

BOUNDS FOR THE DEGREE OF THE ENTRIES OF GENERALIZED INVERSES OF POLYNOMIAL MATRICES (*)

by A. FABIANO and V. MARINO (in Cosenza) (**)

SOMMARIO. - *Si determinano limiti superiori per i gradi degli elementi di alcune inverse generalizzate di matrici polinomiali.*

SUMMARY. - *Bounds for the degrees of the entries of some generalized inverses of polynomial matrices are found.*

1. Introduction.

In recent years many papers have been written to find explicit solutions of the well known Bezout equation

$$(B) \quad P_1 Q_1 + \dots + P_m Q_m = 1$$

where $P_i (i = 1, \dots, m)$ are given polynomials in C^n and $Q_i (i = 1, \dots, m)$ are unknown polynomials to be determined (for a review on this topic see Berenstein and Struppa [1988, 1991] and given references). In view of possible applications this subject seems to be a very interesting one. In fact equation (B) appears in the study of multivariable control systems and in problems concerned with image reconstruction techniques. For instance, in several control system problems, it often arises the need to find the generalized inverse of a given polynomial matrix. For this specific applications it is important to provide bounds for the degrees of the polynomial entries

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(**) Indirizzo degli Autori: Dipartimento di Matematica – Università degli Studi della Calabria – Arcavacata di Rende (CS) (Italy).

of the matrices. Many results have been obtained for providing bounds not only for the polynomials Q_i of equation (B) [Berenstein and Struppa 1984, 1986, 1988, 1991. Brownawell 1987, Gentili and Struppa 1987, Kollar 1988] but also for the entries of a left inverse and for a $\{1\}$ -inverse of a polynomial matrix [Fabiano and Struppa 1990]. The linear algebra of several generalized inverses of suitable matrices has been already developed [Bose and Mitra 1978, Sontag 1980, Bhaskara Rao 1983, Bapat, Bhaskara Rao and Manjunatha Prasad 1990, Manjunatha Prasad, Bhaskara Rao and Bapat 1991]. In this paper, by combining complex analysis with algebraic techniques, we give the bounds for the degrees of the polynomial entries of some generalized inverses for a polynomial matrix.

The paper is organized as follows: Section 2 is devoted to the basic definitions and the basic theorems which will be applied in the sequel. In section 3 we improve the bounds for the degrees of the polynomial entries of a $\{1\}$ -inverse of a polynomial matrix which has been reported in Fabiano and Struppa [1990]. We also provide the same bounds for a $\{1, 2\}$ -inverse of a polynomial matrix A . In the last section we propose a bound for the degree of the numerators of the entries of the Moore-Penrose inverse of A . Finally, bounds for the degree of the numerators of the entries of the group-inverse of a square matrix are also given.

2. Generalized inverses: definitions and basic theorems.

Let \mathbf{K} be a field (\mathbf{R} or \mathbf{C}). Let $A = [a_{ij}]$ be an $m \times n$ matrix over \mathbf{K} . A left (or right) inverse of A is an $n \times m$ matrix X over \mathbf{K} such that $XA = I$, I being the $n \times n$ identity matrix (or $AX = I$, I being the $m \times m$ identity matrix). More generally one can consider the Moore-Penrose equations [Ben-Israel and Greville 1974]:

- 1) $AXA = A$
- 2) $XAX = X$
- 3) $(AX)^t = AX$
- 4) $(XA)^t = XA$

where t denotes the transpose.

If X is an $n \times m$ matrix satisfying equation 1), then X is called a $\{1\}$ -inverse (or $\{g\}$ -inverse) of A . A matrix A which has a $\{1\}$ -inverse

is called *regular*. If X is an $n \times m$ matrix satisfying equations 1) and 2), then X is called a $\{1, 2\}$ -inverse, or weak generalized inverse (W.G.I.), or reflexive g -inverse of A . An $n \times m$ matrix satisfying equations 1) ... 4) is called a *Moore-Penrose inverse* or generalized inverse (G.I.) of A . In general one can denote by $X^{(i, \dots, 1)}$ an $n \times m$ matrix X which satisfies only equations $i), \dots, 1)$ among the whole set of equations 1), ... 4). Usually the Moore-Penrose inverse of a given matrix A is denoted by A^\dagger and it can be shown that, if this last exists, it is necessarily unique.

