

# THE SPECTRUM OF THE HOPF FIBRATION (\*)

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**SOMMARIO.** - Sia  $(M, B, F, G)$  un fibrato, dove  $M$  è lo spazio totale,  $B$  la base,  $F$  la fibra e  $G$  il gruppo di struttura; tutti questi siano varietà compatte. Siano  $g, h, k$  le metriche Riemanniane su  $M, B$  ed  $F$  rispettivamente. Uno dei problemi relativi allo spettro è determinare la relazione tra  $Sp(M, g)$  e  $Sp(B, h)$ . Lo scopo di questo lavoro è di risolvere questo problema per la fibrazione di Hopf.

**SUMMARY.** - Let  $(M, B, F, G)$  be a fibre bundle, where  $M$  is the total space,  $B$  the base,  $F$  the fibre and  $G$  the structure group, which all of them are compact manifolds. Let  $g, h$  and  $k$  be the Riemannian metrics on  $M, B$  and  $F$  respectively. One of the problems of the spectrum is to determine the relation between  $Sp(M, g)$  and  $Sp(B, h)$ . The aim of this paper to solve this problem for the Hopf fibration.

1. We consider the fibre bundle  $(M, B, F, G)$ , where  $M$  is the total space,  $B$  the base,  $F$  the fibre and  $G$  the structure group, which all of them are compact manifolds. We assume that the projection  $\pi : M \rightarrow B$  is a submersion. We suppose that there are Riemannian metrics  $g, h$  and  $k$  on  $M, B$  and  $F$  respectively such that the projection map  $\pi$  is a Riemannian submersion. One of the problems of the spectrum is to determine the relation between  $Sp(M, g)$ ,  $Sp(B, h)$  and  $Sp(F, k)$ .

The whole paper contains three paragraphs. The second paragraph deals with the general theory of the  $Sp(M, g)$ ,  $Sp(B, h)$  and  $Sp(F, k)$ . This relation for the Hopf fibrations is given in the third paragraph.

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2. Let  $(M, B, F, \pi)$  be a fibre bundle

$$\begin{array}{ccc} & (F, k) & \\ i & \downarrow & \\ & (M, g) & \xrightarrow{\pi} (B, h) \end{array}$$

whose total space  $M$ , base manifold  $B$  and fibre  $F$  are compact and there are Riemannian metrics  $g$ ,  $h$  and  $k$  on  $M$ ,  $B$  and  $F$  respectively such that  $\pi$  is a Riemannian submersion. The meaning of Riemannian submersion  $\pi$  is the following. Its derivative  $\pi_*$  at  $x \in M$ , i.e.

$$(\pi_*)_x : T_x(M) \rightarrow T_{\pi(x)}(B)$$

induces an isometry from the orthogonal complement  $H_x$  of  $V_x = \text{Kern } \pi_*$  in  $T_x M$  on  $T_{\pi(x)}(B)$ .

To find the above relation we make some more assumptions. We refer some of them (i) The fibres are totally geodesic manifolds (ii) The fibres are minimal submanifolds (iii) We consider special metrics on  $M$ ,  $B$  and  $F$  which permit to find this relation.

We assume that  $M$  is the Riemannian product of two Riemannian manifolds  $B$  and  $F$  and if  $\pi$  is onto the first factor, then  $\pi$  is a Riemannian submersion with totally geodesic fibres. In this case it is known the relation of the  $Sp(M, g)$ ,  $Sp(B, h)$ ,  $Sp(F, k)$  is given by

$$Sp(M, g) = \{ \lambda + v / \lambda \in Sp(B, h), \quad v \in Sp(F, k) \}$$

and  $m(\lambda + v) = \Sigma m(\lambda) \cdot m(v)$ , where  $m(\lambda + v)$ ,  $m(\lambda)$  and  $m(v)$  are the multiplicities of  $\lambda + v$ ,  $\lambda$  and  $v$  respectively.

First we prove the following proposition.

