FIBER SHAPE THEORY (*)

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SOMMARIO. - *Scopo di questo lavoro è di dare una nozione di "fiber shape" per un arbitrario spazio topologico sopra uno spazio metrizzabile* B.

SUMMARY. - *The purpose of this paper is to exhibit a notion of "fiber shape" for arbitrary topological spaces over metrizable space* B.

1. Introduction. The notion of fiber shape, recently introduced by the author ([Ba$_1$], [Ba$_2$], [Ba$_3$]), M. Clapp and L. Montejano ([CM]), H. Kato ([Ka]), S.C. Metcalf ([Me]), Y. Yagasaki ([Ya]), is a modification of homotopy type of maps ([Sp]).

The purpose of this paper is to exhibit a notion of fiber shape for arbitrary topological spaces over metrizable space B.

In our development we follow the method of ANR_B-resolutions, i.e. resolutions of spaces over B, consisting of ANR_B-spaces ([Do], [Ja], [Sc], [Ya]).

The fiber shape category Sh_B is by definition the general shape category $Sh_{(\mathcal{T},\mathcal{P})}$ ([MS]), where $\mathcal{T} = [Top_B]$ and $\mathcal{P} = [ANR_B]$.

2. Notations and preliminaries. We use the following notations. Let B denote the fixed space of the category C spaces and maps. The space X over B is a pair consisting of a topological space X and a continuous mapping $\pi_X : X \to B$. Let X and Y be spaces over B. A map $f : X \to Y$ is said to be a fiber preserving (f.p.) map, if $\pi_Y \cdot f = \pi_X$. By C_B we denote

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the category of all spaces over B and all f.p. maps.

Let $C = Top$ be the category of all topological spaces and maps.

Two f.p. maps $f : X \to Y$ and $g : X \to Y$ of category Top_B is said to be fiber preserving (f.p.) homotopic, $f \underset{B}{\simeq} g$, if there is a homotopy $H : X \times I \to Y$ from f to g, such that $\pi_Y \cdot H = \pi_{X\times I}$, where $\pi_{X\times I}(x, t) = \pi_X(x)$ for every $x \in X$ and $t \in I$. The relation $\underset{B}{\simeq}$ is an equivalence relation and we denote by $[f]_B$ the homotopy class of f.p. map f. The relation $\underset{B}{\simeq}$ is compatible with the composition. Therefore, one can define the composition of class $[f]_B : X \to Y$ and $[g]_B : Y \to Z$ by composing representatives:

$$[g]_B \circ [f]_B = [g \cdot f]_B .$$

$[Top_B]$ denotes the fiber homotopy category of Top_B. Its objects are all the objects of Top_B and the morphisms are equivalence classes with respect to $\underset{B}{\simeq}$ of morphisms in Top_B. Two spaces over B X and Y are said to be fiber homotopy equivalent, $X \underset{B}{\simeq} Y$, if there exists two f.p. maps $f : X \to Y$ and $g : Y \to X$ such that $g \cdot f \underset{B}{\simeq} 1_X$ and $f \cdot g \underset{B}{\simeq} 1_Y$.

3. Retracts and extensors of spaces over B. Let B be a fixed metrizable space and $\mathcal{M}_B$ the category of all metrizable spaces over B and all f.p. maps.

Let X be a metrizable space over B and Y a subspace of X. A f.p. map $r : X \to Y$ is called a fibrewise retraction if $r \cdot i = 1_Y$, where $i : Y \to X$ is the f.p. inclusion map. In this case the subspace Y is called a fiber retract of X.

A subspace Y of metrizable space X over B is called a fibrewise neighborhood retract of X if there exists a neighborhood U of Y in X and fibrewise retraction $r : U \to Y$.

The space $Y \in \mathcal{M}_B$ is an absolute retract over B (an absolute neighborhood retract over B), if Y has the following property: for any closed f.p. embedding $i : Y \to X \in \mathcal{M}_B$ there exists a fibrewise retraction $r : X \to i(Y)$ (a neighborhood U of $i(Y)$ in X and a fibrewise retraction $r : U \to i(Y)$).

Let AR_B (ANR_B) be the category consisting of all absolute (neighborhood) retracts over B.

By $[AR_B]$ ($[ANR_B]$) we denote the fiber homotopy category of category AR_B (ANR_B).