It is obvious that, if A is a non singular matrix, one finds that the matrix $X = A^{-1}$ satisfies all conditions 1) ... 4). If A is a square matrix one can consider also the following equation

$$5) \quad AX = XA.$$

If X is a matrix which satisfies conditions 1), 2) and 5), then X is called the *group inverse* of A .

For a given matrix A , several theorems exist which permit to discover conditions for the existence of a $X^{(i, \dots, 1)}$ [Ben-Israel and Greville 1974]. If \mathbf{K} denotes some particular ring, then the study of the existence of $X^{(i, \dots, 1)}$ is more complicate. For some particular cases, we have the first careful study of the generalized inverses in a paper of Bose-Mitra [1978]. In fact, by using reduction to the Smith form, they characterize those matrices which admit a $\{1, 2\}$ -inverse in the case of the ring of the integers and of the polynomials in a single variable. Successively, Sontag [1980] gives a complete characterization of the above mentioned matrices in the case of polynomials of several variables. More precisely he proves the

THEOREM 2.1. *The following statements are equivalent for any matrix A over $R = \mathbf{C}[z_1, \dots, z_n]$:*

- a) A has a W.G.I.;
- b) there exist square unimodular matrices P, Q over R such that $A = PA_0Q$, with

$$A_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r denotes the identity matrix of order $r = \text{rank}(A)$;

- c) as a function of (z_1, \dots, z_n) the rank of A is a constant.

For a matrix A defined over an integral domain R , Bhaskara Rao

[1983] gives the necessary and sufficient condition for the existence of a $\{1\}$ -inverse. He states:

THEOREM 2.2. *Let A be an $m \times n$ matrix with $\text{rank}(A) = t$ and let $C_t(A)$ denote the t -th compound matrix of A . Then the following propositions are equivalent:*

- a) A is regular;
- b) $C_t(A)$ is regular;
- c) a linear combination of all the $t \times t$ minor of A is equal to one.

Moreover it is interesting to note that, if $b = [g_{ij}]$ is a $\{1\}$ -inverse of A ,

$$g_{ji} = \frac{\partial}{\partial a_{ij}} \left(\sum_{\alpha} \sum_{\beta} |A_{\beta}^{\alpha}| c_{\alpha\beta} \right)$$

with $c_{\alpha\beta} \in R$ and satisfying the condition:

$$\sum_{\alpha, \beta} c_{\alpha\beta} |A_{\beta}^{\alpha}| = 1$$

and where the notation $\frac{\partial}{\partial a_{ij}} |A_{\beta}^{\alpha}|$ denotes the coefficient of $a_{i,j}$ in the expansion of the determinant $|A_{\beta}^{\alpha}|$, A_{β}^{α} being the (α, β) entry of $C_t(A)$.

Recently Bapat, Bhaskara Rao and Manjunatha Prasad [1990] show that if G is a $\{1, 2\}$ -inverse of a given matrix A and if $c_{\alpha\beta} = |G_{\alpha}^{\beta}|$, then theorem 2-2 holds and gives a way to construct the $\{1, 2\}$ -inverse of A . In fact it is shown the

THEOREM 2.3. *Let $A = [a_{ij}]$ be a $m \times n$ matrix with $\text{rank}(A) = t$ and let $G = [g_{ij}]$ be reflexive g -inverse of A . Then for all i, j*

$$g_{ji} = \sum_{\alpha: i \in \alpha} \sum_{\beta: j \in \beta} |G_{\alpha}^{\beta}| \frac{\partial}{\partial a_{ij}} |A_{\beta}^{\alpha}|$$

where α, β run over all t -elements subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$ respectively.