**PROPOSITION 2.1.** *Let  $(M, g)$  be a compact orientable Riemannian manifold of dimension  $n$ . If  $\lambda$  is an eigenvalue of  $\Delta^{(M, g)}$ , then  $\lambda/\alpha$  is an eigenvalue of  $\Delta^{(M, g')}$ , where the new metric  $g'$  on  $M$  is given  $g' = \alpha g$ ,  $\alpha \in \mathbb{R}_*^+$ .*

*Proof.* We obtain a chart  $(V, \varphi)$  on  $(M, g)$  with local coordinate system  $(x_1, \dots, x_n)$ . The Laplace operator  $\Delta^{(M, g)}$  on  $(V, \varphi)$  takes the form

$$\Delta_{/V}^{(M, g)} f = -\frac{1}{\sqrt{|g|}} \sum_{j=1}^n \frac{\partial}{\partial x_j} g^{jk} \sqrt{|g|} \sum_{k=1}^n \frac{\partial f}{\partial x_k}, \quad (2.1)$$

where  $|g|$  is the determinant of the matrix  $(g_{ij})$  and  $(g^{jk}) = (g_{jk})^{-1}$  and  $(g_{ij})$  is defined by the restriction of  $g$  on  $(V, \varphi)$ , that means

$$g|_V = ds^2 = g_{jk} dx^j dx^k. \quad (2.2)$$

When we change the metric  $g$  as follows  $g' = \alpha g$ ,  $\alpha \in \mathbf{R}_*^+$ , this implies

$$g'_{jk} = \alpha g_{jk}. \quad (2.3)$$

From the definitions of  $|g|$  and  $g^{jk}$  and the relation (2.3) we obtain

$$\sqrt{|g|} = (\alpha)^{-n/2} \sqrt{|g'|}, \quad g^{jk} = \alpha g'^{jk} \quad (2.4)$$

The relation (2.1) by means of (2.4) takes the form

$$\Delta^{(M,g)} f = -\alpha \frac{1}{\sqrt{|g'|}} \sum_{j=1}^n \frac{\partial}{\partial x_j} g'^{ijk} \sqrt{|g'|} \sum_{k=1}^n \frac{\partial f}{\partial x_k} \quad (2.5)$$

which implies

$$\Delta^{(M,f)} f = \alpha \Delta^{(M,g')} f. \quad (2.6)$$

If we assume that  $f$  is an eigenfunction for  $\Delta^{(M,g)}$  with eigenvalue  $\lambda$ , then from (2.6) we get that  $f$  is also an eigenfunction of  $\Delta^{(M,g')}$  with eigenvalue  $\lambda/\alpha$ . q.e.d.

3. In this paragraph we consider the Hopf fibrations which are the following

$$\begin{array}{ccc} S^1 & & S^3 \\ \downarrow & & \downarrow \\ S^{2n+1} \rightarrow \mathbf{P}^n(\mathbf{C}) & & S^{4n+3} \rightarrow \mathbf{P}^n(\mathbf{H}) \end{array} \quad (3.1)$$

We shall study separately each case.

For the first Hopf fibration we have ([2])

$$Sp((S^1, g_0(\tau))) = \{\lambda_0(S^1(\tau)) = 1, \lambda_k(S^1(\tau)) = k^2/\tau^2 : k = 1, 2, \dots\} \quad (3.2)$$

with multiplicity

$$m_0(S^1) = 1, \quad m_k(S^1) = 2 \quad k = 1, 2, \dots, \quad (3.3)$$

where  $S^1(r)$  is the circle of radius  $r$ .

$$Sp(S^{2n+1}(1)) = \{\lambda_k(S^{2n+1}(1)) = k(k+2n) : k = 0, 1, \dots\} \quad (3.4)$$

with multiplicity

$$m_k(S^{2n+1}(1)) = \frac{2(k+n)}{\kappa} \prod_{v=1}^{k-1} \frac{2n+v}{v}. \quad (3.5)$$

We denote by  $S^m(r_1)$  the sphere of radius  $r_1$  and dimension  $m$ .

For the  $\mathbf{P}^n(\mathbf{C})$  we have ([3])

$$Sp((\mathbf{P}^n(\mathbf{C}), g)) = \{\lambda_k((\mathbf{P}^n(\mathbf{C}), g)) = \frac{k(k+n)}{n+1} : k = 0, 1, 2, \dots\} \quad (3.6)$$

with multiplicity

$$m_k((\mathbf{P}^n(\mathbf{C}), g)) = n(n+2k) \prod_{v=0}^{k-1} \left(\frac{n-v}{v+1}\right)^2 \quad (3.7)$$

where the Riemannian metric  $g$  on  $\mathbf{P}^n(\mathbf{C})$  induced by the negative of the Killing-Cartan form on  $SU(n+1)$ .