The space $Y \in \mathcal{M}_B$ is an absolute extensor over B (an absolute neighborhood extensor over B), if has the following property: for any space $X \in \mathcal{M}_B$ and any closed subset $A \subseteq X$, every f.p. map $f : A \to Y$ admits a f.p. extension $\tilde{f} : X \to Y$ ($\tilde{f} : U \to Y$, where U is a neighborhood of A in X).

By AE_B (ANE_B) we denote the category consisting of all absolute (neighborhood) retracts over B.

The following proposition is proved by T. Yagasaki ([Ya]).

PROPOSITION 1 ([Ya]). A metrizable space Y over B is an ANR_B if and only if Y is an ANE_B.

PROPOSITION 2 (Comp.[Ya], proposition 1.1). For every metrizable space X over B there exists an ANE_B-space M over B with weight

$$w(M) \leq \max\{w(X), w(B), \chi_0\}$$

and there exists a f.p. embedding $i : X \to M$ such that $i(X)$ is closed in M.

Let the class of objects of category C is weakly hereditary ([Hu]).

The space Y over B is an absolute (neighborhood) extensor for the category C_B, $Y \in AE(C_B)$ ($Y \in ANE(C_B)$), if has the following property: for any space $X \in C_B$ and any closed subset $A \subseteq X$ every f.p. map $f : A \to Y$ admits a f.p. extension $\tilde{f} : X \to Y$($\tilde{f} : U \to Y$, where U is a neighborhood of A in X).

Generalizing propositions of ([Hu], ch. II) we have the following results.

PROPOSITION 3. If $Y \in AE(C_B)$, then $Y \in ANE(C_B)$; if $X_\alpha \in AE(C_B)$, $\alpha \in \mathcal{A}$ ($X_i \in ANE(C_B)$, $i = \overline{1,n}$), then the product of category C_B $\prod_{\alpha\in\mathcal{A}} X_\alpha \in AE(C_B)$($\prod_{i=1}^{n} X_i \in ANE(C_B)$); if Y is open subspace of space over B $X \in ANE(C_B)$ then $Y \in ANE(C_B)$.

PROPOSITION 4. Let C be the category of normal spaces. If Y_1

and Y_2 be open subsets of space Y over B and $Y = Y_1 \cup Y_2, Y_1, Y_2 \in ANE(C_B)$ $(Y_1, Y_2 \in AE(C_B))$, then $Y \in ANE(C_B)(Y \in AE(C_B))$.

PROPOSITION 5. Let C be the category of normal spaces. Let $Y = Y_1 \cup Y_2$. If Y_1, Y_2 be closed subsets of space Y over B and $Y, Y_1 \cap Y_2 \in ANE(C_B)$ $(Y, Y_1 \cap Y_2 \in AE(C_B))$, then $Y_1, Y_2 \in ANE(C_B)$ $(Y_1, Y_2 \in ANE(C_B))$.

PROPOSITION 6. Let C be the category of completely normal spaces.

Let $Y = Y_1 \cup Y_2$. If Y_1, Y_2 be closed subsets of space Y over B and $Y_1, Y_2, Y_1 \cap Y_2 \in ANE(C_B)$ $(Y_1, Y_2, Y_1 \cap Y_2 \in AE(C_B))$, then $Y \in ANE(C_B)$ $(Y \in AE(C_B))$.

Let $U = \{u_\alpha\}_{\alpha \in A}$ be a covering of space Y. We say that the maps $f, g : X \to Y$ are U-near, if for every $x \in X$ there exists a $u_\alpha \in U$ such that $f(x), g(x) \in u_\alpha$. We say that a homotopy $H : X \times I \to Y$, which connects f and g, is a U-homotopy if for every $x \in X$ there exists a $u_\alpha \in U$ such that $H(x,t) \subseteq u_\alpha$ for all $t \in I$.

PROPOSITION 7. (Comp. [F-Ch], proposition 1.2, Ch. III). Let Y be an ANR_B. Then every open covering U of Y admits an open covering V of Y such that any two V-near f.p. maps $f, g : X \to Y$ from an arbitrary space X over B into space Y over B are f.p. U-homotopic. Moreover, if for a given $x \in X$, $f(x) = g(x)$, then $H(x,t) = f(x)$ for every $t \in I$, where H is a f.p. homotopy from f to g.