We recall now some results which are concerned with the Moore-Penrose inverse A^{\dagger} of a matrix A . When A is defined over a field, the

condition for the existence of A^\dagger are well known. If A is a matrix defined over some particular ring, the existence and the characterization of A^\dagger (or G.I.) are known only for some particular cases. Sontag [1980] gives the characterization of the G.I. of a matrix A defined over the ring R of rational functions with real coefficient and no real poles. Bhaskara Rao [1983] characterizes all those matrices A over $Z[x]$, those over the ring $Z[x_1, \dots, x_n]$ of the polynomials in several variables with integral coefficients and those over the ring $R[x, y]$ of polynomials in the variables x and y over the field of the real numbers, which admit the Moore-Penrose inverse A^\dagger .

The problem of the existence of A^\dagger , when A is a matrix with entries in an integral domain R , has been recently solved by Bapat, Bhaskara Rao and Manjunatha Prasad [1990] with the following

THEOREM 2.4. *Let A be an $m \times n$ matrix with $\text{rank}(A) = t$. Then the following propositions are equivalent:*

- a) A has a Moore-Penrose inverse;
- b) $C_t(A)$ has a Moore-Penrose inverse;
- c) $u = \sum_{\alpha, \beta} |A_\beta^\alpha|^2$ is invertible in R .

Here α, β run over all t -elements subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$ respectively and, as before, $C_t(A)$ denotes the t -th compound matrix of A .

The same authors give also an explicit construction of A^\dagger by stating that the Moore-Penrose inverse, when it exists, is given by $G = [g_{ij}]$, where

$$g_{ji} = \sum_{\alpha: i \in \alpha} \sum_{\beta: j \in \beta} u^{-1} |A_\beta^\alpha| \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha|$$

and

$$u = \sum_{\alpha, \beta} |A_\beta^\alpha|^2. \quad (2.1)$$

Furthermore, Manjunatha Prasad, Bhaskara Rao and Bapat [1991] give the necessary and sufficient conditions for the existence of the group inverse $A^\#$ of a given matrix A defined over an integral domain. More precisely, they state:

THEOREM 2.5. *Let A be an $n \times n$ matrix with $\text{rank}(A) = t$ over an integral domain R . Then the following propositions are equivalent:*

- a) A has a group inverse;
- b) $C_t(A)$ has a group inverse;
- c) $\sum_{\gamma} |A_{\gamma}^{\gamma}|$ is invertible in R ;
- d) $\text{rank}(A) = \text{rank}(A)^2$ and $(A)^2$ is regular.

THEOREM 2.6. *Let A be an $n \times n$ matrix with $\text{rank}(A) = t$ over an integral domain R . Then:*

- a) *If $u = \sum_{\gamma} |A_{\gamma}^{\gamma}|$ is invertible in R , then $G = [g_{ij}]$, defined by $g_{ji} = \sum_{\gamma} u^{-1} \frac{\partial}{\partial a_{ij}} |A_{\gamma}^{\gamma}|$ is a commuting $\{1\}$ -inverse of A , that is G satisfies conditions 1) and 5);*
- b) *If $u = \sum_{\gamma} |A_{\gamma}^{\gamma}|$ is invertible in R , then $G = [g_{ij}]$, defined by*

$$g_{ji} = \sum_{\alpha, \beta} u^{-2} |A_{\alpha}^{\beta}| \frac{\partial}{\partial a_{ij}} |A_{\beta}^{\alpha}| \tag{2.2}$$

is the group inverse $A^{\#}$ of A .