We assume that  $\lambda_k(S^{2n-1}(1))$ ,  $\lambda_k(S^1(r))$  and  $\lambda_k(\mathbf{P}^n(\mathbf{C}))$  are connected by the relation

$$\lambda_k(S^{2n-1}(1)) = \alpha \lambda_k(\mathbf{P}^n(\mathbf{C}), g) + \beta \lambda_k(S^1(r)) \quad (3.8)$$

$\alpha, \beta \in \mathbf{R}$  and  $\forall k \in \mathbf{Z}^+$ , which by means of (3.2), (3.4) and (3.6) takes the form

$$k(k+2n) = \alpha \frac{k(k+n)}{n+2} + \beta \frac{k^2}{r^2} \quad (3.9)$$

which implies

$$\frac{\alpha}{n+2} + \frac{\beta}{r^2} = 1, \quad \alpha \frac{n}{n+2} = 2n. \quad (3.10)$$

From the solution of the system (3.10) we have

$$\alpha = 2(n+2), \quad \beta = -r^2. \quad (3.11)$$

The relation (3.8) by means of (3.11) takes the form

$$\lambda_k(S^{2n+1}(1)) = 2(n+2)\lambda_k((\mathbf{P}^n(\mathbf{C}), g)) - r^2\lambda_k(S^1(r)). \quad (3.12)$$

If we take the radius  $r$  of the circle  $S^1$  in order to have the value

$$r = 1 \quad (3.13)$$

and a new metric  $g'$  on  $\mathbf{P}^n(\mathbf{C})$  which is connected with  $g$  by the relation

$$g' = \frac{1}{2(n+2)}g, \quad (3.14)$$

then (3.12) becomes

$$\lambda_k(S^{2n-1}(1)) = \lambda_k((\mathbf{P}^n(\mathbf{C}), g')) - \lambda_k(S^1(1)) \quad (3.15)$$

where  $\lambda_k((\mathbf{P}^n(\mathbf{C}), g'))$  is the  $k$ -eigenvalue for  $\Delta(\mathbf{P}^n(\mathbf{C}), g')$ .

Now, we can state the following theorem.

**THEOREM 3.1.** *Let  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbf{P}^n(\mathbf{C})$  be the Hopf fibration with fibre a circle. If the radius  $r$  of the circle  $S^1$  is given by  $r = 1$  and the metric  $g'$  on  $\mathbf{P}^n(\mathbf{C})$  is the product by  $1/2(n+2)$  of the metric  $g$  which is induced by the negative of the Killing-Cartan form on  $SU(n+1)$ , then the  $k$ -eigenvalue of  $S^{2n+1}(1)$  is the difference between the  $k$ -eigenvalue of  $(\mathbf{P}^n(\mathbf{C}), g)$  and the  $k$ -eigenvalue of  $S^1(1)$ .*

From the relation (3.12) after some estimates we obtain

$$\lambda_k((\mathbf{P}^n(\mathbf{C}), g)) = \frac{1}{2(n+2)}\lambda_k(S^{2n+1}(1)) + \frac{r^2}{2(n+2)}\lambda_k(S^1(r)), \quad (3.16)$$

which can be written

$$\lambda_k((\mathbf{P}^n(\mathbf{C}), g)) = \lambda_k(S^{2n+1}(\sqrt{1/2(n+2)})) + \lambda_k(S^1(\sqrt{1/2(n+2)})). \quad (3.17)$$

Therefore we have the corollary.

**COROLLARY 3.2.** *The  $k$ -eigenvalue of  $\mathbf{P}^n(\mathbf{C})$  with metric  $g$  induced by the negative of the Killing-Cartan form on  $SU(n+1)$  is the*

sum of  $k$ -eigenvalue of  $S^{2n+1}(\sqrt{1/2(n+2)})$  and the  $k$ -eigenvalue of  $S^1(\sqrt{1/2(n+2)})$ .