T. Yagasaki in ([Ya]) showed the following proposition.

PROPOSITION 8. ([Ya]). Let $Y \in ANR_B$. Let A be a closed subspace of metrizable space X over B. Let $f, g : X \to Y$ be f.p. maps and let $H : A \times I \to Y$ be a homotopy over B from $f_{|A}$ to $g_{|A}$. Then there exists a neighborhood U of A in X and a homotopy over B $H : U \times I \to Y$ from $f_{|U}$ to $g_{|U}$.

Let $C(I, Y)$ be a function space with compact-open topology. It is known that if Y is a metric space then the compact-open topology on $C(I, Y)$ agrees with the topology induced by the metric:

$$d(\varphi, \psi) = \sup\{d(\varphi(t), \psi(t)) \mid \varphi, \psi \in C(I, Y), t \in I\}.$$

Let Y be a space over B. Consider the subspace $C_B(I,Y)$ of space $C(I,Y)$: $C_B(I,Y) = \{\varphi \in C(I,Y) | \pi_Y \ \varphi = \text{const}\}$.

Let $\pi_{C_B(I,Y)} : C_B(I,Y) \to B$ be a map given by

$$\pi_{C_B(I,Y)}(\varphi) = \pi_Y(\varphi(t)), \quad t \in I .$$

Consequently, the pair consisting of space $C_B(I,Y)$ and map $\pi_{C_B(I,Y)}$ is the space over B.

PROPOSITION 9. Let Y be an ANE_B-space. Then the space $C_B(I,Y)$ over B is an ANE_B-space.

These proposition are used in sections 4 and 5.

4. Resolutions of spaces over B. An inverse system of category Top_B is a collection $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathcal{A}\}$ of spaces X_α over B indexed by a directed set $\mathcal{A}$ and f.p. maps $p_{\alpha\alpha'} : X_{\alpha'} \to X_\alpha$ for each pair $\alpha \leq \alpha'$, such that $p_{\alpha\alpha'} \cdot p_{\alpha'\alpha''} = p_{\alpha\alpha''}$ for every $\alpha \leq \alpha' \leq \alpha''$ and $p_{\alpha\alpha} = 1_{X_\alpha}$ for every $\alpha \in \mathcal{A}$.

A morphism $\underline{f} = \{f_\beta, \varphi\} : \underline{X} \to \underline{Y} = \{Y_\beta, q_{\beta\beta'}, \mathcal{B}\}$ of inverse system of category Top_B consists of a function $\varphi : \mathcal{B} \to \mathcal{A}$ and f.p. maps $f_\beta : X_{\varphi(\beta)} \to Y_\beta$ such that whenever $\beta \leq \beta'$ then there is an index $\alpha \geq \varphi(\beta), \varphi(\beta')$ for which $f_\beta \cdot p_{\varphi(\beta)\alpha} = q_{\beta\beta'} \cdot f_{\beta'} \cdot p_{\varphi(\beta')\alpha}$.

Two morphisms $\underline{f} = \{f_\beta, \varphi\}, \underline{g} = \{g_\beta, \psi\} : \underline{X} \to \underline{Y}$ are said to be equivalent, $\underline{f} \underset{B}{\simeq} \underline{g}$, provided that for each $\beta \in \mathcal{B}$ there is an $\alpha \in \mathcal{A}$, $\alpha \geq \varphi(\beta), \psi(\beta)$, such that $f_\beta \cdot p_{\varphi(\beta)\alpha} = g_\beta \cdot p_{\psi(\beta)\alpha}$.

Let pro-Top_B be a category, whose objects are the inverse system $\underline{X}$ of category Top_B and whose morphisms are equivalence classes $[\underline{f}]_B$ relative to $\underset{B}{\simeq}$ of morphisms $\underline{f} : \underline{X} \to \underline{Y}$.

A morphism $\underline{p} = \{p_\alpha\}_{\alpha\in\mathcal{A}} : X \to \underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathcal{A}\}$ from a rudimentary system (X) to an inverse system $\underline{X}$ consists of f.p. maps $p_\alpha : X \to X_\alpha$, $\alpha \in \mathcal{A}$, such that $p_\alpha = p_{\alpha\alpha'} \cdot p_{\alpha'}$, $\alpha \leq \alpha'$.

DEFINITION 1. Let X be a space over B and $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathcal{A}\}$ be an inverse system of category Top_B. We say that $\underline{p} : X \to \underline{X}$ is a resolution over B of space X over B provided that it satisfies the following two conditions:

R_B 1) Let $P \in ANR_B$ and U be open covering of P and $h : X \to P$ a f.p. map. Then there exist a $\alpha \in A$ and a f.p. map $f : X_\alpha \to P$ such that h and $f \cdot p_\alpha$ are U-near.