3. Bounds for the entries of the $\{1\}$ -inverses and of the $\{1, 2\}$ -inverses matrices.

The starting point of our investigation is the final remark of Fabiano, Stuppa [1990]. First we observe that it is possible to improve the bounds given in theorems 1 and 2 by Fabiano, Stuppa [1990]. To be more precise we state:

THEOREM 3.1. *Let $A = [a_{ij}]$ be an $m \times r$ matrix whose entries are polynomials in n variables, $m \geq r$ and suppose that A has a constant rank $t < r$. Then A has a $\{1\}$ -inverse $G = [g_{ij}]$ such that the degrees of the polynomial entries satisfy:*

$$\text{deg}(g_{ij}) \leq d^{nt} + d^{t-1} = d^{t-1} (d^{t(n-1)+1} + 1)$$

where $d = \max_{i,j} \text{deg}(a_{ij})$.

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_t)$, $\beta = (\beta_1, \dots, \beta_t)$, with $1 \leq \alpha_j \leq m$ and $1 \leq \beta_j \leq r$, be increasing multindices, and let $|A_\beta^\alpha|$ be the $t \times t$ minor of A corresponding to the rows and to the columns $(\alpha_1, \dots, \alpha_t)$, $(\beta_1, \dots, \beta_t)$ respectively. Since $\text{rank}(A) = t$, using theorem 2.2, we can find polynomials $c_{\alpha\beta}$ such that

$$\sum_{|\alpha|=|\beta|=t} c_{\alpha\beta} |A_\beta^\alpha| = 1$$

with $\max_{\alpha, \beta} \deg(c_{\alpha\beta}) \leq d^{nt}$ [Brownawell 1987]. (For details on this formula, see Bhaskara Rao [1983], Fabiano, Struppa [1990]). Bhaskara Rao [1983] shows that a $\{1\}$ -inverse $G = [g_{ij}]$ of A can be constructed by taking

$$g_{ji} = \sum_{\alpha, \beta} c_{\alpha\beta} \frac{\partial}{\partial a_{ij}} |A_\beta^\alpha|.$$

Fabiano, Struppa [1990] show that

$$\deg(g_{ij}) \leq d^{nt} d^{t-1} = d^{(n+1)t-1}.$$

In fact it is sufficient to give the bound

$$\deg(g_{ij}) \leq d^{nt} + d^{t-1} = d^{t-1} (d^{t(n-1)+1} + 1).$$

REMARK 1. This bound can also replace the bound which appears in theorem 1 given by Fabiano and Struppa [1990].

In an analogous way now we give bounds for the polynomial entries of $\{1, 2\}$ -inverses. It holds the following

THEOREM 3.2. *Let $A = [a_{ij}]$ be a $m \times n$ polynomial matrix with constant rank $(A) = t$ and let $G = [g_{ij}]$ be a $\{1, 2\}$ -inverse of A . Then the entries g_{ij} satisfy*

$$\deg(g_{ij}) \leq d^{nt} + d^{t-1}$$

where $d = \max_{i,j} \deg(a_{ij})$.

Proof. Theorem 2.3 gives the construction of the entries g_{ij} of a $\{1, 2\}$ -inverse G of A :

$$g_{ji} = \sum_{\alpha:i \in \alpha} \sum_{\beta:j \in \beta} c_{\alpha\beta} \frac{\partial}{\partial a_{ij}} |A_{\beta}^{\alpha}|$$

where $c_{\alpha\beta} = |G_{\alpha}^{\beta}|$. Therefore we can state the bounds $\deg(g_{ji}) \leq d^{nt} + d^{t-1}$.

4. On the Moore-Penrose inverse and on the group inverse of a polynomial matrix.

We now want to calculate the bounds for the degrees of the polynomials which appear as the entries of the matrix A^{\dagger} and of the matrix $A^{\#}$ in the case that $A = [a_{ij}]$ is a matrix with elements in an integral domain R , in particular when $R = \mathbb{C}[z_1, \dots, z_n]$ is the ring of the polynomials in the variables z_1, \dots, z_n with coefficients in \mathbb{C} .

