are just the first steps of fractal analysis. Functional analysis must enter, appropriate function spaces on fractal measures must be defined. One should not aim at the rather particular deterministic fractals – which now serve well as a first model for the development of the proper tools – but at the more realistic random fractals.

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