R_B 2) Let $P \in ANR_B$ and U be open covering of P. Then there exists an open covering U' of P with the following property: if $\alpha \in A$ and $f, f' : X_\alpha \to P$ are f.p. maps such that the f.p. maps $f \cdot p_\alpha$ and $f' \cdot p_\alpha$ are U'-near, then there exists a $\alpha' \geq \alpha$ such that f.p. maps $f \cdot p_{\alpha\alpha'}$ and $f' \cdot p_{\alpha\alpha'}$ are U-near.

If in a resolution over B $\underline{p} : X \to \underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ of space X over B each X_α is an ANR_B, then we say that $\underline{p}$ is an ANR_B-resolution over B.

The next theorem is essential to the construction of the fiber shape category.

THEOREM 1. Every space X over metrizable space B admits an ANR_B-resolution over B.

In the proof of theorem 1 we shall need the following lemma.

LEMMA 1. Let $f : X \to Y$ be a f.p. map from topological space X over B to an ANR_B-space Y over B. Then there exists an ANR_B-space Z over B of weight

$$w(Z) \leq \max\{w(X), w(B), \chi_0\}$$

and there exist f.p. maps $g : X \to Z$ and $h : Z \to Y$ such that $f = h \cdot g$.

THEOREM 2. Let $\underline{p} : X \to \underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ be a resolution over B. If every space X_α over B is a normal space, then $\underline{p}$ has the following property:

(A1) Let $\alpha \in A$ and let G be an open set in X_α which contains $Cl(p_\alpha(X))$. Then there exists a $\alpha' \geq \alpha$ such that $p_{\alpha\alpha'}(X_{\alpha'}) \subseteq G$.

THEOREM 3. Every resolution over B $\underline{p} : X \to \underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ of space X over B has the following property:

(A2) For every normal covering U of X there is a $\alpha \in \mathcal{A}$ and a normal covering V of X_α such that $(p_\alpha)^{-1}(V)$ refines U.

In the proof of theorem 2 we shall need the following lemma.

LEMMA 2. For every normal covering U of a space X over B there exist an ANR_B-space P, an open covering W of P and a f.p. map $f : X \to P$ such that the covering $f^{-1}(W)$ refines U.

THEOREM 4. Let $\underline{p} : X \to \underline{X}\{X_\alpha, p_{\alpha\alpha'}\mathcal{A}\}$ be a morphism of pro-Top_B from space X over B to an inverse system $\underline{X}$ over B. If $\underline{p}$ has properties (A1) and (A2), then $\underline{p}$ is a resolution over B of X.

COROLLARY 1. Let M be an ANR_B-space and $X \subseteq M$ an arbitrary closed subset of M. Let $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathcal{A}\}$ be the system which consists of all the open neighborhoods X_α of X in M and let $\underline{p} = \{p_\alpha\}_{\alpha\in\mathcal{A}} : X \to \underline{X}$ consists of f.p. inclusions $p_\alpha : X \to X_\alpha$. Then $\underline{p}$ is an ANR_B-resolution over B of space X over B.

5. The fiber shape category Sh_B. In this section we give the construction of the fiber shape category. Our construction, given here, is based on the notion of resolution over B of topological spaces over metrizable space B (comp. [MS], ch. 1).

DEFINITION 2. Let X be a topological space over B, $[\underline{X}] = \{X_\alpha, [p_{\alpha\alpha'}], \mathcal{A}\}$ an inverse system in $[Top_B]$ and $[\underline{p}] = \{[p_\alpha]\}_{\alpha\in\mathcal{A}} : X \to [\underline{X}]$ a morphism of pro-$[Top_B]$. We call $[\underline{p}]$ an expansion over B of space X over B provided it has the following properties:

E_B 1) For every ANR_B-space P over B and f.p. map $h : X \to P$ there is a $\alpha \in \mathcal{A}$ and a f.p. map $f : X_\alpha \to P$ such that $f \cdot p_\alpha \underset{B}{\simeq} h$.

2) If $f, f' : X_\alpha \to P$ are f.p. maps, $P \in ANR_B$ and $f \cdot p_\alpha \underset{B}{\simeq} f' \cdot p_\alpha$, then there is a $\alpha' \geq \alpha$ such that $f \cdot p_{\alpha\alpha'} \underset{B}{\simeq} f' \cdot p_{\alpha\alpha'}$.