In this section we will consider a variant of the Moore-Penrose inverse and group inverse. Indeed, Theorem 2.4 shows that a Moore-Penrose inverse essentially never exists in a polynomial ring, since the request $u = \sum_{\alpha,\beta} |A_{\beta}^{\alpha}|^2$ being invertible in R is never satisfied. However, we now discuss a matrix G with entries in $\mathbb{C}(z_1, \dots, z_n)$ which satisfies conditions 1), ..., 4) of §2. We will (improperly) refer to this matrix as the Moore-Penrose inverse of A . Analogous considerations allow us to refer to a matrix G with entries in $\mathbb{C}(z_1, \dots, z_n)$ which satisfies conditions 1), 2), 5) of §2 as the group inverse of A .

REMARK. It is obvious that the matrix A^{\dagger} , if it exists, is also a $\{1\}$ -inverse, a $\{1, 2\}$ -inverse, and so on, of A . Therefore, we can utilize the known bounds on the entries of G just by looking at G as a $\{1\}$ -inverse, a $\{1, 2\}$ -inverse, and so on.

If the conditions for the existence of A^{\dagger} holds, just by following an analogous procedure to the one used by Fabiano and Struppa [1990], we can state:

THEOREM 4.1. *Let $A = [a_{ij}]$ be an $n \times m$ matrix with polynomial entries; suppose that A has a constant rank t ; suppose that $u = \sum_{\alpha, \beta} |A_{\beta}^{\alpha}|^2$ is an invertible element in $\mathbb{C}(z_1, \dots, z_n)$. Then the Moore-Penrose inverse $A^{\dagger} = [g_{ij}]$ of A has rational entries satisfying the condition*

$$\deg(N_{ji}) \leq d^{mt} + d^{t-1}$$

where $d = \max_{i,j} \deg(a_{ij})$ and N_{ji} denotes the numerator of the rational function g_{ji} .

Proof. Theorem 2.4 and formula (2.1) allow to explicitly construct the entries of A^{\dagger} . In fact, from (2.1), we recognize that the entries g_{ji} are rational function in (z_1, \dots, z_n) ; then it is possible for the numerator N_{ji} of g_{ji} to give the bounds

$$\deg(N_{ji}) \leq d^{mt} + d^{t-1}.$$

The study of the poles of the entries g_{ij} will be the object of further investigations.

If A is an $n \times n$ matrix with constant rank t over an integral domain, then theorem 2.5 states that the group inverse $A^{\#}$ of A exists if and only if $\sum_{\gamma} |A_{\gamma}^{\gamma}|$ is a non-zero constant in R when γ runs over all the t -element subsets of $\{1, \dots, n\}$. Thus we can state:

THEOREM 4.2. *Let $A = [a_{ij}]$ be an $n \times n$ matrix with constant rank t over the integral domain $R = \mathbb{C}[z_1, \dots, z_n]$. Let $u = \sum_{\gamma} |A_{\gamma}^{\gamma}|$ be non identically zero. Then the group inverse $A^{\#} = [g_{ji}]$ has rational entries satisfying*

$$\deg(N_{ji}) \leq d^{mt} + d^{t-1}$$

where N_{ji} denotes the numerator of the rational function g_{ji} and $d = \max_{i,j} \deg(a_{ij})$.

Proof. Theorem 2.5 and formula (2.2) allow to explicitly construct the entries of the matrix $A^{\#}$. Firstly we observe that $A^{\#}$ is a $\{1\}$ -inverse commuting with A . Then it is possible to found coefficient $c_{\alpha\beta}$ such that

$$\sum_{|\alpha|=|\beta|=t} c_{\alpha\beta} |A_{\beta}^{\alpha}| = 1$$

$$(c_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta \text{ and } c_{\alpha\alpha} = u^{-1})$$

with $\max_{\alpha,\beta} \deg(c_{\alpha\beta}) < d^{nt}$ [Brownawell 1987].

Finally, formula (2.2) gives

$$\deg(N_{ji}) \leq d^{nt} + d^{t-1}.$$

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