Now, we consider the other Hopf fibration for which we have ([2], [3])

$$Sp(S^3(1)) = \{\lambda_k(S^3(1)) = k(k+2)/k = 0, 1, \dots\}, \quad (3.18)$$

with multiplicity

$$m_k(S^3) = (k+1)^2 \quad (3.19)$$

$$Sp(S^{4n+3}(1)) = \{\lambda_k(S^{4n+3}(1)) = k(k+4n+2)/k = 0, 1, \dots\} \quad (3.20)$$

with multiplicity

$$m_k(S^{4n+3}) = \frac{2(k+2n+1)}{\kappa} \prod_{v=1}^{k-1} \frac{4n+2+v}{v} \quad (3.21)$$

$$Sp(\mathbf{P}(\mathbf{H})) = \{\lambda_k((\mathbf{P}^n(\mathbf{H}), g)) = \frac{k(k+2n+1)}{2(n+2)}/k = 0, 1, \dots\} \quad (3.22)$$

with multiplicity

$$m_k(\mathbf{P}^n(\mathbf{H})) = \frac{2k+2n+1}{2n+1} \prod_{v=1}^{\kappa} \frac{2n+v}{v} \prod_{s=1}^{\kappa} \frac{2n-1+s}{s} \quad (3.23)$$

where the Riemannian metric  $g$  on  $\mathbf{P}^n(\mathbf{H})$  is induced by the negative of the Cartan-Killing form on  $Sp(n+1)$ .

We assume that there exists a relation of the form

$$\lambda_k(S^{4n+3}(1)) = \alpha \lambda_k(\mathbf{P}^n(\mathbf{H}), g) + \beta \lambda_k(S^3(1)), \quad (3.24)$$

where  $\alpha, \beta \in \mathbf{R}$ , which by means (3.18), (3.20) and (3.22), takes the form

$$k(k+4n+2) = \alpha \frac{k(k+2n+1)}{2(n+2)} + \beta k(k+2). \quad (3.25)$$

Applying the same method as in the first case we obtain

$$\lambda_k(S^{4n+3}(1)) = \frac{8n(n-2)}{2n-1} \lambda_k((\mathbf{P}^k(\mathbf{H}), g)) - \frac{2n+1}{2n-1} \lambda_k(S^3(1)) \quad (3.26)$$

or

$$\lambda_k(S^{4n+3}(1)) = \lambda_k((\mathbf{P}^n(\mathbf{H}), g')) - \lambda_k(S^3(\sqrt{(2n+1)/(2n-1)})) \quad (3.27)$$

where

$$g' = \frac{2n-1}{8n(n+2)}g. \quad (3.28)$$

Now, we can state the following theorem

**THEOREM 3.3.** *Let  $S^3 \rightarrow S^{4n+3} \rightarrow \mathbf{P}^n(\mathbf{H})$  be the Hopf fibration with fibre a sphere of three dimension and radius  $r = \sqrt{(2n+1)/(2n-1)}$  and the metric  $g'$  on  $\mathbf{P}^n(\mathbf{H})$  is the product by  $(2n-1)/8n(n+2)$  of the metric  $g$  induced by the negative of the Killing-Cartan form on  $Sp(n+1)$ , then the  $k$ -eigenvalue of  $S^{4n+3}(1)$  is the difference between the  $k$ -eigenvalue of  $(\mathbf{P}^n(\mathbf{H}), g')$  and the  $k$ -eigenvalue of  $S^3(\sqrt{(2n+1)/(2n-1)})$ .*

From the relation (3.26), we obtain

$$\lambda_k((\mathbf{P}^n(\mathbf{H}), g)) = \frac{2n-1}{8n(n+2)}\lambda_k S^{4n+2}(1) + \frac{2n+1}{8n(n+2)}\lambda_k(S^3(1)) \quad (3.29)$$

or

$$\begin{aligned} \lambda_k((\mathbf{P}^n)(\mathbf{H}), g)) &= \lambda_k(S^{4n+3}(\sqrt{(2n-1)/8n(n+2)})) \\ &+ \lambda_k(S^3(\sqrt{(2n+1)/8n(n+2)})). \end{aligned} \quad (3.30)$$

Now, we can state the corollary

**COROLLARY 3.4.** *The  $k$ -eigenvalue of  $\mathbf{P}^n(\mathbf{H})$  with metric  $g$  induced by the negative of the Killing-Cartan form on  $Sp(n+1)$  is the sum of the  $k$ -eigenvalue of  $S^{4n+3}(\sqrt{(2n-1)/8n(n+2)})$  and the  $k$ -eigenvalue of  $S^3(\sqrt{(2n+1)/(2n+1)})$ .*

## REFERENCES

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