If all $X_\alpha \in ANR_B$ then $[\underline{p}]$ is called an $[ANR_B]$-expansion over B.

The main result of section 5 is the following theorem.

THEOREM 5. Let X be a topological space over metrizable space B. Then every resolution over B $p : X \to \underline{X}$ of X induces expansion over B $[\underline{p}] : X \to [\underline{X}]$ of X.

In the proof of theorem 5 we need the following lemma.

LEMMA 3. Let X be a topological space over metrizable space B, $P, P' \in ANR_B$, $f : X \to P'$ a f.p. map, $h_0, h_1 : P \to P$ two f.p. maps such that $h_0 \cdot f \underset{B}{\simeq} h_1 \cdot f$. Then there exists an ANR_B-space P'' over B and f.p. maps $f' : X \to P''$, $h : P'' \to P$ such that $h \cdot f' = f$ and $h_0 \cdot h \underset{B}{\simeq} h_1 \cdot h$.

The proof of lemma 3 uses the proposition 9.

COROLLARY 2. The homotopy category $[ANR_B]$ is a dense subcategory ([MS], ch. 1), of the homotopy category $[Top_B]$. The fiber shape category Sh_B of topological spaces over metrizable space B is by definition the abstract shape category $Sh_{(\mathcal{T},\mathcal{P})}$ ([MS], ch. 1), where $\mathcal{T} = [Top_B]$, $\mathcal{P} = [ANR_B]$.

The objects of the fiber shape category are all topological spaces over metrizable space B. The morphism of Sh_B from space X over B to space Y over B are given by triples $([\underline{p}], [\underline{q}], [\underline{f}])$, where $[\underline{p}] : X \to [\underline{X}]$, $[\underline{q}] : Y \to [\underline{Y}]$ are $[ANR_B]$-expansions of X and Y respectively and $[\underline{f}] : [\underline{X}] \to [\underline{Y}]$ is a morphism of pro-$[Top_B]$. In order to define a fiber shape morphism $F : X \to Y$ one chooses ANR_B-resolutions over B $p : X \to \underline{X}$ and $\underline{q} : Y \to \underline{Y}$, which exist by theorem 1 and one chooses a morphism $[\underline{f}] : [\underline{X}] \to [\underline{Y}]$ of pro-$[Top_B]$.

By theorem 1 of ([MS], ch.1, 2.1) for every morphism $[f] : X \to Y$ of $[Top_B]$ and for $[ANR_B]$-expansions $[\underline{p}] : X \to [\underline{X}], [\underline{q}] : Y \to [\underline{Y}]$, there is a unique morphism $[\underline{f}] : [\underline{X}] \to [\underline{Y}]$ of pro-$[ANR_B]$ such that $[\underline{q}] \cdot [f] = [\underline{f}] \cdot [\underline{p}]$.

If we put $Sh_B(X) = \underline{X}$ and $Sh_B([f]) = [\underline{f}]$ we obtain a covariant functor $sh_B : [Top_B] \to Sh_B$. We call sh_B the fiber shape functor.

COROLLARY 3. If $X \underset{B}{\simeq} Y$, then $sh_B(X) = sh_B(Y)$.

For the category metrizable spaces over metrizable space B and f.p. maps a fiber shape category has been previously announced by T. Yagasaki ([Ya]), who used fiber version of the R.H. Fox approach to shape ([Fo]).

THEOREM 6. The fiber shape category Sh_B coincides with the fiber shape category of Yagasaki [Ya] on the category of metrizable spaces over metrizable space B.

6. Fiber shape retracts. Let X be a metrizable space over metrizable space B. A subspace $Y \subseteq X$ is said to be a fiber shape retract if there exist a fiber shape morphism $R : X \to Y$ such that $R \cdot I = 1d_Y$, where $I : Y \to X$ is the inclusion fiber shape morphism and $1d_Y$ is the identity fiber shape morphism. A fiber shape morphism $R : X \to Y$ is called a fiber shape retraction.

A subspace Y of space X over B is said to be a neighborhood fiber shape retract if it is a fiber shape retract of a neighborhood U of Y in X.

A space Y over B is said to be an absolute fiber shape retract for metrizable spaces over B ($Y \in ASR_B$) provided that for each f.p. homeomorphism φ mapping Y onto a closed subspace $\varphi(Y)$ of a space X over B the space $\varphi(Y)$ is a fiber shape retract of X.

A space Y over B is said to be an absolute neighborhood fiber shape retract for metrizable space over B ($Y \in ANSR_B$) if for every f.p. homeomorphism φ mapping Y onto a closed subspace of a space X over B there is a neighborhood U of the space $\varphi(Y)$ in X such that $\varphi(Y)$ is a fiber shape retract of U.

Let P be a contractible space over $B(|Fe - Ch|)$. A space X over B has a trivial fiber shape, $sh_B(X) = 0$, if $sh_B(X) = sh_B(P)$.

THEOREM 7. A metric space Y over B is an ASR_B if and only if $sh_B(Y) = 0$.

THEOREM 8. A metric space Y over B is an $ANSR_B$ if and only if it is fiber shape dominated by a space $P \in ANR_B$.

7. Extensions of fiber homotopy functor's. Let Grp be the category of all groups and all homeomorphisms.

DEFINITION 3. We say that a covariant (contravariant) functor $H : Top_B \to Grp$ is continuous at $X \in Top_B$ provided that it satisfies the following condition:

if $\underline{p} = \{p_\alpha\}_{\alpha\in\mathcal{A}} : X \to \underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathcal{A}\}$ is a resolution over B of $X \in Top_B$ then $H(\underline{p}) = \{H(p_\alpha)\}_{\alpha\in\mathcal{A}} : H(X) \to H(\underline{X}) = \{H(X_\alpha), H(p_{\alpha\alpha'}), \mathcal{A}\}$ is an inverse (a direct) limit.

We say that H is continuous provided H is continuous at any $X \in Top_B$.

Let $K : ANR_B \to Grp$ be a covariant (contravariant) functor satisfying the fiber homotopy axiom, i.e. if $f \underset{B}{\simeq} g$, then $K(f) = K(g)$.

In order to define extensions of functor K we consider $[ANR_B]$-expansions $[\underline{p}] = \{[p_\alpha]\}_{\alpha\in\mathcal{A}} : X \to [\underline{X}] = \{X_\alpha, p_{\alpha\alpha'}, \mathcal{A}\}$ of every $X \in Top_B$.

Since K satisfies the fiber homotopy axiom, we obtain an inverse (a direct) system

$$K([\underline{X}]) = \{K(X_\alpha), K(p_{\alpha\alpha'}, \mathcal{A}\} .$$

We can now define the groups $\check{K}(X)(\hat{K}(X))$ of every $X \in Top_B$. By definition

$$\check{K}(X) = \underleftarrow{\lim}\, K([\underline{X}]) \qquad (\hat{K}(X) = \underrightarrow{\lim}\, K([\underline{X}])) .$$

Let $[q] : Y \to [\underline{Y}]$ be an $[ANR_B]$-expansion of $Y \in Top_B$.

By E_B 1) and E_B 2) every f.p. map $f : X \to Y$ induces a morphism $[\underline{f}] = \{f_\beta, \varphi\} : [\underline{X}] \to [\underline{Y}]$.

Consequently, $K([\underline{f}]) = \{K(f_\beta), \varphi\} : K([\underline{X}]) \to K([\underline{Y}])$ $(K([\underline{f}]) = \{K(f_\beta), \varphi\} : K([\underline{Y}]) \to K([\underline{Y}]))$ forms a system morphism. Applying the inverse (direct) limit functor, one obtains the homomorphism of groups:

$$\check{K}(f) = \underleftarrow{\lim}([\underline{f}]) : \check{K}(X) \to \check{K}(Y)(\hat{K}(f) =$$

$$\underrightarrow{\lim}\, K([f]) : \hat{K}(Y) \to \hat{K}(X)) .$$

THEOREM 9. (Comp. [Do]). If $K : ANR_B \to Grp$ be a covariant (contravariant) functor satisfying the fiber homotopy axiom, then there exists the covariant (contravariant) continuous, functor $\check{K} : Top_B \to Grp$ ($\hat{K} : Top_B \to Grp$) which is an extension of K and satisfies the fiber homotopy axiom.

THEOREM 10. (Comp. [Do]). Let $H : Top_B \to Grp$ be a continuous covariant (contravariant) functor satisfying the fiber homotopy axiom. Let $K : ANR_B \to Grp$ be the restriction of H to ANR_B. Then H and $\check{K}$ ($\hat{K}$) are naturally equivalent.

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